Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2019, 2019:1
ISSN: 2052-5338

FORTY FIVE YEARS WITH FIXED POINT THEORY

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Abstract. In the last forty five years, we have obtained a large number of results in various branches of fixed point theory (simply, FPT). Our main aim is to introduce only some of them in the chronological order. Each of them belongs to one of the following branches with the indicated periods: Topological FPT (1972-1975); Metric FPT (around 1980); Analytical FPT (around 1992); and Abstract KKM Theory (after 2000). Especially, the last part surveys fixed point theorems developed in the frame of our KKM theory on abstract convex spaces.

Keywords: abstract convex space; KKM map; Fan-Browder map; LΓ-space; better admissible class $\mathfrak{B}$; fixed point; coincidence point.

2010 AMS Subject Classification: 47H10, 49J53, 54C60, 54H25, 55M20.

1. INTRODUCTION

In the last forty five years, we have obtained a large number of results in various branches of fixed point theory (simply, FPT). We also developed the KKM theory on abstract convex spaces originated from the Knaster-Kuratowski-Mazurkiewitz (simply, KKM) theorem in 1929. In this theory, we also obtained many fixed point theorems for various types of multimaps.

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Received January 19, 2019

Our main aim in this survey is to introduce only some of our results on fixed points of various types of multimaps in the chronological order. Each of them belongs to one of the following branches with the indicated periods:

Topological FPT — 1972-1975

Metric FPT — around 1980

Analytical FPT — around 1992

The KKM Theory — after 2000

Especially, the final section surveys fixed point theorems developed in the frame of our KKM theory on abstract convex spaces.

2. Topological FPT — 1972-1975

Let us begin with the following notations in this section:

X — a topological space

X^k — a k-fold Cartesian product of X, $k \in \mathbb{N}$

Γ — a group of permutations of factors of X^k

Definition. *The* Γ*-product* X_Γ *of* X is the orbit space of X^k on which Γ acts. A map $F : X \multimap X_\Gamma$ is called a *symmetric product multimap* of X w.r.t. Γ or a Γ*-product multimap.*

Let us recall the following well-known result:

Theorem 2.1. (Lefschetz) *If X is a compact ANR and $f : X \to X$ is a continuous map whose Lefschetz number is not zero, then f has a fixed point.*

This theorem has a large number of generalizations for multimaps.

In 1972-1975, we also generalized as follows under the guidance of the late Professor Jan Jaworowski in Indiana University, Bloomington, Indiana:

Definition. A point $x \in X$ is a *fixed point* of a *symmetric product multimap* $F : X \multimap X_\Gamma$ if x is a coordinate of an element of $F(x)$.

The following type of results was obtained under appropriate preparation:

Theorem 2.2. ([22]) *If X is a compact ANR and $F : X \multimap X_\Gamma$ is a u.s.c. multimap such that $F(x)$ is acyclic for each $x \in X$ and its Lefschetz number is not zero, then F has a fixed point.*

Some further related works were given in [23].

William A. Kirk [17] recalled as follows: Jan Jawoworski (1928-2013) received his Ph.D. in 1955 from the Polish Academy of Sciences under the supervision of Karol Borsuk. (Borsuk's other students include Samuel Eilenberg and Andrzej Granas.) One of Jaworowki's students, Sehie Park, is also well known in fixed point theory.

3. METRIC FPT — AROUND 1980

From the late 1970's to the early 1990's we had engaged to study the metric FPT. Especially, in 1979-1993, we made eleven joint papers with Billy E. Rhoades. In [45], we collected briefly the contents of all of the joint papers. As he recalled "Our collaboration has ceased only because our research interests have moved in different directions."

In 2007, Rhoades [49] recalled the contents of his historical Transactions paper [48] and gave comments on our works as follows ([49], pp.12–13):

"Sehie Park also observed that fixed point theorems for many contractive definitions used the same proof technique. In 1980 [24] he proved the following two theorems, where $O(u) := \{u, Tu, T^2u, \dots\}$.

Theorem 3.1. ([24]) *Let T be a selfmap of a metric space (X,d). If there exists a point $u \in X$ and a $\lambda \in [0,1)$ such that $\overline{O}(u)$ complete and*

$$(*) \qquad d(Tx,Ty) \leq d(x,y)$$

holds for any $x, y = Tx$ in $O(u)$, then $\{T^n u\}$ converges to some $\xi \in X$, and

$$d(T^i u, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, fu) \;\; for \;\; i \geq 1.$$

Further, if f is orbitally continuous at ξ or if $(*)$ holds for any $x,y \in \overline{O}(u)$, then ξ is a fixed point of T.

