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ON FURTHER EXAMPLES OF PARTIAL KKM SPACES

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Abstract. A partial KKM space is an abstract convex space satisfying the abstract form of the KKM theorem. In the last decade we have studied a large number of properties and examples of such spaces. In the present article we introduce some new examples of partial KKM spaces such as $\mathscr{N}\mathscr{R}$-metric spaces of Amini-Fakhar-Zafarani [1], some generalizations of them, modular function spaces of Khamsi - Latif - Al-Sulami [12], tvs-cone metric spaces of Simić [27], abstract convexity spaces of Xiang-Xia-Chen [31], modular spaces of Shabanian-Vaezpour [26], convex metric spaces of Takahashi [29] studied by Jafari-Farajzadeh-Moradi-Khanh [9]. We also note that circular metric spaces of Chaipunya-Kumam [2] are not known to be partial KKM spaces.

Keywords: abstract convex space; KKM theorem; partial KKM space; Horvath spaces; ϕ_A-space.

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1. Introduction

A partial KKM space is an abstract convex space satisfying the abstract form of the KKM theorem, and a KKM space is the one also satisfying open-valued version of the form. We had shown that such spaces have a large number of important properties and interesting examples.

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The definitions, some basic facts, and some of typical examples of abstract convex spaces or KKM spaces are shown in our previous works [13-25].

Recently we introduced some relatively new examples of the KKM spaces such as hyperbolic spaces, complete continuous midpoint metric spaces, metric spaces with global nonpositive curvature (NPC), complete finite dimensional Riemannian manifolds, the $\mathbb{B}$-spaces, and some others. We showed that many of their properties can be deduced by following our previous works [13,14].

In the present article, we introduce some new examples of partial KKM spaces such as $\mathscr{N}\mathscr{R}$-metric spaces of Amini-Fakhar-Zafarani [1], certain generalizations of them, modular function spaces of Khamsi - Latif - Al-Sulami [12], tvs-cone metric spaces of Simić [27], abstract convexity spaces of Xiang-Xia-Chen [31], modular spaces of Shabanian-Vaezpour [26], and convex metric spaces of Takahashi [29] studied by Jafari-Farajzadeh-Moradi-Khanh [9]. However, circular metric spaces of Chaipunya-Kumam [2] are not known to be partial KKM spaces.

This article is organized as follows: Section 2 introduces the basic facts on abstract convex spaces, KKM maps, and multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$. In Section 3, we simply recall characterizations and properties of (partial) KKM spaces, and introduce one of the most general KKM type theorems. Section 4 devotes to introduce known examples of KKM spaces.

Section 5 devotes to introduce $\mathscr{N}\mathscr{R}$-metric spaces and certain generalizations of them. Sections 6 to 11 are given by the chronological order of the subject matters. Especially, in Section 10, we note that circular metric spaces are not known to be partial KKM spaces. In other sections, we give new examples of partial KKM spaces as listed above.

2. Preliminaries

We follow our previous works [13, 14] and the references therein.

Definition 2.1. An *abstract convex space* $(E,D;\Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D, such that, for any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E,D;\Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E;\Gamma) := (E,E;\Gamma)$.

Definition 2.2. Let $(E,D;\Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E of E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E,Z)$ [resp. $F \in \mathfrak{KO}(E,Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E,D;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E,E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Our KKM theory concerns with the study of partial KKM spaces and their applications.

Example 2.4. For typical subclasses of KKM spaces, see [14,18,21] and the references therein. We need the following:

Recall that a topological space X is *contractible* if there exist a continuous function $H : [0,1] \times X \to X$ and a point $x_0 \in X$, such that, for any $x \in X$, $H(0,x) = x_0$ and $H(1,x) = x$. Any star-shaped set, and consequently, any convex set in a topological vector space is contractible.

But there is a more general concept. A nonempty topological space X is *homotopically trivial* if for any natural number n and any continuous function $f : \vartheta\Delta_n \to X$, defined on the boundary of the standard n-dimensional simplex Δ_n, there exists its continuous extension $g : \Delta_n \to X$.

