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Several variational relation problems in abstract convex spaces

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Abstract

Luc [5] initiated to study variational relations, which are unifying approaches to various models of equilibrium theory and variational inclusions. In our previous work [13], some of Luc's results were extended to abstract convex spaces. As a continuation of [13], in this article, we obtain some abstract space versions of known results on generalized KKM maps and variational relation problems appeared in the papers of Park and Lee [14], Balaj and Luc [2], Luc, Sarabi and Soubeyran [6], Lin [4], and Balaj [1], in the chronological order.

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1 Introduction

The KKM theory is originated from the celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem in 1929. In 1961-1984, Ky Fan investigated various results in the theory on Hausdorff topological vector spaces. His results were elaborated and extended by many authors for various types of general spaces. Since 2006, such results have been unified and abstracted by our KKM theory on abstract convex spaces. For the history of such research, see [10].

Almost independently to such progress, D. T. Luc [5] in 2008 began to work on the variational relations in order to present a unifying approach to study various models of equilibrium theory and variational inclusions. Since then a score of authors have published quite large numbers of related works on a model for many problems in optimization, equilibrium theory, variational inclusions or variational inequalities in Hausdorff topological vector spaces. However, such models also appeared in various types of abstract convex spaces. Moreover, we found that the most works on variational relations do not reflect recent development of the KKM theory of abstract convex spaces.

In our previous work [13], we follow the pioneering work of Luc [5] and show that some of his results can be extended to our abstract convex spaces. As the continuation of [13], in this article, we are going to obtain some abstract space versions of known results on generalized KKM maps and variational relation problems appeared in several papers such as Park and Lee [14], Balaj and Luc [2], Luc, Sarabi and Soubeyran [6], Lin [4], and Balaj [1], in the chronological order.

This article is organized as follows: In Section 2, definitions and some basic facts on abstract convex spaces are introduced. Section 3 deals with generalized KKM maps introduced by Park and Lee [14]. In Section 4, we improve some results of Balaj and Luc [2] on mixed variational relation problems. Section 5 deals with Luc, Sarabi and Soubeyran [6] on the existence of solutions in variational relation problems without convexity. In Section 6, Lin's work [4] on variational relation problems is investigated. Finally, Section 7 deals with a generalization of a basic result of Balaj [1] on three types of variational relation problems.

2 Abstract convex spaces

Recall the following in [7-11] and the references therein.

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \to 2^E$ a multimap with nonempty values $\Gamma_N := \Gamma(N)$ for $N \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_N : N \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \to 2^Z$ with nonempty values, if a multimap $G : D \to 2^Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y\in N} G(y) \qquad \text{for all} \quad N \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \to 2^E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. A multimap $F : E \to 2^Z$ to a set Z is called *a $\mathcal{K}$-map* if, for a KKM map $G : D \to 2^Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathcal{K}(E, Z) := \{F : E \to 2^Z \mid F \text{ is a } \mathcal{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a *$\mathcal{KC}$-map* is defined for closed-valued maps G, and a *$\mathcal{KO}$-map* for open-valued maps G. In this case, we denote $F \in \mathcal{KC}(E, Z)$ [resp. $F \in \mathcal{KO}(E, Z)$].

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathcal{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathcal{KC}(E, E) \cap \mathcal{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Now the following diagram for triples $(E, D; \Gamma)$ is well-known:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3 Park and Lee 2001 [14]

In the KKM theory, many authors adopted the concept of generalized KKM maps and applied to extend or refine already well-known results. In [14], the authors give a unified account for such maps in generalized convex spaces. Their results include the KKM type theorems and characterizations of generalized KKM maps [3]. They also deduce an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

Their definition can be extended to abstract convex spaces as follows:

Definition 3.1. Let $(X, D; \Gamma)$ be an abstract convex spaces and Y be a nonempty set such that, for each $N \in \langle Y \rangle$, there exists a function $\lambda_N : N \to D$. Then a new abstract convex spaces $(X, N; \Lambda)$ induced by Γ and N is defined by the following

$$\Lambda(J) := \Gamma(\lambda_N(J)) \quad \text{for each } J \subset N.$$

Moreover, a multimap $T : Y \to 2^X$ (called a *generalized KKM map*) reduces to a KKM map on $(X, N; \Lambda)$ for each $N \in \langle Y \rangle$ satisfying $\Lambda(J) \subset T(J)$ for each $J \subset N$.