Theorem 3.2. ([24]) *Let T be a selfmap of a metric space (X,d). If*

(i) *there exists a point $u \in X$ such that the orbit $O(u)$ has a cluster point $\xi \in X$,*

(ii) *T is orbitally continuous at ξ and $T\xi$, and*

(iii) *T satisfies*

$$d(Tx,Ty) < d(x,y)$$

for each $x, y = Tx \in \overline{O}(u),\ x \neq y$,

then ξ is a fixed point of T.

These theorems contain as special cases a number of papers involving contractive conditions not covered by my Transaction paper."

And then Billy E. Rhoades added an example of an application of Theorem 3.1, not previously published. He continues as follows:

"In 1980 Sehie Park [25] constructed a table of contractive conditions of Meir-Keeler type, which extended the list in my Transactions paper."

Let f be a selfmap of a metric space (X,d). Given $x \in X$, let $O(x) = \{f^n x : n \in \mathbb{N}\}$ and $\overline{O(x)}$ be its closure. A point $x \in X$ is said to be *regular* for f if diam $O(x) < \infty$. Given $x,y \in X$, let

$m(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}$,

$\delta(x,y) = \mathrm{diam}\{O(x) \cup O(y)\}$ whenever x and y are regular.

We list contractive type conditions to be considered.

(A) For any $x,y \in X, x \neq y$,

(Ad) $d(fx,fy) < d(x,y)$. [Edelstein]

(Am) $d(fx,fy) < m(x,y)$. [Rhoades]

(Aδ) if x and y are regular, $d(fx,fy) < \delta(x,y)$.

(B) Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x,y \in X$,

(Bd) $\varepsilon < d(x,y) < \varepsilon + \delta$ implies $d(fx,fy) < \varepsilon$. (Meir-Keeler)

(Bm) $\varepsilon < m(x,y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$.

(Bδ) $\varepsilon < \delta(x,y) < \varepsilon + \delta$ implies $d(fx, fy) < \varepsilon$.

(C) Given $\varepsilon > 0$, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for any $x, y \in X$,

(Cd) $\varepsilon < d(x,y) < \varepsilon + \delta_0$ implies $d(fx, fy) \leq \varepsilon_0$.

(Cm) $\varepsilon < m(x,y) < \varepsilon + \delta_0$ implies $d(fx, fy) \leq \varepsilon_0$.

(Cδ) $\varepsilon < \delta(x,y) < \varepsilon + \delta_0$ implies $d(fx, fy) \leq \varepsilon_0$. (Hegedüs-Szilágyi).

(D) There exists a nondecreasing right continuous function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$,

(Dd) $d(fx, fy) \leq \phi(d(x,y))$. (Browder)

(Dm) $d(fx, fy) \leq \phi(m(x,y))$. (Danes)

(Dδ) $d(fx, fy) \leq \phi(\delta(x,y))$ if x, y are regular. (Kasahara)

(E) There exists $\alpha \in [0,1)$ such that for any $x, y \in X$,

(Ed) $d(fx, fy) \leq \alpha d(x,y)$. (Banach)

(Em) $d(fx, fy) \leq \alpha m(x,y)$. (Ciric, Massa)

(Eδ) $d(fx, fy) \leq \alpha \delta(x,y)$ if x, y are regular. (Hegedüs)

Then we have the following diagram:

$$
\begin{array}{ccccccccc}
\text{(Ad)} & \Leftarrow & \text{(Bd)} & \Leftarrow & \text{(Cd)} & \Leftarrow & \text{(Dd)} & \Leftarrow & \text{(Ed)} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{(Am)} & \Leftarrow & \text{(Bm)} & \Leftarrow & \text{(Cm)} & \Leftarrow & \text{(Dm)} & \Leftarrow & \text{(Em)} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{(A}\delta\text{)} & \Leftarrow & \text{(B}\delta\text{)} & \Leftarrow & \text{(C}\delta\text{)} & \Leftarrow & \text{(D}\delta\text{)} & \Leftarrow & \text{(E}\delta\text{)}
\end{array}
$$

Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X,d).

Theorem 3.3. ([25]) *Let f be a continuous compact selfmap of a metric space X satisfying* (Aδ). *Then f has a unique fixed point, and furthermore, for any $\alpha \in (0,1)$ there exists a metric ρ in $M(X)$ relative to which f satisfies* (Ed) *with the Lipschitz constant α.*

Theorem 3.4(Cδ). ([25]) *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that* (1) *$O(u)$ has a regular cluster point $p \in X$, and* (2) *the condition* (Cδ) *holds on $O(u) \cup O(p)$. Then f has a unique fixed point p in $O(u)$ and $f^n u \to p$.*

Theorem 3.5(Cδ). ([25]) *Let f be a selfmap of a complete metric space X. If* (Cδ) *holds for all regular points $x, y \in X$, then f has a unique fixed point $p \in X$, and $f^n x \to p$ for any regular point $x \in X$.*

Jungck first gave a fixed point theorem for commuting selfmaps f and g of a complete metric space X satisfying the conditions $gX \subset fX$, f is continuous, and

(Ed)$'$ $d(gx, gy) \leq \alpha d(fx, fy),\ \alpha \in [0, 1)$.