Definition 2.5. A triple $(X \supset D;\Gamma)$ is called an H-*space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X;\Gamma) := (X,X;\Gamma)$ is called a c-space by Horvath [6,7].

In case Γ is a family of homotopically trivial sets, then $(X \supset D;\Gamma)$ will be called a *Horvath space* [22] which is more general than H-spaces and becomes the following type of spaces.

Definition 2.6. A *generalized convex space* or a G-convex space $(X,D;\Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A\rangle$.

Actually, G-convex spaces have a large number of subclasses and very useful to applications, and were subject of hundreds of papers of other authors. Consequently, many authors tried to extend, imitated, reformulated or modified the concept to such as so called FC-spaces, FWC-spaces, GFC-spaces, L-spaces, M-spaces, W-spaces, simplicial spaces, spaces having property (H), convexity structures satisfying the H-condition, etc. Such spaces are unified by the following:

Definition 2.7. A ϕ_A-*space* $(X,D;\{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with $|A| = n+1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X,D;\Gamma)$ becomes an abstract convex space.

Now we have the following well-known diagram for triples $(E,D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde's convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space

$\Longrightarrow$ KKM space $\Longrightarrow$ Partial KKM space

$\Longrightarrow$ Abstract convex space.

This diagram shows main subclasses of abstract convex spaces.

3. Most general KKM type theorem

Recall that, in [13], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann-Sion minimax theorem, the von Neumann-Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [13] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [14], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [14] unifies and enlarges previously known several proper examples of such statements for particular types of partial KKM spaces.

Such basic results are consequences of a certain simple KKM type theorem.

Now we prepare to introduce one of the most general forms of the KKM type theorems.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. in 2010 showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems due to ourselves for abstract convex spaces:

Theorem C. *Let* $(E,D;\Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E,Z)$*, and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $K = Z$*;*

(ii) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ *for some* $M \in \langle D\rangle$*; or*

(iii) *for each* $N \in \langle D\rangle$*, there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$*,* $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$*.*

Note that this generalizes a large number of KKM type theorems in the literature. It is interesting to find any exceptional KKM type theorem which is not included in Theorem C.

4. Examples of KKM spaces

In our previous works [21-25] and others, we listed known concrete examples of KKM spaces other than the subclasses in the diagram in Section 2 as follows:

(1) Hyperconvex metric spaces of Aronszajn and Panitchpakdi (1956)

(2) Hyperbolic metric spaces — Kirk (1982), Reich and Shafrir (1990)

(3) Transfer FS convex map — Tian (1993) — KKM map

(4) Topological semilattices — Horvath and Llinares-Ciscar (1996)

(5) E-convex spaces — Youness (1999)

(6) Bayoumi's KKM spaces — Bayoumi (2003)

(7) Γ-convex spaces — Zafarani (2004)

(8) $\mathbb{R}$-tree — Kirk and Panyanak (2007)

(9) Horvath's convex space — Horvath (2008)

(10) $\mathbb{B}$-spaces — Briec and Horvath (2008)

(11) Connected linearly ordered spaces — Park (2007)

(12) Extended long line L^* — Park (2008)

(13) Complete continuous midpoint metric spaces — Horvath (2009)

(14) Metric spaces with global nonpositive curvature (NPC) — Niculescu - Rovenţa (2009)

(15) R-KKM spaces — Sankar Raj and Somasundaram (2012)

(16) KKM spaces of Chaipunya-Kumam — Chaipunya and Kumam (2015)

(17) Complete finite dimensional Riemannian manifolds — Park (2019)

The readers are encouraged to try to find new examples of partial KKM spaces or KKM spaces.

From now on, we introduce new partial KKM spaces by the chronological order, except Section 10. The numbers attached to Definitions, Lemmas, Theorems, and Corollaries are the ones in their original sources.

5. $\mathscr{N}\mathscr{R}$-METRIC SPACES AND GENERALIZATIONS

For any abstract convex spaces, we can define KKM maps. However, such spaces may not (partial) KKM spaces. Some authors obtained results on such cases. In this section, we give some examples related metric type spaces.

