$\mathbb{B}$-SPACES ARE KKM SPACES

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ABSTRACT. A subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if for all $x_1,\ x_2 \in B$ and all $t \in [0,1]$ one has $tx_1 \vee x_2 \in B$. These sets were first investigated in [2]. In this paper, we show that any finite dimensional $\mathbb{B}$-space is a KKM space, that is, a space satisfying the abstract form of the celebrated Knaster-Kuratowski-Mazurkiewicz theorem appeared in 1929 and its open-valued version. Therefore, a $\mathbb{B}$-space satisfies a large number of the KKM theoretic results appeared in the literature.

1. Introduction

$\mathbb{B}$-convexity is an abstract convexity type. In [2], W. Briec and C. D. Horvath introduced $\mathbb{B}$-convexity and its structure. Then various works on $\mathbb{B}$-convex sets, $\mathbb{B}$-convex functions, separation in $\mathbb{B}$-convexity, halfspaces and Hahn-Banach like properties in $\mathbb{B}$-convexity were published. Its applications to the mathematical economy via data envelopment analysis were also given. For the literature, see the references of [14].

In this article, we show that any finite dimensional $\mathbb{B}$-space is a KKM space, that is, a space satisfying the abstract form of the celebrated Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem appeared in 1929 and its open-valued version. Note that Horvath's c-spaces (or H-spaces) are well-known in the history of the KKM theory originated from the KKM theorem; see [13].

Let us recall the early study on $\mathbb{B}$-convexity as noted by Briec and Horvath [3]. After the introduction of $\mathbb{B}$-convexity by Briec and Horvath [2] in 2004, for $\mathbb{B}$-convex sets, separation and Hahn-Banach like theorems were obtained by Briec, Horvath and Rubinov [4] in 2005, and Adilov and Rubinov [1] in 2006. In 2008, Briec and Horvath [3] showed that all the basic results related to fixed point theorems are available in $\mathbb{B}$-convexity, and established Ky Fan inequality, existence of Nash equilibria, and existence of equilibria for abstract economies in the framework of $\mathbb{B}$-convexity. Monotone analysis, or analysis on Maslov semimodules in [6, 7, 8] is the natural framework for these results. From this point of view Max-Plus convexity and $\mathbb{B}$-convexity are isomorphic Maslov semimodules structures over isomorphic semirings. Therefore all the results of [3] hold in the context of Max-Plus convexity.

In the present article, we are mainly concerned with the KKM theoretic results on $\mathbb{B}$-convexity in [3]. We show that $\mathbb{B}$-convex sets are c-spaces, a particular form

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of our KKM spaces, and that some of typical results in [3] are consequences of our KKM theoretic results for abstract convex spaces in [10, 12].

This article is organized as follows: Section 2 is a preliminary on our abstract convex spaces. In Section 3, we introduce the basic facts on $\mathbb{B}$-convexity given by Briec and Horvath [3]. In Section 4, we show that $\mathbb{B}$-spaces are KKM spaces and, hence they satisfy many KKM theoretic results of KKM spaces in our [10] with corrections in [12]. Especially, in Section 4, various types of general KKM theorems are given for $\mathbb{B}$-spaces. Finally, some important results in [10, 12] are applied to $\mathbb{B}$-spaces, namely, the Fan-Browder fixed point property, the Ky Fan type minimax inequality, and a generalized Nash-Fan type equilibrium theorem.

2. Abstract convex spaces

We follow our previous work [13] and the references therein:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D, such that, for any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y\in N} G(y) \qquad \text{for all} \quad N \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. A multimap $F : E \multimap Z$ to a topological space Z is called *a* $\mathfrak{K}$-*map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

For typical examples of KKM spaces and $\mathfrak{K}$-maps, see [10, 13] and the references therein. We need the following:

Definition 2.5. A triple $(X \supset D; \Gamma)$ is called an *H-space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

In case $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a c-space by Horvath in his earlier works; see [13].