Similarly, we can consider other conditions ( )$'$ just imitating (Ed)$'$.

In 1999, Liu [21] stated as follows:

" On the other hand, the following open questions were raised by Park [25]:

1. Are there other counterexamples of the implications between various conditions in (Ed)-(Aδ)?

2. Are there any extensions of Theorem 3.4 (Cδ) to the conditions (Bm) and (Bδ)?"

From his Summary: "We answer two fixed-point questions of Park by constructing ten non-trivial examples and prove some fixed-point theorems for general contractive type mappings which, in turn, generalize, improve, and unify some results due to Fisher, Hegedüs, Hegedüs and Szilágyi, Hikida, Kasahara, Park, Park and Rhoades, and others." (MR1694853 (2000b:54055)).

4. Analytical FPT — around 1992

According to Lassonde [20], a *convex space X* is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $N \subset X$ there is a compact convex set $L_N \subset X$ such that $L \cup N \subset L_N$.

We begin with the following general coincidence theorem in [26].

Theorem 4.1. ([26]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $S: D \multimap Y$, $T: X \multimap Y$ multimaps, $F: X \to ka(Y)$ a u.s.c. multimap, and K a nonempty compact subset of Y. Suppose that*

(1) *for each $x \in D$, $Sx \subset Tx$ and Sx is open;*

(2) *for each $y \in F(X)$, $T^{-}y$ is convex;*

(3) *$\overline{F(X)} \cap K \subset S(D)$; and*

(4) *for each $N \in \langle D \rangle$, there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \backslash F^{+}(K)$ implies $Fx \subset S(L_N \cap D)$.*

Then T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$.

Here $ka(Y)$ and $kc(X)$ denote the class of compact acyclic subsets of Y and nonempty compact convex subsets of X, resp. This theorem is a far-reaching generalization of the Fan-Browder fixed point theorem and has numerous applications as shown in [26].

We have the following generalized fixed point theorem:

Theorem 4.2. ([26]) *Let X and C be nonempty convex subsets of a Hausdorff locally convex t.v.s. E. Let $F: X \to ca(X+C)$ be a compact u.s.c. multifunction. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) $C = \{0\}$.

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x}+C) \neq \emptyset$.

Here *ca* denotes the class of closed acyclic subsets.

The following is the case $C = \{0\}$ generalizing the well-known Himmelberg fixed point theorem.

Corollary 4.3. ([26]) *Let X be a nonempty convex subset of a Hausdorff locally convex t.v.s. E. Let $F: X \multimap X$ be a compact acyclic map. Then F has a fixed point.*

This corollary has been applied to many problems by several authors.

The class of u.s.c. multimaps on t.v.s. is further extended as follows: For a subset X of a t.v.s. E, a map $F : X \multimap E$ is said to be

(i) *upper demicontinuous* (*u.d.c.*) if for each $x \in X$ and open half-space H in E containing $F(x)$, there exists an open neighborhood N of x in X such that $F(N) \subset H$;

(ii) *upper hemicontinuous* (*u.h.c.*) if for each $h \in E^*$ and for any real α, the set $\{x \in X \mid \sup hF(x) < \alpha\}$ is open in X; and

(iii) *generalized u.h.c.* if for each $p \in E^*$, the set $\{x \in X \mid \sup pF(x) \geq p(x)\}$ is closed in X.

In our earlier work [27], we unified a large number of generalizations of the Kakutani theorem to maps of the above-mentioned types.

Let $cc(E)$ denote the set of nonempty closed convex subsets of a t.v.s. E and $kc(E)$ the set of nonempty compact convex subsets of E.

Let X be a nonempty convex subset of a vector space E. The *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* $\mathrm{Bd}\, X = \mathrm{Bd}_E X$ of X is the complement of $\mathrm{Int}_E X$ in the closure $\overline{X}$. It is known that $\delta_E(X) \subset \mathrm{Bd}\, X$ and in general $\delta_E(X) \neq \mathrm{Bd}\, X$.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, resp., are defined as follows:

$$I_X(x) := x + \bigcup_{r>0} r(X - x), \quad O_X(x) := x + \bigcup_{r<0} r(X - x).$$

For $p \in E^*$ and $U, V \subset E$, let

$$d_p(U,V) = \inf\{|p(u-v)| \ : \ u \in U, \ v \in V\}.$$

The following is the main theorem in [30]:

Theorem 4.4. ([30]) *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in E^*$,*

(0) $p|_X$ *is continuous on* X*;*

(1) $X_p := \{x \in X \mid \inf pF(x) \leq p(x)\}$ *is closed in* X*;*

(2) $d_p(F(x), \overline{I_X(x)}) = 0$ *for every* $x \in K \cap \delta_E(X)$*; and*

(3) $d_p(F(x), \overline{I_L(x)}) = 0$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$.