In 2005, motivated by Khamsi's earlier work [10] on hyperconvex metric spaces, Amini-Fakhar-Zafaran [1] defined for a bounded subset A of a metric space (M,d) as follows:

(i) $\mathrm{co}(A) = \bigcap\{B \subseteq M : B$ is a closed ball in M such that $A \subset B\}$.

(ii) $\mathscr{A}(M) = \{A \subseteq M : A = \mathrm{co}(A)\}$, i.e. $A \in \mathscr{A}(M)$ if and only if A is an intersection of closed balls containing A. In this case, we say that A is an *admissible set* in M.

(iii) A is called *subadmissible*, if for each $D \in \langle A \rangle$, $\mathrm{co}(D) \subseteq A$.

They obtained a Schauder type fixed point theorem for metric spaces as follows:

Corollary 2.4. [1] *Let (M,d) be a metric space and X a nonempty subadmissible subset of M. Suppose that the identity map $1_X : X \to X$ belongs to $\mathfrak{KC}(X,X)$, then any continuous map $f : X \to X$ such that* $\mathrm{cl} f(X)$ *is compact, has a fixed point.*

In our terminology, *if a nonempty subadmissible subset X is a partial KKM space, every continuous compact selfmap has a fixed point.*

The authors stated that Fakhar and Zafarani have shown that those multifunctions defined on G-convex spaces which are closed, compact and acyclic valued have the KKM property. However we noted earlier that the multimap class $\mathfrak{A}_c^\kappa$ has the KKM property.

In the following we give some examples of the metric spaces X for which it is assumed that the identity mapping $1_X : X \to X$ belongs to $\mathfrak{KC}(X,X)$; that is, X is assumed to be a partial KKM space.

Definition 2.9. [1] We say that (M,d) is an *$\mathscr{N}\mathscr{R}$-metric space*, if there exists a closed convex subset (E,ρ) of a completely metrizable Hausdorff topological vector space (V,ρ) in which

$$\rho(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \leq \max(\rho(x_1,y_1), \rho(x_2,y_2))$$

$$\text{for each } x_1,x_2,y_1,y_2 \in E,\ \alpha+\beta = 1,\ \alpha,\beta > 0$$

such that (M,d) can be isometrically embedded into (E,ρ) and there exists a nonexpansive retraction $r : E \to M$.

Every hyperconvex metric space is an $\mathscr{N}\mathscr{R}$-metric space, and every $\mathscr{N}\mathscr{R}$-metric space is a generalized convex space in the sense of Park; see [1].

Later in 2010, Turkoglu-Abuloha-Abdeljawad [30] defined KKM maps in cone metric spaces, and defined *$\mathscr{N}\mathscr{R}$-cone metric spaces* to obtain some fixed point theorems and hence generalized the results of Amini-Fakhar-Zafarani [1] in 2005.

The following is an example:

Corollary 17. [30] *Let* (M,d) *be a cone metric space and* X *a nonempty subadmissible subset of* M. *Suppose that* X *is a partial KKM space (that is, the identity map* $1_X \in \mathfrak{KC}(X,X)$*), then any continuous compact map* $f : X \to X$ *has a fixed point.*

They showed that, in every $\mathscr{N}\mathscr{R}$-cone metric space (M,d) and for any subadmissible subset X of M, the identity mapping belongs to $\mathfrak{KC}(X,X)$. This gives an example of a partial KKM space.

Moreover, in 2010, Khamsi and Hussain [11] discussed some recent results about KKM maps in cone metric spaces, and also discussed the fixed point existence results of multimaps defined on such metric spaces. In particular they showed that most of the new results are merely copies of the classical ones and do not necessitate the underlying Banach space nor the associated cone.