Park and Lee [14] gave many examples of the above definition for G-convex spaces. Here we give only one example:

Example 3.2. Let X and Y be convex subsets of topological vector spaces E and F, resp. A map $G : X \to 2^F$ is called a generalized KKM map by Chang and Zhang [3], if for any finite set $\{x_1, \ldots, x_n\} \subset X$, there exists a finite set $\{y_1, \ldots, y_n\} \subset Y$ such that any finite subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$, $1 \leq k \leq n$, we have $\text{co}\{y_{i_1}, \ldots, y_{i_k}\} \subset \bigcup_{j=1}^k G(x_{i_j})$. Note that any KKM map $G : X \to 2^E$ is a generalized KKM map, and a counterexample ensuring the converse does not hold was given there.

The following generalizes [14, Theorem 2], which was stated for G-convex spaces:

Theorem 3.3. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set, and $T : Y \to 2^X$ a map with closed [resp. open] values.*

(i) *If T is a generalized KKM map, then the family of its values has the finite intersection property.*

(ii) *The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.*

Proof. (i) Let $N \in \langle Y \rangle$ and $T|_N : N \to 2^X$ be a KKM map having closed [resp. open] values on $(X, N; \Lambda)$, that is,

$$\Lambda(J) \subset T(J) \quad \forall J \subset N.$$

Let $N = \{y_i\}_{i=1}^n$, $z_i = \lambda_N(y_i) \in D$, and $G(z_i) = T(y_i)$ for each $i = 1, \ldots, n$. Then

$$\Gamma(\lambda_N(J)) \subset G(\lambda_N(J)) \quad \forall J \subset N.$$

Hence $G : \lambda_N(N) \to 2^X$ is a KKM map with closed [resp. open] values on $(X, N; \Gamma|_{\langle \lambda_N(N) \rangle})$ which is a (partial) KKM space [resp. KKM space]. Hence $\{G(z_i)\}_{i=1}^n = \{T(y_i)\}_{i=1}^n$ has the finite intersection property.

(ii) Suppose that $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. For any $N \in \langle Y \rangle$, by assumption, we have an $x^* \in \bigcap_{y \in N} T(y) \neq \emptyset$. Define a function $\lambda_N : N \to D = X$ by $\lambda_N(y) = x^*$ for all $y \in N$. Then for any nonempty subset J of N, we have

$$\Gamma_{\lambda_N(J)} = \Gamma_{\{x^*\}} = \{x^*\} \subset \bigcap_{y \in N} T(y) \subset T(J).$$

Therefore, T is a generalized KKM map. □

Based on this theorem, other results in [14] can be improved.

Corollary 3.4. *Let $(X, D; \Gamma)$ be a partial KKM space, and $T : D \to 2^X$ a multimap with closed values. If T is a generalized KKM map such that $T(y_0)$ is compact for at least one $y_0 \in D$, then $\bigcap_{y \in D} T(y) \neq \emptyset$.*

This type of results is quite popular; one of recent ones appears in Balaj [1]; see Section 7. Other results in [14] also can be generalized by using Definition 3.1.

4 Balaj and Luc 2010 [2]

Balaj and Luc [2] exploited the method of variational relations to establish existence of solutions to a general inclusion problem (VI). The result is applied to variational relation problems in which several relations are simultaneously considered. Particular cases of variational inclusion and intersection of set-valued maps are also discussed.

In Section 2 of [2], some details on the KKM maps are given. It can be modified as follows:

"In the classical sense a multimap $G : Y \to 2^Y$, where Y is a convex set in a topological vector space, is called KKM if for every finite subset N of Y, its convex hull $\mathrm{co}(N)$ is contained in the image $G(N)$. When $G : Y \to 2^X$, in which X and Y are taken from spaces with different structure, a generalized KKM property comes in force and yields also the finite intersection property of the family $\{\mathrm{cl}(G(y)) : y \in Y\}$ (here "cl" denotes the closure). Below we exploit two generalizations of KKM maps to derive existence of solutions to (VI), the first one belongs to Park and the second one seems to be new and generalizes the concept of KKM maps by Chang and Zhang [3].