E	$f : K \longrightarrow K$		$F : K \longrightarrow 2^K$	
I	*Brouwer*	*1912*	*Kakutani*	*1941*
II	*Schauder*	*1927, 1930*	*Bohnenblust and Karlin*	*1950*
III	*Tychonoff*	*1935*	*Fan*	*1952*
			Glicksberg	*1952*
IV	*Fan*	*1964*	*Granas and Liu*	*1986*
	$f : K \longrightarrow E$		$F : K \longrightarrow 2^E$	
I	*Bohl*	*1904*		
	Knaster, Kuratowski and Mazurkiewicz	*1929*		
II	*Rothe*	*1938*		
III	*Halpern*	*1965*	*Browder*	*1968*
	Fan	*1969*	*Fan*	*1969*
	Reich	*1972*	*Glebov*	*1969*
	Sehgal and Singh	*1983*	*Halpern*	*1970*
			Cellina	*1970*
			Reich	*1972, 1978*
			Cornet	*1975*
			Lasry and Robert	*1975*
			Simons	*1986*
IV	*Halpern and Bergman*	*1968*	*Granas and Liu*	*1986*
	Kaczynski	*1983*	*Park*	*1988, 1991*
	Roux and Singh	*1989*		
	Sehgal, Singh and Whitfield	*1990*		
			$F : X \longrightarrow 2^E$	
II			*Ding and Tan*	*1992*
III			*Fan*	*1984*
			Shih and Tan	*1987, 1988*
			Jiang	*1988*
IV			*Park*	*1992, 1993*
			Yuan, Smith, and Lou	*1998*

(II) *Suppose that for each* $p \in E^*$,

(0)$'$ $p|_X$ *is continuous on* X*;*

(1)$'$ $X_p := \{x \in X \mid \sup pF(x) \geq p(x)\}$ *is closed in* X*;*

(2)$'$ $d_p(F(x), \overline{O_X(x)}) = 0$ *for every* $x \in K \cap \delta_E(X)$*; and*

(3)$'$ $d_p(F(x), \overline{O_L(x)}) = 0$ *for every* $x \in X \setminus K$.

Then there exists an $x \in X$ *such that* $x \in F(x)$. *Further, if* F *is u.h.c., then* $F(X) \supset X$.

Remark. The major particular forms of Theorem 4.4 can be adequately summarized by the preceding enlarged version of the diagrams previously given in [26,30].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex t.v.s., and IV for t.v.s. having sufficiently many linear functionals. Moreover, f stands for single-valued maps and F for multimaps; and K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex subset of E satisfying certain coercivity conditions with respect to $F : X \multimap E$ with certain boundary conditions.

In fact, Theorem 4.4 implies all of the fixed point theorems in the diagram. Note that, in the diagram, Bohl's theorem in 1904 was well-known to be equivalent to the Brouwer fixed point theorem in 1912.

5. From the KKM Theory — After 2000

Recently we established the KKM theory on abstract convex spaces, whose realm is very broad. In this section, we introduce certain fixed point results related to the theory.

Let $\langle D \rangle$ be the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition. ([35,38,40-44]) An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with values $\Gamma_A := \Gamma(A) \neq \emptyset$ for $A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is defined and denoted by $\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset E$.

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if $\forall N \in \langle D' \rangle$, we have $\Gamma_N \subset X$ (that is, $\mathrm{co}_\Gamma D' \subset X$).

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$ and let $(E; \Gamma) := (E, E; \Gamma)$.

Example. We give only some well-known examples of abstract convex spaces:

1. The triple $(\Delta_n \supset V; \mathrm{co})$ in the original KKM theorem [19], where V is the set of vertices of the n-simplex Δ_n.

2. A triple $(X \supset D;\ \mathrm{co})$ for a t.v.s. E such that $\mathrm{co}\, D \subset X \subset E$. Fan's celebrated KKM lemma [3] is for $(E \supset D; \mathrm{co})$.

3. A *convex space* $(X \supset D;\ \mathrm{co})$ [28-31], where X is a subset of a vector space, $\mathrm{co}\, D \subset X$, and $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle D \rangle$ equipped with the Euclidean topology. This is due to Lassonde for $X = D$; see [20]. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

4. An H-*space* $(X \supset D; \Gamma)$ [31,46], where $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is a c-space due to Horvath [11-14] or an H-space of Bardaro and Ceppitelli.

5. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park [31-34,46,47]; see references therein.

6. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ with a family $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with $|A| = n+1$.

Every ϕ_A-space can be made into a G-convex space; see [38,39].

7. A convexity space $(E, \mathscr{C})$ in the classical sense is an abstract convex space whenever E is given a topology; see [50], where the bibliography lists 283 papers.

8. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree M, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_M(A)$; see Kirk and Panyanak [18]. Then $(M, X; \Gamma)$ is an abstract convex space.

9. The convexity space due to Horvath [14,15].

10. A $\mathbb{B}$-space due to Briec and Horvath [1].

Note that each of 3-10 has a large number of concrete examples.

Let $(E, D; \Gamma)$ be an abstract convex space.

Definition. A *KKM map* $G : D \multimap E$ is the one satisfying

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \forall A \in \langle D \rangle.$$

Example. Granas in 1981 gave examples of KKM maps as follows:

(i) *Variational problems*. Let C be a convex subset of a vector space E and $\phi : C \to \mathbb{R}$ is a convex function. Then $G : C \multimap C$ is defined by

$$G(x) = \{y \in C \mid \phi(y) \leq \phi(x)\} \text{ for } x \in C.$$

(ii) *Best approximation*. Let C be a convex subset of a vector space E, p a seminorm on E, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid p(f(y) - y) \leq p(f(y) - x)\} \text{ for } x \in C.$$

(iii) *Variational inequalities*. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, C a convex subset of H, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid \langle f(y), y - x \rangle \leq 0\} \text{ for } x \in C.$$

Example. For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \text{ for each } A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space.

Definition. $G : D \multimap Z$ is a *KKM map* w.r.t. $F : E \multimap Z$ whenever

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \forall A \in \langle D \rangle.$$

A *KKM map* $G : D \multimap E$ is a KKM map w.r.t. 1_E.

A $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] $F : E \multimap Z$ is the one satisfying, for any closed [resp., open]-valued KKM map $G : D \multimap Z$ w.r.t. F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

We denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

A *KKM space* $(E,D;\Gamma)$ is an abstract convex space satisfying the *KKM principle* $1_E \in \mathfrak{KC}(E,D,E) \cap \mathfrak{KO}(E,D,E)$.

A *partial KKM space* is the one satisfying the *partial KKM principle* $1_E \in \mathfrak{KC}(E,D,E)$.

In our recent work [41,43], we studied foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces.

Example.

1. Every G-convex space is a KKM space [32].

2. A connected linearly ordered space $(X,\leq)$ can be made into a KKM space [44].

3. The extended long line L^* is a KKM space $(L^* \supset D;\Gamma)$ with the ordinal space $D := [0,\Omega]$. But L^* is not a G-convex space [43].

4. A complete $\mathbb{R}$-tree $(M \supset X;\Gamma)$ satisfies the partial KKM principle, where X is a closed convex subset; see Kirk and Panyanak [18]. Later we found that $(M \supset X;\Gamma)$ is a KKM space.

5. For Horvath's convexity space $(X,\mathscr{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X;\Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathscr{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [15].

Now the KKM theory steps in the new era and becomes the study of partial KKM spaces. We have the following diagram for triples $(E,D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s.

$\Longrightarrow$ Lassonde type convex space $\Longrightarrow$ H-space

$\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

In the KKM theory [40,43], it is routine to reformulate the KKM principle to the following equivalent forms:

- Fan type matching property

- An intersection property

- Geometric or section properties

- The Fan-Browder type fixed point theorem

- Existence theorem of maximal elements, and others.

Any of such statements can be used to characterize the KKM spaces.

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type. From this, we can deduce the following:

Theorem 5.1. *An abstract convex space* $(X,D;\Gamma)$ *is a KKM space iff for any map* $G : D \multimap X$ *satisfying*

(1.1) *G has closed* [*resp., open*] *values; and*

(1.2) *G is a KKM map,*

$\{G(z)\}_{z\in D}$ *has the finite intersection property.*

Further, if

(1.3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some* $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

From this theorem, in [41,43], we can deduce the following equivalents of the partial KKM principle:

- Analytic alternatives (a basis of various equilibrium problems)
- Fan type minimax inequalities
- Variational inequalities, and others.

Consequently, for a compact abstract convex space $(X;\Gamma)$, we deduced 15 theorems from any of the partial KKM principle. Moreover, we noticed there that, for a compact G-convex space $(X;\Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem.