Theorem 4.3. [11] *Let* (M,d) *be a metric type space and* X *a nonempty subadmissible subset of* M. *Suppose that* X *is a partial KKM space (that is, the identity map* $1_X \in \mathfrak{KC}(X,X)$*), then any continuous compact map* $f : X \to X$ *has a fixed point.*

Furthermore, in 2011, Hussain and Shah [8] introduced *cone b-metric spaces* and defined KKM maps on them. The following is a sample result in [8]:

Theorem 4.4. [8] *Let* (M,D) *be a cone b-metric space and* X *a nonempty subadmissible subset of* M. *Suppose that* X *is a partial KKM space (that is, the identity map* $1_X \in \mathfrak{KC}(X,X)$*). Then any continuous compact map* $f : X \to X$ *has a fixed point.*

Note that, in the last two papers treated in this section, their authors assumed the existence of partial KKM spaces, but did not give any concrete example of them.

6. Modular function spaces of Khamsi - Latif - Al-Sulami

In modular function spaces, Khamsi - Latif - Al-Sulami [12] introduced KKM-maps and proved an analogue to Ky Fan's fixed point theorem.

They introduced $\mathfrak{R}$, a modular function space L_ρ and its convex subset, and defined as follows:

Definition 3.1. [12] Let $\rho \in \mathfrak{R}$ and $C \subset L_\rho$ be nonempty. A multimap $G : C \multimap L_\rho$ is called a KKM map if

$$conv(\{f_1, \dots, f_n\}) \subset \bigcup_{1 \le i \le n} G(f_i)$$

for any $f_1, \dots, f_n \in C$.

Theorem 3.1. [12] *Let $\rho \in \mathfrak{R}$. Let $C \subset L_\rho$ be nonempty and $G : C \multimap L_\rho$ be a KKM-map such that for any $f \in C$, $G(f)$ is nonempty and finitely ρ-closed. Then, the family $\{G(f) : f \in C\}$ has the finite intersection property.*

This shows that $(L_\rho, C; conv)$ is a partial KKM space. Therefore, it satisfies all results in our 2010 article [14]. Consequently, we have rich information on such spaces.

Their method is similar, but much complicated, to Khamsi's earlier paper on hyperconvex metric spaces in 1996 [10].

7. TVS-CONE METRIC SPACES OF SIMIĆ

In 2010, A. Sonmez [28] proved that a cone metric space is paracompact when the underlying cone is normal, and K. P. Chi and T. V. An [4] proved Dugundji's extension theorem for the normal cone metric space. Motivated by these, Simić [27] proved this in the frame of the tvs-cone spaces in which the cone does not need to be normal, and gave examples to illustrate the results.

Simić obtained the following:

Theorem 2.3. [27] *Let $(X, |||\cdot|||)$ be a cone normed space over a solid cone P and let $e \in \operatorname{int} P$. Let (X, τ_d) be a topological space with topology τ_d induced on X by the cone metric $d : X \times X \to \mathbb{R}$ with $d(x,y) = |||x-y|||$, and let q_e be the corresponding Minkowski functional of $[-e,e]$. Then, on X there exists a norm $||\cdot|| : X \to \mathbb{R}$ with $||\cdot|| = q_e \circ |||\cdot|||$, such that the topologies induced on X by cone norm $|||\cdot|||$ and norm $||\cdot||$ are equivalent. In other words, the spaces $(X, ||\cdot||)$ and $(X, |||\cdot|||)$ have the same collections of open, closed, and compact sets, and also the same convergent and Cauchy sequences, and the same continuous functions. Moreover, for arbitrary $x, y \in X$, if $|||x||| \preccurlyeq |||y|||$, then $||x|| \le ||y||$.*

Since X can be regarded as a normed space, Simić [27] obtained the following:

Corollary 2.2. [27] *Let $(X,|||\cdot|||)$ be a cone normed space, K be a nonempty subset of X, and $H : K \multimap X$ be a KKM map with closed values. If $H(x)$ is compact for at least one $x \in K$, then $\bigcap_{x\in K} H(x) \neq \emptyset$.*

With the routine reason, $(X,K;\mathrm{co})$ is a partial KKM space.