Now we have the following well-known diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Now we prepare to introduce one of the most general forms of the KKM type theorems.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems in [11, 13] for abstract convex spaces:

Theorem 2.6. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that one of the following coercivity conditions hold:*
   (i) *$K = Z$;*
   (ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or*
   (iii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

This theorem includes a large number of generalizations of the original KKM theorem in 1929.

3. $\mathbb{B}$-spaces

In this section, we follow basic facts on $\mathbb{B}$-convexity given by Briec and Horvath [3].

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \min\{x_i\} \geq 0\}$. For $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n_+$, tx is the usual multiplication by a scalar; for x and y in $\mathbb{R}^n_+$ we let $x \vee y$ be the element of $\mathbb{R}^n_+$ defined by $(x \vee y)_j = \max\{x_j, y_j\}$. One can easily see that:

(A) *$(x, y) \mapsto x \vee y$ is associative, commutative and idempotent, and also continuous, and $x \vee 0 = x$ for all $x \in \mathbb{R}^n_+$.*

(B) *For $t \in \mathbb{R}_+$, the map $t \mapsto tx$ is continuous and order preserving, and for all t_1, t_2 in $\mathbb{R}_+$ and for all x and y in $\mathbb{R}_+$, $(t_1 t_2)x = t_1(t_2 x)$ and $t(x \vee y) = (tx) \vee (ty)$.*

A *finite-dimensional* $\mathbb{B}$*-space* is, by definition, a subset X of $\mathbb{R}^n_+$ such that:

(BS) $0 \in X,\ \forall t \geq 0$ and $\forall x \in X,\ tx \in X$ and $\forall x, y \in X,\ x \vee y \in X$.

For a subset B of X the following properties are equivalent (see [3]):

(B1) $\forall x, y \in B$ and $\forall t \in [0, 1],\ tx \vee y \in B$;

(B2) $\forall x_1, \ldots, x_m \in B$ and $\forall t_1, \ldots, t_m \in [0, 1]$ such that $\max_{1 \leq i \leq m}\{t_i\} = 1$, $t_1 x_1 \vee \cdots \vee t_m x_m = \vee t_i x_i \in B$.

A subset of X for which (B1), or (B2), holds is said to be $\mathbb{B}$-*convex*. Clearly, (B1) holds for increasing sets (S is increasing if $x \leq y$ and $x \in S$ implies $y \in S$). A set S is called a *radial set* if, for all $t \in]0, 1]$ and all $x \in S,\ tx \in S$. A radial set S which is also a semilattice ($x, y \in S$ implies $x \vee y \in S$), is $\mathbb{B}$-convex.

In a $\mathbb{B}$-space, increasing sets and radial sets which are also semilattices are $\mathbb{B}$-convex. Sets of the form $\prod_{i=1}^m [a_i, b_i]$ are $\mathbb{B}$-convex in $\mathbb{R}^n_+$.

Since an arbitrary intersection of $\mathbb{B}$-convex sets is $\mathbb{B}$-convex, an arbitrary set $S \subset X$ is always contained in a smallest $\mathbb{B}$-convex subset of X, we call that set the $\mathbb{B}$-*convex hull* of S, it is denoted by $[[S]]$.

From (B2) one has the following characterization (see [3], Fig. 1):

The $\mathbb{B}$-convex hull of S is the set of all elements of the form $t_1 x_1 \vee \cdots \vee t_m x_m$ with $x_i \in S$ and $\max_{1 \leq i \leq m}\{t_i\} = 1,\ t_i \in [0, 1]$.

The following is given in [3]:

Lemma 3.1. *$\mathbb{B}$-convex sets and $\mathbb{B}$-convex hulls are contractible.*

4. $\mathbb{B}$-spaces are KKM spaces

For any finite dimensional $\mathbb{B}$-space X with a nonempty subset $D \subset X$, let $\Gamma_A := [[A]]$ be the $\mathbb{B}$-convex hull of $A \in \langle D \rangle$.

