A Panoramic View of the Realm of Ky Fan's 1952 Lemma

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Abstract

It is well-known that the Brouwer fixed point theorem, the weak Sperner combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem are mutually equivalent and have nearly one hundred equivalent formulations and several thousand applications. Recently, Nyman and Su [23] choose a particular form of Fan's 1952 Lemma and called it "Fan's $N+1$ lemma". They showed that the $N+1$ lemma is equivalent to Borsuk-Ulam theorem, LSB theorem, and Tucker's lemma, and directly implies the weak Sperner lemma. Therefore "Fan's $N+1$ lemma" leads equivalents of the KKM theorem and their applications. In this article, we consider an imaginary realm consisting of consequences of the $N+1$ lemma. Our aim in this article is to give a panoramic view of this realm.

1. Introduction

It is well-known that the Brouwer fixed point theorem in 1912, the weak Sperner combinatorial lemma in 1928, and the Knaster-Kuratowski-Mazurkiewicz (KKM) covering theorem in 1929 are equivalent each other and have nearly one hundred equivalent formulations; see Park [24]. It is also well known that several thousand articles are concerned with these three theorems.

In 1945, Tucker [33] discovered a very interesting combinatorial lemma which is based on the non-retraction theorem. By its use, he gave elementary and elegant proofs of various well-known topological properties of the n-sphere, such as the antipodal-point theorem of Borsuk-Ulam, that of Lusternik-Schnirelmann, and many others for the particular case $n = 2$.

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In 1952, Ky Fan [6] obtained a generalization of Tucker's combinatorial lemma, together with two new antipodal-point theorems generalizing the theorems of Borsuk-Ulam and Lusternik-Schnirelmann.

An *octahedral subdivision* of an n-sphere S^n in $\mathbb{R}^{n+1}$ is the subdivision of S^n into 2^{n+1} n-simplices by $n+1$ arbitrarily chosen orthogonal hyperplanes in $\mathbb{R}^{n+1}$ passing through the center of S^n.

Ky Fan's combinatorial lemma (Fan [6]) *Let K be a symmetric barycentric subdivision of the octahedral subdivision of the n-sphere S^n. Suppose that each vertex of K is assigned a label from $\{\pm1, \pm2, \ldots, \pm m\}$ in such a way that* (i) *labels at antipodal vertices sum to zero and* (ii) *labels at adjacent vertices do not sum to zero. Then there are an odd number of n-simplices whose labels are of the form $\{k_0, -k_1, k_2, \ldots, (-1)^n k_n\}$, where $1 \leq k_0 < k_1 < \cdots < k_n \leq m$. In particular, $m \geq n+1$.*

Later it was known that the Borsuk-Ulam theorem, the Lusternik-Schnirelmann-Borsuk (LSB) theorem, and Tucker's lemma are another triumvirate of equivalent results and imply the Brouwer theorem, the weak Sperner lemma, and the KKM theorem.

Sixty-one years later, Nyman and Su [23] choose a particular form of Fan's 1952 Lemma [6] and called it "Fan's $N+1$ lemma". They showed that $N+1$ lemma is equivalent to Borsuk-Ulam theorem, LSB theorem, and Tucker's lemma, and directly implies the weak Sperner lemma. Also recall that Fan's 1952 paper was extended to his 1999 paper [10].

Let us consider an imaginary realm consisting of consequences and applications of Fan's 1952 lemma. Since the non-retraction theorem implies Tucker's lemma, some of the consequences of the non-retraction theorem belong to this realm. Our aim in this article is to give a panoramic view of this realm.

The present author was happy to have had to review the Nyman-Su article in 2013 for Mathematical Reviews of the AMS; see MR3935127. This survey article is motivated by that review.

2. Old Mathematical Trinity

In 1910, the following Brouwer fixed point theorem appeared:

Theorem (Brouwer [2]) *A continuous map from an n-simplex to itself has a fixed point.*

In this theorem, an n-simplex can be replaced by the unit ball $\mathbb{B}^n$ or any compact convex subset in $\mathbb{R}^n$ without affecting its conclusion. This theorem appeared as Satz 4 in [2]. At the end of this article, Brouwer himself added "Amsterdam, Juli 1910".

In 1928, Sperner [30] obtained the following combinatorial lemma and its applications. For historical remarks of Sperner himself was given in [31].