8. Abstract convexity spaces of Xiang-Xia-Chen

Xiang-Xia-Chen in 2016 [31] introduced an abstract convexity structure via an upper semicontinuous multi-valued map and established some generalized versions of the KKM lemma. By employing these general KKM lemmas, they derived some generalizations of minimax inequalities. Our aim in [19] was to show that most results in [31] are either consequences of known ones or can be stated in more general forms in the frame of abstract convex spaces in the sense of ourselves.

Definition 4.1. [31] Let Y be a compact set of a topological space, and let $q : \Delta_n \multimap Y$ be a multimap. If for each continuous map $p : Y \to \Delta_n$ (called a simplex mapping), there exists some point $x_0 \in p \cdot q(x_0)$ then we say that q has a fixed point property with respect to Δ_n and simplex mappings.

Lemma 4.2.[**4.3.**] [31] *Let Y be a metric space [resp. compact space], and let $\{F_0,F_1,\ldots,F_n\}$ be a family of closed subsets of Y. If there exists an upper semicontinuous mapping $q : \Delta_n \multimap Y$ such that*

$$q(\Delta_J) \subset \bigcup_{j\in J} F_j \ \ [\text{resp. } q(\Delta_J) \subset \mathrm{co}_{\mathscr{C}}\{y_j : j \in J\}], \ \ \forall J \subset N,$$

and q has a fixed point property with respect to Δ_n and simplex mappings. Then $\bigcap_{i=0}^n F_i \neq \emptyset$.

The present author obtained the following:

Theorem 5.2. [19] *Let $(E,D;\Gamma)$ be an abstract convex space, $G : D \multimap E$ be a KKM map with closed [resp. open] values. Suppose that for each finite subset $\{y_0,y_1,\ldots,y_n\} \subset D$, there exists an upper [resp. a lower] semicotinuous map $F : \Delta_n \multimap E$ such that*

$$F(\Delta_J) \subset \mathrm{co}_\Gamma\{y_j : j \in J\} \ \ \forall J \subset N = \{0,1,\ldots,n\}.$$

Then $\{G(y) : y \in D\}$ has the finite intersection property.

Consequently, spaces having abstract convexity structures of [31] are all KKM spaces. Moreover, Theorem 5.2 [19] generalizes and characterizes Park's ϕ_A-spaces.

9. MODULAR SPACES OF SHABANIAN-VAEZPOUR

Shabanian and Vaezpour [26] presented a modular version of KKM and generalized KKM mappings, and then established a characterization of generalized KKM mappings in modular spaces. Also they proved an analogue to KKM principle in modular spaces. Moreover, as an application, they gave some sufficient conditions which guarantee existence of solutions of minimax problems in which they got Fan's minimax inequality in modular spaces.

They began to define a modular ρ on a real linear space X and the corresponding modular space X_ρ given by

$$X_\rho = \{x \in X : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

Lemma 3.2. [26] *The mapping $f : \Delta_n \to X_\rho$, defined as*

$$f(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n} t_i x_i$$

is continuous for each $x_0, \ldots, x_n \in X_\rho$ and $n \in \mathbb{N}$ where ρ is a modular on X.

This shows that X_ρ is a ϕ_A-space of Park and hence a KKM space. Therefore, it satisfies all 25 results of our 2010 article [14].

Corollary 3.4. [26] *Let ρ be a modular on Y, X be a nonempty set of Y_ρ, and $G : X \multimap Y_\rho$ be a closed-valued map. If G is KKM, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

This shows again that $(Y_\rho, X; \mathrm{co})$ is a partial KKM space and hence, satisfies the large number of statements in [14].

Theorem 3.5. [26] *Let ρ be a modular on Y, X be a nonempty set, and $G : X \multimap Y_\rho$ be a map with closed values. Moreover, suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\bigcap_{x \in X} G(x) \neq \emptyset$ if and only if the mapping G is a generalized KKM mapping.*

Note also that a generalized KKM map of Chang-Zhang type can be regarded simply a KKM map in the abstract convex theory; see [20].