Lemma 4.1. *$(X, D; \Gamma)$ can be made into an H-space and hence a KKM space.*

Proof. For any $A \in \langle D \rangle$, let $\Gamma_A = \Gamma(A) = [[A]]$. Then each Γ_A is contractible by Lemma 3.1. Moreover, $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. Therefore $(X, D; \Gamma)$ is an H-space, and hence a KKM space by our KKM theory. □

By putting $E = Z$ and $F = 1_E$ in Theorem 2.6, we immediately have the following form of the KKM theorem in the setting of a $\mathbb{B}$-space:

Theorem 4.2. *Let $(X, D; \Gamma)$ be as in Lemma 4.1, and $G : D \multimap X$ a map such that*

(1) *G is a closed-valued KKM map; and*
(2) *there exists a nonempty compact subset K of X such that one of the following coercivity conditions hold:*
   (i) *$K = X$;*
   (ii) *$\bigcap\{G(y) \mid y \in N\} \subset K$ for some $N \in \langle D \rangle$; or*
   (iii) *for each $N \in \langle D \rangle$, there exists a closed compact $\mathbb{B}$-convex subset L_N of X containing some $D' \subset D$ such that $N \subset D'$, and*

$$L_N \cap \bigcap_{y \in D'} G(y) \subset K.$$

Then we have

$$K \cap \bigcap_{y \in D} G(y) \neq \emptyset.$$

The following is a simple observation:

Theorem 4.3. *Let X be a $\mathbb{B}$-space and $K \subset X$ a $\mathbb{B}$-convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is closed. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*

Moreover, if there exists $x_0 \in K$ such that $G(x_0)$ is compact, then

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

Proof. By Lemma 4.1, a $\mathbb{B}$-convex subset $K \subset M$ is a KKM space. Hence, by the definition itself, the conclusion follows. □

For open-valued KKM map, we have the following:

Theorem 4.4. *Let X be a $\mathbb{B}$-space and $K \subset X$ a $\mathbb{B}$-convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is open. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*

Proof. By Lemma 4.1, K is a KKM space. Hence, by the definition itself, the conclusion follows. □

The following consequence of Theorems 4.3 and 4.4 is given in ([3], Corollary 2.2):

Corollary 4.5 ($\mathbb{B}$-KKM)**.** *Let X be a $\mathbb{B}$-space and $A_0, \dots, A_m$ subsets of X, either all open in X or all closed in X. If there exist points $a_i \in A_i$ such that for all set of indices $\{i_0, \dots, i_k\}$ one has $[[a_{i_0}, \dots, a_{i_k}]] \subset \bigcup_{j=0}^{k} A_{i_k}$, then $A_0 \cap \dots \cap A_m \neq \emptyset$.*

This shows that $\mathbb{B}$-spaces are KKM spaces.

5. Further results in the KKM theory

Recall that, in [9], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann-Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [9] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [10], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [10] unifies and enlarges previously known several proper examples of such statements for particular types of partial KKM spaces.

Recall that some corrections on [10] were made in [12].

For any $\mathbb{B}$-space X with a nonempty set $D \subset X$, the KKM space $(X, D; \Gamma)$ satisfies all results in [10, 12].

We give some known KKM theoretic results related to $\mathbb{B}$-spaces which are consequences of the preceding results in the present article:

The following is ([12], (V)) for KKM spaces:

Theorem 5.1 (The Fan-Browder fixed point property). *Let $(E, D; \Gamma)$ be a KKM space. Let $S : E \multimap D$, $T : E \multimap E$ be maps satisfying*

1. *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
2. *$S^-(z)$ is open [resp., closed] for each $z \in D$; and*
3. *$E = \bigcup_{z\in N} S^-(z)$ for some $N \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$.

The following is ([3], Theorem 2.7):

Corollary 5.2 (Fan-Browder's Fixed Point Theorem). *Let B be a compact $\mathbb{B}$-convex set and $\Phi : B \multimap B$ a multimap with nonempty $\mathbb{B}$-convex values and open fibers. Then there exists $x \in B$ such that $x \in \Phi x$.*

From this, the authors of [3] deduced a generalized Fan-Browder fixed point theorem, a Kakutani type fixed point theorem, and a generalized Kakutani type fixed point theorem for compact $\mathbb{B}$-convex sets.