Lemma (Sperner [30]) *Let K be a simplicial subdivision of an n-simplex $v_0v_1\cdots v_n$. To each vertex of K, let an integer be assigned in such a way that whenever a vertex u of K lies on a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ ($0\le k\le n$, $0\le i_0\le i_1\le\cdots\le i_k\le n$), the number assigned to u is one of the integers $i_0,i_1,\cdots,i_k$. Then the total number of those n-simplices of K, whose vertices receive all $n+1$ integers $0,1,\cdots,n$, is odd. In particular, there is at least one such n-simplex.*

The particular case is usually called the *weak Sperner lemma.*

In 1929, Knaster, Kuratowski, and Mazurkiewicz [19] obtained the following so-called KKM theorem from the weak Sperner lemma and applied it to a new proof of the Brouwer fixed point theorem:

Theorem (KKM [19]) *Let $A_i\,(0\le i\le n)$ be $n+1$ closed subsets of an n-simplex $p_0p_1\cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1}\cdots p_{i_k}\subset A_{i_0}\cup A_{i_1}\cup\cdots\cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1}\cdots p_{i_k}$ ($0\le k\le n$, $0\le i_0<i_1<\cdots<i_k\le n$), then $\bigcap_{i=0}^{n}A_i\neq\emptyset$.

In fact, those three theorems are regarded as a sort of mathematical trinity. All are extremely important and have many applications. See Park [24].

Brouwer

1974 $\swarrow$ $\nwarrow$ 1929

weak Sperner $\underset{1929}{\longrightarrow}$ KKM

3. New Mathematical Triumvirate

Similarly to the mathematical trinity introduced by Brouwer, KKM, and Sperner, relatively new Borsuk-Ulam theorem, the Lusternik-Schnirelmann-Borsuk (LSB) theorem, and Tucker's lemma are another triumvirate of equivalent results. In each of these triples, the first is an algebraic topology result, the second is a set-covering result, and the third is a combinatorial result.

Borsuk-Ulam Theorem (Borsuk [1]) *Let $h:S^n\to\mathbb{R}^n$ be a continuous function such that $h(-x)=-h(x)$ for all $x\in S^n$. Then there exists $w\in S^n$ such that $h(w)=0$.*

The following LSB covering theorem originated by Lusternik and Schnirelman in 1930 is equivalent to the Borsuk-Ulam theorem. See Borsuk [1]:

Statistics from Google Scholar (March 5, 2017):

Brouwer Fixed Point Theorem	31,100
Sperner's Lemma	5,150
KKM	80,400
KKM theorems	6,290
KKM applications	11,200
KKM general convex spaces	3,220
KKM guideline [not related]	612
KKM equilibrium problems	4,020
KKM minimax	1,820
KKM fixed point theorem	4,730
KKM space	5,060
KKM fixed point	8,610
KKM mappings in metric	2,120
KKM maps	11,500
KKM inequalities	4,760
KKM property	6,650
KKM type theorems	4,220
KKM type theorems coincidences	1,150
KKM type theorems convex spaces	2,380
KKM type theorems fixed point	3,400
KKM metric spaces	3,010

Lusternik-Schnirelman-Borsuk Theorem [1] *Let $C_1, \ldots, C_{n+1}$ be a collection of closed sets that cover S^n. Then at least one of the sets must contain a pair of antipodal points.*

From Freund-Todd [13]:

Let B^n denote the n-ball $\{x \in \mathbb{R}^n : ||x|| \leq 1\}$, where $||x||$ is the l_1-norm $\sum_i |x_i|$, and let S^{n-1} denote its boundary $\{x \in \mathbb{R}^n : ||x|| = 1\}$. We will call *special* a centrally symmetric triangulation of B^n that refines the octahedral subdivision. The following result was proved for $n = 2$ in [33] and for the general case in [20, pp.134-141] from a fundamental non-existence theorem:

Tucker's Combinatorial Lemma (Tucker [33]) *Let the vertices of a special triangulation T of B^n be assigned labels from $\{\pm 1, \ldots, \pm n\}$. If antipodal vertices of T on S^{n-1} receive complementary labels (labels that sum to zero), then T contains a complementary 1-simplex whose vertices have complementary labels.*

From Nyman-Su [23]:

Let Σ^n be a polyhedral version of the n-sphere, the set of all points in $\mathbb{R}^{n+1}$ of distance 1 from the origin in the L_1 norm:

$$\Sigma^n = \{(x_1, \ldots, x_{n+1}) : \sum |x_i| = 1\}.$$

A *triangulation* is a subdivision by simplices that either meet face-to-face or not at all. Each simplex is the affine hull of its *vertices*; these are the *vertices of the triangulation*. A triangulation of Σ^n is *symmetric* if, when σ is a simplex of the triangulation, then $-\sigma$ is a simplex as well.