10. Circular metric spaces of Chaipunya and Kumam

In 2013, Chaipunya-Kumam [2] studied some topological nature of circular metric spaces and deduced some fixed point theorems for maps satisfying the KKM property. They also investigated the solvability of a variant of a quasi-equilibrium problem as an application.

Definition 2.1. [2] Let X be a nonempty set. A function $\delta : \mathbb{R}^+ \times X \times X \to [0,\infty]$ is said to be a *circular metric* if the following conditions are satisfied:

(C1) $\delta_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(C2) $\delta_\lambda(x,y) = \delta_\lambda(y,x)$ for all $(\lambda,x,y) \in \mathbb{R}^+ \times X \times X$;

(C3) For any $\lambda > 0$ and $x,y,z \in X$, we can find $\mu \in (0,\lambda)$ such that

$$\delta_\lambda(x,y) \leq \delta_\mu(x,z) + \delta_{\lambda-\mu}(z,y).$$

In this case, the pair (X,δ) is called a *circular metric space*.

If the inequality in (C3) holds for all $\mu \in (0,\lambda)$, we say that (X,δ) is a modular metric space.

After investigating topological properties of circular metric spaces, Chaipunya-Kumam [2] adopted the terminology of Khamsi [10] and studied some KKM property as follows:

Definition 3.4. [2] Let M be a circular metric space and X be a subadmissible subset of M. A multimap $G : X \multimap M$ is said to be a *KKM map* if for each $A \in \langle X \rangle$ we have

$$\mathrm{ad}(A) := \bigcap\{B \subset X : B \text{ is a closed ball in } X \text{ such that } B \supset A\} \subset G(A).$$

The following are typical results:

Theorem 3.9. [2] *Let (M,δ) be a circular metric space, X be a nonempty subadmissible subset of M and $F \in \mathfrak{KC}(X,X)$. If F is closed and firmly compact, then F has a fixed point.*

Corollary 3.12. [2] *Let (M,δ) be a circular metric space, Y be a topological space and X be a nonempty subadmissible subset of M. Suppose that $f : X \to X$ is continuous and $\overline{f(A)}$ is bounded and compact for all nonempty bounded subset A of X. If X is a partial KKM space (that is, $1_X \in \mathfrak{KC}(X,X)$), then f has a fixed point.*

Here Chaipunya-Kumam [2] assumed the existence of a partial KKM space.

11. CONVEX METRIC SPACES OF TAKAHASHI BY JAFARI ET AL.

Jafari et al. [9] deals with equilibrium problems in the setting of metric spaces with a continuous convex structure. They extend Fan's 1984 KKM theorem to convex metric spaces of Takahashi [29] in order to employ some weak coercivity conditions to establish existence results for suitable local Minty equilibrium problems, where the involved bifunctions are φ-quasimonotone. By an approach which is based on the concept of the strong φ-sign property for bifunctions, they obtain existence results for equilibrium problems which generalize some results in the literature.

FROM TEXT: The third approach to avoid linear structures was due to Takahashi [29] for metric spaces. Namely, he introduced a W-convex structure as follows. A metric space (X,d) with a mapping $W : X \times X \times [0,1] \to X$ such that, for every $u \in X$,

$$d(u,W(x,y;t)) \leq td(u,x)+(1-t)d(u,y) \quad \text{for } x,y \in X,$$

is called a *convex metric space* and is denoted by (X,W,d). A subset K of a convex metric space X is said to be *convex* if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$.

In order to extend the notions of the KKM map and the proper quasimonotonicity of a bifunction to convex metric spaces, they extend the notion of the convex combination of a finite number of elements of a convex metric space X by applying induction.

Definition 2.1. [9] Let $x_1,x_2,\ldots,x_n$ be a finite number of elements of a convex metric space X and $t_1,t_2,\ldots,t_n \in [0,1]$ be such that $\sum_{i=1}^n t_i = 1$. Defined the convex combination of elements $x_1,x_2,\ldots,x_n$ as $\tilde{W}(x_1,x_2,\ldots,x_n;t_1,t_2,\ldots,t_n)$ case by case. For details, see [9].