Further applications of the partial KKM principle are given in [40,41] as follows:

- Best approximations
- The von Neumann type minimax theorem
- The von Neumann type intersection theorem
- The Nash type equilibrium theorem
- The Himmelberg fixed point theorem for KKM spaces
- Weakly KKM maps [42]

The KKM principle is equivalent to the Fan-Browder type fixed point theorem:

Theorem 5.2. *An abstract convex space* $(X,D;\Gamma)$ *satisfies the partial KKM principle iff for any maps* $S : D \multimap X,\ T : X \multimap X$ *satisfying*

(2.1) $S(z)$ *is open for each* $z \in D$*;*

(2.2) *for each* $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$*; and*

(2.3) $X = \bigcup_{z\in M} S(z)$ *for some* $M \in \langle D\rangle$*,*

T *has a fixed point* $x_0 \in X$.

Moreover, $(X,D;\Gamma)$ *is a KKM space iff the above condition also holds for any map* $S : D \multimap X$ *such that*

(2.1)$'$ $S(z)$ *is closed for each* $z \in D$

instead of (2.1).

Corollary 5.3. *Let* $(X,D;\Gamma)$ *be a compact space satisfying the partial KKM principle and* $S : X \multimap D$, $T : X \multimap X$ *two maps such that*

(1) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*

(2) $X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.

Then T *has a fixed point* $x_0 \in X$.

From Corollary 5.3, we can easily obtain the following Browder theorem in 1968 [2]:

Corollary 5.4. (Browder [2]) *Let* K *be a nonempty compact convex subset of a Hausdorff t.v.s. Let* T *be a map of* K *into* 2^K*, where for each* $x \in K$, $T(x)$ *is a nonempty convex subset of* K.

Suppose further that for each y in K, $T^{-}(y)$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.

Note that Browder's result is a reformulation of Fan's geometric lemma [3] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

Browder [2] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems.

The Fan-Browder type fixed point theorem is used by Borglin and Keiding in 1976 and Yannelis and Prabhakar in 1983 to the existence of maximal elements in mathematical economics. We give a generalization of their result as follows:

Corollary 5.5. *Let $(X,D;\Gamma)$ be a compact partial KKM space and $S : X \multimap D$, $T : X \multimap X$ two maps such that*

(1) *$S^{-}(z)$ is open for each $z \in D$;*

(2) *for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*

(3) *for each $x \in X$, $x \notin T(x)$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) = \emptyset$.

We introduce particular subclasses or subsets of abstract convex uniform spaces. We assume all uniform spaces are Hausdorff for simplicity.

Definition. An *abstract convex uniform space* $(E,D;\Gamma;\mathscr{U})$ is an abstract convex space having a basis $\mathscr{U}$ of a uniform structure of E.

Definition. A KKM uniform space $(E \supset D;\Gamma;\mathscr{U})$ is an LΓ-*space* whenever D is dense in E and, for each $U \in \mathscr{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Example 1. In particular, for G-convex spaces or H-spaces $(E \supset D; \Gamma; \mathscr{U})$ (where each Γ_A is contractible), we can define LG-spaces [35] or LH-spaces, resp.

2. For a c-space $(X;\Gamma)$, an LΓ-space reduces to an LC-space ($l.c.$-spaces) [13,14]. Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X;\Gamma)$ with $\Gamma_A = \mathrm{co}A$ for $A \in \langle X \rangle$. For other examples, see [13,14]. A singleton is not necessarily Γ-convex in an LΓ-space.

3. A G-convex space $(X \supset D; \Gamma)$ is called a *metric* LG-*space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x,C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of metric LC-spaces due to Horvath [13].

4. (Horvath [14]) Any hyperconvex metric space (H,d) is a complete metric LC-space $(H;\Gamma)$.

Let $(E \supset D; \Gamma; \mathscr{U})$ be an abstract convex uniform space.

Definition. A subset X of E is *of the Zima type* or *of the Zima-Hadžić type* whenever $D \cap X$ is dense in X and, $\forall U \in \mathscr{U} \;\; \exists V \in \mathscr{U}$ such that, $\forall N \in \langle D \cap X \rangle$ and any Γ-convex subset A of X, we have

$$A \cap V[z] \neq \emptyset \;\forall z \in N \Rightarrow A \cap U[x] \neq \emptyset \;\forall x \in \Gamma_N.$$

Let X be a topological space.

Definition. A Φ-*map* (or a *Fan-Browder map*) $H : X \multimap E$ is a map having a map $G : X \multimap D$ such that

(i) $\forall x \in X,\;\; \mathrm{co}_\Gamma G(x) \subset H(x)$ [that is, $H(x)$ is Γ-convex relative to $G(x)$];

(ii) $X = \bigcup\{\mathrm{Int}\, G^-(y) \mid y \in D\}$.

Let $(E, D; \Gamma; \mathscr{U})$ be an abstract convex uniform space.