Define an *m-labelling* to be a function ℓ that assigns to each vertex v one of $2m$ possible integers: $\{\pm 1, \pm 2, \ldots, \pm m\}$. A symmetric triangulation of Σ^n has an *antisymmetric* labelling if $\ell(-v) = -\ell(v)$ for all vertices v. A labelling has a *complementary edge* if some adjacent pair of vertices has labels that sum to zero, e.g., $\{+i, -i\}$. Call a simplex *alternating* if its vertex labels are distinct in magnitude and alternate signs, when arranged in order of increasing value. So the labels have the form

$$\{k_1, -k_2, k_3, \ldots\} \text{ or } \{-k_1, k_2, -k_3, \ldots\}$$

when $1 \leq k_1 < k_2 < k_3 < \ldots$. The first kind is called *positive alternating* and the second is *negative alternating*, based on the sign of k_1.

Tucker's Lemma ([33]) *Let T be a symmetric triangulation of Σ^n with an n-labelling that is anti-symmetric. Then T has a complementary edge.*

Tucker's lemma is a combinatorial equivalent of the Borsuk-Ulam theorem [1] and has many applications.

Nyman and Su [23] derived the following from the main result of Fan [6]:

Fan's $N+1$ Lemma *Let T be a symmetric triangulation of Σ^n with an $(n+1)$-labelling that is anti-symmetric and has no complementary edge. Then T has a positive alternating n-simplex.*

Fan's original lemma with m-labellings [6] is more general than the above lemma and implies the Borsuk-Ulam theorem through Tucker's lemma.

The main results of Nyman and Su [23] are the following:

Theorem 1. *Fan's $N+1$ Lemma is equivalent to the Borsuk-Ulam Theorem.*

Theorem 2. *Fan's $N+1$ Lemma implies the weak Sperner lemma.*

There is a beautiful diagram [23, Figure 1] showing connections between the topological, set-covering, and combinatorial results:

BU $\Longleftrightarrow$ LSB (Borsuk [1])

Tucker $\Longrightarrow$ BU, LSB (Tucker [33], Freund-Todd [13])

BU $\Longrightarrow$ Tucker (Freund [12])

BU $\Longrightarrow$ Brouwer (Su [32])

BU $\Longrightarrow$ Sperner (Nyman-Su [23])

LSB $\Longrightarrow$ KKM

Theorem 1: Fan's $N+1 \Longleftrightarrow$ BU
Theorem 2: Fan's $N+1 \Longrightarrow$ weak Sperner

Statistics from Google Scholar (March 5, 2017):

Tucker's Lemma	17,600
Tucker's Lemma Borsuk-Ulam theorem	197
Borsuk-Ulam theorem	3,660
Borsuk-Ulam Tucker's Lemma	262
Borsuk-Ulam Brouwer fixed point	328
Borsuk-Ulam fixed point theorem	636
Lusternik-Schnirelman	3,510
Lusternik-Schnirelman theorem	3,090

4. Equivalent Formulations of the Brouwer fixed point theorem

The following are equivalents of the Brouwer theorem stated in [24] and added some more:

1883 Poincaré's theorem
1904 Bohl's non-retraction theorem
1912 Brouwer's fixed point theorem
1928 Sperner's combinatorial lemma
1929 The Knaster-Kuratowski-Mazurkiewicz theorem
1930 Caccioppoli's fixed point theorem
1930 Schauder's fixed point theorem
1935 Tychonoff's fixed point theorem
1937 von Neumann's intersection lemma
1940 Intermediate value theorem of Bolzano-Poincaré-Miranda

1941 Kakutani's fixed point theorem

1950 Bohnenblust-Karlin's fixed point theorem

1950 Hukuhara's fixed point theorem

1952 Fan-Glicksberg's fixed point theorem

1954 Steinhaus' chessboard theorem

1955 Main theorem of mathematical economics on Walras equilibria of Gale (1955), Nikaido (1956), and Debreu (1959)

1960 Kuhn's cubic Sperner lemma

1961 Fan's KKM lemma

1961 Fan's geometric or section property of convex sets

1964 Debrunner-Flor's variational equality

1966 Fan's theorem on sets with convex sections

1966 Hartman-Stampacchia's variational inequality

1967 Browder's variational inequality

1967 Scarf's intersection theorem

1968 Fan-Browder's fixed point theorem

1969 Fan's best approximation theorems

1972 Fan's minimax inequality

1972 Himmelberg's fixed point theorem

1973 Shapley's generalization of the KKM theorem

1976 Tuy's generalization of the Walras excess demand theorem

1984 Fan's matching theorems

1997 Horvath-Lassonde's intersection theorem

1998 Greco-Moschen's KKM type theorem

It is known that many generalizations of the results in the above list are also equivalent to the Brouwer fixed point theorem. We introduce some articles on this matter.