In 1984 Fan extended his 1961 KKM-Fan lemma in order to relax the compactness condition and obtained the following result which is known as Fan's 1984 KKM theorem.

Lemma 3.2. [9] *Let C be a nonempty subset of a Hausdorff topological vector space Y and $F : C \multimap Y$ be a map such that*

(i) *F is closed-valued;*

(ii) *F is a KKM map;*

(iii) *there exists a nonempty compact convex subset B of C such that $\bigcap_{x\in B} F(x)$ is compact.*

Then, $\bigcap_{x\in C} F(x) \neq \emptyset$.

Definition 3.2. [9] Let K be a convex subset of a convex metric space X and $F : K \multimap X$ be a map. We say that F is a *weak KKM map* if, for every $n \in \mathbb{N}$ and $x_1,\dots,x_n \in K$,

$$\mathrm{conv}(\{x_1,\dots,x_n\}) \subseteq \bigcup_{i=1}^{n} (\mathrm{cl}F(x_i) \cup \{x_i\}).$$

The following statement is the mentioned extension of Fan's 1984 KKM theorem:

Theorem 3.6. [9] *Let X be a convex metric space and K be a non-empty convex subset of X. Suppose that a multimap $F : K \multimap X$ satisfies the following conditions:*

(i) *F is intersectionally closed;*

(ii) *F is a weak KKM map;*

(iii) *there exists a non-empty compact convex subset B of K such that $\bigcap_{x\in B} \mathrm{cl}F(x)$ is compact.*

Then, $\bigcap_{x\in K} F(x) \neq \emptyset$.

COMMENTS: Since W and $\tilde{W}$ are continuous, $\tilde{W}$ can be regarded a continuous function $\phi_A : \Delta_{n-1} \to X$ for $A \in \langle X\rangle$ with $|A| = n$. Therefore, X can be regarded as a ϕ_A-space and hence a KKM space. Therefore, every convex metric space of Takahashi satisfies all results in our 2010 article [14].

The following is the correct form of Ky Fan's 1984 KKM theorem [5]:

Theorem 4. [5] *In a topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1,x_2,\dots,x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a non-empty subset X_0 of X such that the intersection $\bigcap_{x\in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

It is already well-known that the coercivity condition in Theorem 4 [5] is a particular form of the condition (iii) of our Theorem C.

The following are our new results:

Lemma 1. *Let E be a convex metric space and D a nonempty subset of X. Then $(E,D;\mathrm{conv})$ is a KKM space.*

As we observed already $(E,D;\mathrm{conv})$ can be regarded as a ϕ_A-space.

By putting $E=Z,\ F=1_E\in\mathfrak{KC}(E,E)$ in Theorem C, we have

Theorem 2. *Let $(E,D;\Gamma)$ be as in Lemma 1, and $G:D\multimap E$ a map such that*

(1) *G is a closed-valued KKM map; and*

(2) *there exists a nonempty compact subset K of E such that either*

(i) *$K=E$;*

(ii) *$\bigcap\{G(y)\mid y\in M\}\subset K$ for some $M\in\langle D\rangle$; or*

(iii) *for each $N\in\langle D\rangle$, there exists a compact convex subset L_N of E relative to some $D'\subset D$ such that $N\subset D'$, and*

$$L_N\cap\bigcap_{y\in D'}G(y)\subset K.$$

Then we have

$$K\cap\bigcap_{y\in D}G(y)\neq\emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $K\cap\bigcap\{G(y)\mid y\in D\}\neq\emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y)\mid y\in D\}\neq\emptyset$.*

By putting $G(x)=\mathrm{cl}F(x)\cup\{x\}$, our Theorem 2 reduces to Theorem 3.6 [4]. Moreover, note that the weak KKM maps are senseless.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A. Amini, M. Fakhar, and J. Zafarani, *KKM mappings in metric spaces*, Nonlinear Anal. **60** (2005), 1045–1052.