Definition. A subset Z of E is a Φ-*set* whenever $\forall U \in \mathscr{U}$, $\exists H : Z \multimap E$ a Φ-map such that $\mathrm{Graph}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

Definition. A subset K of E is *Klee approximable* if $\forall U \in \mathscr{U}$, $\exists h : K \to E$ a continuous function satisfying

(1) $(x, h(x)) \in U \ \forall x \in K$;

(2) $h(K) \subset \Gamma_N \ \exists N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n+1$ such that $h = \phi_N \circ p$.

Definition. K is *Klee approximable into* $X \subset E$ whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

Definition. A subset X of E is *admissible* (in the sense of Klee) if every compact subset K of X is Klee approximable into E.

The mutual relations among the various subclasses of abstract convex uniform spaces are as follows as in Park [36,44]:

Lemma 5.6. *In the class of abstract convex uniform spaces $(X, D; \Gamma; \mathscr{U})$, the following hold:*

(1) *Any $L\Gamma$-space is of the Zima-Hadžić type.*

(2) *Every nonempty subset of an $L\Gamma$-space is locally Γ-convex whenever every singleton is Γ-convex.*

(3) *Any nonempty subset of a locally Γ-convex space is a Φ-set.*

(4) *Any Zima-Hadžić type subset of an abstract convex uniform space such that every singleton is Γ-convex is a Φ-set.*

(5) *Every G-convex Φ-space is admissible. More generally, every nonempty compact Φ-subset of a G-convex space is Klee approximable.*

The open-valued KKM principle is also useful to deduce very general fixed point theorems.

The following example is the main result of this section:

Theorem 5.7. ([44]) *Let $(X \supset D; \Gamma; \mathscr{U})$ be a KKM uniform space and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ-convex values. If $T(X)$ is of the Zima-Hadžić type, then T has a fixed point $x_* \in T(x_*)$.*

Corollary 5.8. ([44]) *Let* $(X \supset D;\Gamma;\mathscr{U})$ *be an* LΓ*-space and* $T : X \multimap X$ *a compact u.s.c. map with closed* Γ*-convex values. Then* T *has a fixed point* $x_0 \in X$.

Example. 1. For an LG-space $(X,D;\Gamma)$, this is given in [33]. Many particular forms were stated there.

2. The extended long line L^* is a compact LΓ-space; see [37]. Now it has *the fixed point property*. This is a proper example of Corollary 5.8 which is not for an LG-space.

Corollary 5.9. (Himmelberg [10]) *Let* X *be a convex subset of a locally convex Hausdorff topological vector space* E *and* $T : X \multimap X$ *a compact u.s.c. multimap with nonempty closed convex values. Then* T *has a fixed point* $x_0 \in T(x_0)$.

Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Rhee, and others. For the literature, see [31].

In order to give simple proofs of von Neumann's intersection lemma and the minimax theorem, Kakutani obtained the following generalization of the Brouwer theorem to multimaps:

Corollary 5.10. (Kakutani [16]) *If* $x \mapsto \Phi(x)$ *is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of closed convex subsets of S, then there exists an* $x_0 \in S$ *such that* $x_0 \in \Phi(x_0)$.

Equivalently,

Corollary 5.10.* (Kakutani [16]) *Corollary 5.9 is also valid even if S is an arbitrary bounded closed convex set in an Euclidean space.*

As Kakutani noted, Corollary 5.10 readily implies the following von Neumann Lemma in 1937, and later it is known that those two results are equivalent.

Corollary 5.11. (von Neumann [16]) *Let K and L be two compact convex sets in the Euclidean spaces* $\mathbb{R}^m$ *and* $\mathbb{R}^n$*, and let us consider their Cartesian product* $K \times L$ *in* $\mathbb{R}^{m+n}$*. Let U and V be two closed subsets of* $K \times L$ *such that for any* $x_0 \in K$ *the set* U_{x_0}*, of* $y \in L$ *such that* $(x_0, y) \in U$*, is nonempty, closed and convex and such that for any* $y_0 \in L$ *the set* V_{y_0}*, of all* $x \in K$ *such that*

$(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. We adopt the above formulation from Kakutani [16].

We have the following Horvath type fixed point theorem [14]:

Theorem 5.12. *Let $(E, D; \Gamma; \mathscr{U})$ be an abstract convex uniform space, and $F \in \mathfrak{KC}(E, D, E)$ be a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property. Further if F is closed, then it has a fixed point.*

From Theorem 5.12 as in [14], we have the following generalization of the Schauder-Tychonoff-Hukuhara fixed point theorem:

Corollary 5.13. *Let $(E, D; \Gamma; \mathscr{U})$ be an abstract convex uniform space satisfying the partial KKM principle. If $f : E \to E$ is a continuous function such that $\overline{f(E)}$ is a compact Φ-set in E, then f has a fixed point.*

Theorem 5.12 and Corollary 5.13 contain a large number of known fixed point theorems since there are so many Φ-sets.