Horvath and Lassonde [16] introduced some intersection theorems of KKM-type, Klee-type, Helly-type which are also equivalent to the Brouwer fixed point theorem.

Park and Jeong [26] collected equivalent statements closely related to Euclidean spaces, n-simplices or n-balls. Among them are the weak Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

In 2014, Idzik et al. [17] collected three sets of equivalents of the Brouwer fixed point theorem. The first set covers some classic results connected to surjectivity property of continuous functions under proper assumptions on their boundary behavior. The second covers some results related to the Himmelberg fixed point theorem. The third loop involves equivalence of the existence of economic equilibrium and the Brouwer fixed point theorem.

See also [37].

5. Some Typical Theorems

In this section, we introduce some typical examples related to the preceding list of equivalents of the Brouwer fixed point theorem.

J. von Neumann [35] obtained the following minimax theorem independently to Brouwer or Sperner, and this is one of the basic theorems in the game theory developed by himself:

Theorem (von Neumann [35]) *Let $f(x,y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0,y) \le \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x,y_0) \ge \beta$ is convex, then we have*

$$\max_{x\in K}\min_{y\in L} f(x,y) = \min_{y\in L}\max_{x\in K} f(x,y).$$

This was later extended to the following intersection lemma by himself:

Lemma (von Neumann [36]) *Let K and L be two compact convex sets in the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let us consider their Cartesian product $K\times L$ in $\mathbb{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0,y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x,y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

In order to give simple proofs of the intersection lemma and the minimax theorem of von Neumann, Kakutani obtained the following generalization of the Brouwer fixed point theorem for multi-valued maps.

A *multimap* or a *map* $F : X \multimap Y$ is a function $F : X \to 2^Y$ from a set X into the power set of a set Y, and $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Theorem (Kakutani [18]) *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

An extended real-valued function $f : X \to \overline{\mathbb{R}}$ defined on a convex set X is said to be *quasiconcave* [resp. *quasiconvex*] whenever, for any $r \in \mathbb{R}$, $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is convex.

The first remarkable generalization of the minimax theorem of von Neumann is Nash's theorem [21,22] on equilibrium for non-cooperative games. Ky Fan [8, Theorem 4] reformulated this theorem in the following elegant form:

Theorem (Nash [21,22]) *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\Pi_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \Pi_{j\neq i} X_j$, $f_i(x_1, \cdots, x_{i-1},$ $x_i, x_{i+1}, \cdots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \Pi_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

A milestone on the history of the KKM theory was erected by Ky Fan [7]. He extended the KKM theorem to infinite dimensional topological vector spaces and various applications of it:

Lemma (Fan [7]) *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually called KKMF Theorem and applied by a large number of authors in various fields in mathematical sciences. He also obtained the geometric or section property of convex sets, and it is equivalent to his KKMF theorem.

In 1966, Hartman and Stampacchia introduced the following variational inequality and applied it to partial differential equations:

Lemma (Hartman-Stampacchia [14]) *Let K be a compact convex subset in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$(f(u_0), v - u_0) \geq 0 \quad \text{for } v \in K,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Browder [3] obtained a reformulation of geometric lemma of Fan [7] in the convenient form of a fixed point theorem using the Brouwer fixed point theorem and the partition of unity argument. This is the following Fan-Browder fixed point theorem:

Theorem (Browder [3]) *Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

This was also later known to be equivalent to the Brouwer theorem.

For a topological space X , a function $f : X \to \overline{\mathbb{R}}$ is said to be *lower* [resp. *upper*] *semicontinuous* (l.s.c.) ([resp. u.s.c.]) whenever, for any $r \in \mathbb{R}$, $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open set.

Fan also obtained the following minimax inequality from his KKMF theorem:

Theorem (Fan [9]) *Let X be a compact convex set in a Hausdorff topological vector space. Let f be a real function defined on $X \times X$ such that:*

(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X.*

Then the minimax inequality

$$\min_{y\in X} \sup_{x\in X} f(x, y) \le \sup_{x\in X} f(x, x)$$

holds.

6. Historical remarks

In this section, we introduce some informative articles with their abstracts in the chronological order. These articles are supplementary sources for the history of the main subject of the present paper:

Cohen [4] : A proof of the strong form of the Sperner lemma was given.

Let $T^n = (a_0, \dots, a_n)$ be an n-simplex and K^n be a triangulation of its closure. By a labeling of K^n we mean a function $L(e_i)$ which assigns to each vertex e_i of the triangulation K^n a vertex of its carrier $L(e_i) \in T^n$. An n-simplex of such a labeled triangulation $T_i^n = (e_0, \dots, e_n)$ is said to be complete if the vertices $L(e_i)$, $i = 0, \dots, n$ are distinct.