The following is given as ([12], Theorem 4):

Theorem 5.3 (Minimax inequality). *Let $(E, D; \Gamma)$ be a partial KKM space, $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ extended real-valued functions, and $\gamma \in \overline{\mathbb{R}}$ such that*

1. *for each $z \in D$, $\{y \in E \mid f(z, y) \le \gamma\}$ is closed;*
2. *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in E \mid g(x, y) > \gamma\}$; and*
3. *the map $G : D \multimap E$ defined by $G(z) := \{y \in E \mid f(z, y) \le \gamma\}$ has a coercivity condition as in Theorem 2.6.*

Then

- (i) *there exists a $y_0 \in E$ such that $f(z, y_0) \le \gamma$ for all $z \in D$; and*

(ii) *if* $\gamma := \sup_{x\in E} g(x,x)$, *then we have*

$$\inf_{y\in E}\sup_{z\in D} f(z,y) \le \sup_{x\in E} g(x,x).$$

The following is ([3], Theorem 3.1):

Corollary 5.4 (Ky Fan's Inequality). *Let B be a $\mathbb{B}$-convex set and $F: B\times B \to \mathbb{R}$ a map such that:*

(1) $\forall x \in B,\ F(x,x) \le 0.$
(2) $\forall x \in B,\ y \mapsto F(x,y)$ *is lower-semicontinuous.*
(3) $\exists x_0 \in B$ *such that* $\{y \in B : F(x_0,y) \le 0\}$ *is compact.*
(4) $\forall x, y_1, y_2 \in B, \forall t \in [0.1]\ F(x, ty_1 \vee y_2) \le \max\{F(x,y_1), F(x,y_2)\}.$

Then, there exists $y_0 \in B$ such that $\sup_{x\in B} F(x,y_0) \le 0$.

The following is ([10], (XXIV)) where $X_{-i} := \prod_{j\ne i} X_j$:

Theorem 5.5 (Generalized Nash-Fan type equilibrium theorem). *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a finite family of compact abstract convex spaces such that $(E;\Gamma) = (\prod_{i\in I} X_i;\Gamma)$ is a partial KKM space and, for each $i \in I$, let $f_i, g_i : E = X_{-i}\times X_i \to \mathbb{R}$ be real functions such that*

(0) $f_i(x) \le g_i(x)$ *for each* $x \in E$;
(1) *for each* $x_{-i} \in X_{-i}$, $x_i \mapsto g_i[x_{-i}, x_i]$ *is quasiconcave on* X_i;
(2) *for each* $x_{-i} \in X_{-i}$, $x_i \mapsto f_i[x_{-i}, x_i]$ *is u.s.c. on* X_i; *and*
(3) *for each* $x_i \in X_i$, $x_{-i} \mapsto f_i[x_{-i}, x_i]$ *is l.s.c. on* X_{-i}.

Then there exists a point $\hat{x} \in E$ such that

$$g_i(\hat{x}) \ge \max_{y_i\in X_i} f_i[\hat{x}_{-i}, y_i] \quad \text{for all } i \in I.$$

The following is ([3], Theorem 3.2):

Corollary 5.6 (Existence of Nash Equilibria). *Let B_i, $i = 1,\dots,m$, be compact $\mathbb{B}$-convex sets and $f_i : \prod_{j=1}^m B_j \to \mathbb{R}$, $i = 1,\dots,m$, continuous maps such that, for all i and for all $x \in \prod_{j=1}^m B_j$ the map $y_i \mapsto f_i(x|y_i)$ is $\mathbb{B}$-quasiconcave on B_i. Then, there exists $x^* = (x_1^*,\dots,x_m^*) \in \prod_{j=1}^m B_j$ such that, for all $i \in \{1,\dots,m\}$,*

$$f_i(x^*) = \max_{y_i\in B_i} f_i(x^*|y_i).$$

Here $(x^*|y_i)$ is obtained from x^* by replacing x_i by y_i.

Remark. In this section we compared only a small number of results in [3] and [10, 12]. However, many results in [10] can be applicable to $\mathbb{B}$-convex sets.

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