Corollary 5.14. *Let $(E \supset D; \Gamma; \mathscr{U})$ be an abstract convex uniform space and $F \in \mathfrak{KC}(E, D, E)$ be a closed compact map. If every singleton of X is Γ-convex and $F(X)$ is of the Zima-Hadžić type, then F has a fixed point.*

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, D, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This definition works for G-convex spaces or ϕ_A-spaces. We have the following:

Theorem 5.15. *Let $(E,D;\Gamma;\mathscr{U})$ be an abstract convex uniform space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X,D,Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property.*

Further if F is closed and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

In [36 and many], we gave some of our previous results which are direct consequences of Theorem 5.15 as follows.

Since 1992, we introduced and supplied a lot of examples of the class $\mathfrak{A}_c^\kappa$ in [27,28]. Up to now, many authors used the class $\mathfrak{A}_c^\kappa$, but no one could find any new example of maps in that class.

In 1993 [27] and 1994 [28], we obtained the following with a different method:

Corollary 5.16. *Let X be a compact convex subset of a t.v.s. E on which its dual E^* separates points. Then any map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Corollary 5.17. *Let X be a convex subset of a locally convex t.v.s. E and $F \in \mathfrak{A}_c^\sigma(X,X)$. If F is compact, then it has a fixed point.*

In 1997, the author introduced the 'better' admissible class $\mathfrak{B}$ and noticed that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$:

Corollary 5.18. ([29]) *Let X be a convex subset of a locally convex t.v.s. E. Then every compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Corollary 5.19. ([30)] *Let E be a t.v.s. and X an admissible (in the sense of Klee) convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In 1998 [30], we listed more than sixty papers in chronological order, from which we could deduce particular forms of Corollary 5.19.

The following form of the main theorem of [34] in 2004 follows from Theorem 5.15:

Corollary 5.20. ([34]) *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

For $X = Y = E$, Theorem 5.15 reduces to the following main theorem of [30] in 2007:

Corollary 5.21. ([36]) *Let $(X, D; \Gamma; \mathscr{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X,X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property. Further if F is closed and compact, then F has a fixed point $x_0 \in X$ (that is, $x_0 \in F(x_0)$).*

Corollary 5.21 contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of admissible spaces, Φ-spaces, sets of the Zima-Hadžić type, locally G-convex spaces, and LG-spaces; see [36].

Mutual relations among those subclasses and some related results on approximable maps, Kakutani maps, acyclic maps, Φ-maps, and others are investigated in [36].

The following is a consequence of Theorem 5.15:

Corollary 5.22. *Let X and Y be subsets of a t.v.s. E such that $X \subset Y$ and $F : Y \multimap Y$ a map.*

(1) *If $F|_X \in \mathfrak{B}(X,Y)$ and $F(X)$ is Klee approximable into X, then $F|_X$ has the almost fixed point property (that is, for any $V \in \mathscr{V}$, $F|_X$ has a V-fixed point $x_V \in X$ satisfying $F(x_V) \cap (x_V + V) \neq \emptyset$).*

(2) *Further if F is closed and $F|_X$ is compact, then F has a fixed point.*

Note that, in (1), E is not necessarily Hausdorff. Corollary 5.22 would be better than [44, Theorem 2.2]. In [44], it should be $\mathfrak{B} = \mathfrak{B}^p$.

Finally, recall that there are several hundred published works on the KKM theory and analytical fixed point theory; and we can cover only an essential part of them. For the more historical background for the related fixed point theory, the reader can consult with [31]. For more involved or generalized versions of the results in this paper, see [20,28,37] for convex spaces, [11-14,46] for H-spaces, [31-34,36,47] for G-convex spaces, and [38-43] for abstract convex spaces.

Acknowledgement. This article was first presented as One-Hour Invited Talk at *International Workshop on Fixed Point Theory and Applications* (KIAS Open Research Station Program)(at Chonpook National Univ., Jeonju), June 19-24, 2016. Moreover, this was also presented at *In ternational Conference on Mathematical Sciences and Applications* (ICMSA-2018), Guru Ghasidas University, Bilaspur, CG, India, February 23-25, 2018, as the Plenary Speech at the Opening Ceremony.

This article was prepared for one-hour talk and is not perfect to cover all works by ourselves. Especially, more detailed version for Section 5 was already given in S. Park, *Applications of the KKM theory to fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 50(1) (2011), 21–69, and other publications.

Conflict of Interests

The authors declare that there is no conflict of interests.

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