The Strong Sperner Lemma *Any labeled triangulation K^n of T^n contains an odd number of complete n-simplices.*

FREUND, R.M. [11,12] : The author demonstrates two new dual lemmas on the n-dimensional cube, and use a generalized Sperner lemma to prove a generalization of the KKM covering lemma on the simplex. It is shown that Tucker's lemma can be derived directly from the Borsuk-Ulam theorem. A report is presented on the interrelationships between these results, Brouwer's FPT, and the existence of stationary points on the simplex.

SU, F.E. [32] : A direct proof of the Brouwer fixed point theorem from the Borsuk-Ulam antipodal theorem.

FAN, K. [10] : Obtained an antipodal theorem extending a result in his paper [6] (which strengthens slightly the antipodal theorems of LSB and BU). His new theorem implies a fixed point theorem.

SIMMONS, F.W. AND F.E. SU [28] : In this paper the authors show how theorems of Borsuk-Ulam and Tucker can be used to construct a *consensus-halving*: a division of an object into two portions so that each of n people believes the portions are equal. Moreover, the division takes at most n cuts, which is best possible. This extends prior work using methods from combinatorial topology to solve fair division problems. Several applications of consensus-halving are discussed.

PRESCOTT, T. AND F.E. SU [27] : The authors present a proof of Ky Fan's combinatorial lemma on labellings of triangulated spheres that differs from earlier proofs in that it is constructive. The authors slightly generalize the hypotheses of Fan's lemma to allow for triangulations of S^n that contain a flag of hemispheres. As a consequence, they can obtain a constructive proof of Tucker's lemma that holds for a more general class of triangulations than the usual version.

DE LONGUEVILLE, M. AND R.T. ZIVALJEVIĆ [5] : This article is concerned with a general scheme on how to obtain constructive proofs for combinatorial theorems that have topological proofs so far. To this end the combinatorial concept of Tucker-property of a finite group G is introduced and its relation to the topological Borsuk-Ulam-property is discussed. Applications of the Tucker-property in combinatorics are demonstrated.

HOHTI, A. [15] : The author proves an extension of the well-known combinatorial-topological lemma of E. Sperner to the case of infinite-dimensional cubes. It is obtained as a corollary to an infinitary extension of the Lebesgue Covering Dimension Theorem.

PARK, S. [25] : The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its 'open' version. In this paper, he clearly derives a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, he adds more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

van Dalen, D. [34] : It is by now common knowledge that in 1911 Brouwer gave mathematics a miraculous tool, the fixed point theorem, and that later in life, he disavowed it. It usually came as a shock when he replied to the question "is the fixed point theorem correct?" with a point blank "no". This rhetoric exchange deserves some elucidation. At the time that Brouwer did his revolutionary topological work, he had suspended his constructive convictions for the time being. He was well aware that he was using the principle of the excluded middle, indeed in 1919, he remarked that "In my philosophy-free mathematical papers I have regularly used the old methods, while at the same time attempting to deduce only those results, of which I could hope that they would find a place and be of value, if necessary in a modified form, in the new doctrine after the carrying out of a systematic construction of intuitionistic set theory". And in the case of the fixed point theorem we are presented with exactly such a result. From the intuitionistic point of view the theorem is not correct because the fixed point that is promised can in general not be found, that is to say, approximated.

Idzik, A., W. Kulpa, and P. Mackowiak [17] : The third section contains the before mentioned equivalents. The first set of equivalent forms covers some classic results connected to surjectivity property of continuous functions under proper assumptions on their boundary behavior. Then the authors show some results for multifunctions which are related to the Himmelberg fixed point theorem. The third loop involves equivalence of the existence of economic equilibrium and the Brouwer theorem. The last loop studies relations among simplex coverings, Maynard Smith equilibrium and Nash equilibrium.

Yu, J., N.-F. Wang, and Z. Yang [37] : The authors show that Kakutani and Brouwer's fixed-point theorems can be obtained by using Nash equilibrium theorem directly. The corresponding set-valued problems, such as Kakutani's fixed-point theorem, Walras' equilibrium existence theorem (set-valued excess demand function) and generalized variational inequality, can be derived from the Nash equilibrium theorem, with the aid of an inverse of the Berge maximum theorem. For the single-valued situation, they derive Brouwer's fixed-point theorem, Walras' equilibrium existence theorem (single-valued excess demand function), KKM lemma and variational inequality from the Nash equilibrium theorem directly, without any recourse.

We can continue more and more, but will stop here.

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