[2] P. Chaipunya and P. Kumam, *Topological aspects of circular metric spaces and some observations on the KKM property towards quasi-equilibrium problems*, J. Inequal. Appl. 2013 (2013), 343.

[3] T.-H. Chang and C.-L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl. **203** (1996), 224–235.

[4] K. P. Chi and T. Van An, *Dugundji's theorem for cone metric spaces*, Appl. Math. Lett. **24** (2011), 387-390.

[5] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.

[6] C. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[7] C. D. Horvath, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[8] N. Hussain and M. H. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math. Appl. **62** (2011), 1677–1684.

[9] S. Jafari, A. P. Farajzadeh, S. Moradi, and P. Q. Khanh, *Existence results for φ-quasimonotone equilibrium problems in convex metric spaces*, Optimization **66**(3) (2017) 293-310.

[10] M. A. Khamsi, *KKM and Ky Fan theorems in hyperconvex metric spaces*, J. Math. Anal. Appl. **204** (1996), 298–306.

[11] M. A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **73** (2010), 3123–3129.

[12] M. A. Khamsi, A. Latif and H. Al-Sulami, *KKM and Ky Fan theorems in modular function spaces*, Fixed Point Theory Appl. 2011 (2011), 57.

[13] S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. **69** (2008), 4352–4364.

[14] S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[15] S. Park, *A genesis of general KKM theorems for abstract convex spaces: Revisited*, J. Nonlinear Anal. Optim. **4**(1) (2013), 127–132.

[16] S. Park, *Comments on "Some remarks on Park's abstract convex spaces"*, Nonlinear Anal. Forum 20 (2015), 161–165.

[17] S. Park, *Making new KKM spaces from old*, Nonlinear Funct. Anal. Appl. **20**(4) (2015), 561–577.

[18] S. Park, *New examples of KKM spaces*, Nonlinear Anal. Forum **21**(1) (2016), 23–35.

[19] S. Park, *Remarks on abstract convexity spaces of Xiang et al.*, Nonlinear Funct. Anal. Appl. **21**(2) (2016), 249-261.

[20] S. Park, *A unified approach to generalized KKM maps,* J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **55**(1) (2016), 1–20.

[21] S. Park, *Various examples of the KKM spaces*, (presented at IWNAO2018), J. Nonlinear Convex Anal. (2019), to appear.

[22] S. Park, *From Hadamard manifolds to Horvath spaces*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **58**(1) (2019), 1–24.

[23] S. Park, *Riemannian manifolds are KKM spaces*, Adv. Theory Nonlinear Anal. Appl. **3**(2) (2019), 64–73.

[24] S. Park, *$\mathbb{B}$-spaces are KKM spaces*, J. Nonlinear Convex Anal. **20**(4) (2019), 739-746.

[25] S. Park, *Extending the realm of Horvath spaces*, J. Nonlinear Convex Anal.–Takahashi Issue 2019-03-04.

[26] S. Shabanian and S. M. Vaezpour, *The KKM theorem in modular spaces and applications to minimax inequalities,* Bull. Malays. Math. Sci. Soc. **39** (2016), 921–931.

[27] S. Simić, *A note on Stone's, Baire's, Ky Fan's and Dugundji's theorem in tvs-cone metric spaces*, Appl. Math. Lett. **24** (2011), 999–1002.

[28] A. Sonmez, *On paracompactness in cone metric spaces*, Appl. Math. Lett. **23** (2010), 494–497.

[29] W. Takahashi, *A convexity in metric space and nonexpansive mappings, I*, Kodai Math. Sem. Rep., Dept. of Math., Tokyo Institute of Technology **22** (1970), 142–149.

[30] D. Turkoglu, M. Abuloha, and T. Abdeljawad, *KKM mappings in cone metric spaces and some fixed point theorems*, Nonlinear Anal. **72** (2010), 348–353.

[31] S. Xiang, S. Xia and J. Chen, *KKM lemmas and minimax inequality theorems in abstract convexity spaces*, Fixed Point Theory Appl. 2013 (2013), 209.