In the following, we cite definitions, propositions, and a theorem in the beginning part, and give comments on them."

Definition 4.1. ([2]) Let X and Y be convex sets in topological vector spaces. Let $G,\ F : Y \to 2^X$ be multimaps. We say that

(a) G is KKM with respect to F in the sense of Park (or F-KKM(a) for short) if for every finite subset N of Y, one has $F(\mathrm{co}(N)) \subset G(N)$, in which case Y is assumed convex.

(b) Γ is F-KKM(b) if for every finite set $\{y_1, \dots, y_n\} \subset Y$, there exist $x_i \in F(y_i)$ such that for every index set $I \subset \{1, \dots, n\}$, one has $\mathrm{co}\{x_i : i \in I\} \subset G(\{y_i : i \in I\})$, in which case X is assumed convex.

For abstract convex spaces $(X; \Gamma)$ or $(Y; \Gamma)$, we note that a F-KKM(a) map is generalized to a KKM map with respect to F as in Section 2. Moreover, a F-KKM(b) map is extended to a generalized KKM map as in Section 3 in view of Park and Lee [14]. In fact, for each $y \in N \in \langle Y \rangle$, we can choose $\lambda_N(y) \in F(y)$.

In this section, we improve and generalize main results of Balaj and Luc [2]:

Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \to 2^Z$ with nonempty values, a multimap $G : D \to 2^Z$ is a *KKM map* with respect to F as in Section 2.

Let us present some conditions for G to be KKM with respect to F:

Proposition 4.2. *The map G is KKM with respect to F if and only if for every $z \in Z$, one has inclusion*

$$\mathrm{co}_\Gamma(D \setminus G^-(z)) \subset E \setminus F^-(z).$$

In particular, if $E = D$, each of the following conditions is sufficient for G to be KKM with respect to F:

(i) *The map $z \mapsto E \setminus G^-(z)$ has Γ-convex values and F is a submap of G, that is, $F(x) \subset G(x)$ for every $x \in E$.*

(ii) *The map $z \mapsto E \setminus F^-(z)$ has Γ-convex values and F is a submap of G.*

Proof. Assume that G is KKM with respect to F. Let $z \in Z$ and $x \in \Gamma_N$ with $N = \{y_i\}_{i=1}^n \in \langle D \setminus G^-(z) \rangle$. Since $F(\Gamma_N) \subset G(N) = \bigcup_{i=1}^n G(y_i)$ and $z \notin G(y_i)$ for all i by the hypothesis, z does not belong to $F(\Gamma_N)$. In particular it does not belong to $F(x)$, and hence $x \in E \setminus F^-(z)$.

Conversely, assume $\mathrm{co}_\Gamma(D \setminus G^-(z)) \subset D \setminus F^-(z)$ for all $z \in Z$. Let $y_1, \dots, y_n \in D$ and $x \in \mathrm{co}_\Gamma\{y_1, \dots, y_n\}$. Let $z \in F(x)$. We have to show that z belongs to $G(\{y_1, \dots, y_n\})$.

Suppose to the contrary that this is not true. Then, for each index i, $z \notin G(y_i)$ which yields $y_i \in D \setminus G^-(z)$. By the hypothesis $x \in E \setminus F^-(z)$, i.e. $z \notin F(x)$; a contradiction.

Further, under (i), for every $z \in Z$ one has $F^-(z) \subset G^-(z)$, and therefore $\mathrm{co}_\Gamma(E \setminus G^-(z)) = E \setminus G^-(z) \subset E \setminus F^-(z)$. By the first part, G is KKM with respect to F. Under (ii) one has $\mathrm{co}_\Gamma(E \setminus G^-(z)) \subset \mathrm{co}_\Gamma(E \setminus F^-(z)) = E \setminus F^-(z)$ and yields the same conclusion. □

This proposition generalizes [2, Proposition 3] for W-KKM(a).

In [2], it is defined that the map F that makes the family $\{\mathrm{cl}\, G(y) : y \in Y\}$ to have the finite intersection property whenever G is F-KKM(a), is said to have the KKM property. In our terminology, this means F is a $\mathcal{KC}$ map.

Regarding the KKM property, Balaj and Luc [2] summarized some known sufficient conditions as follows:

Proposition 4.3. ([2]) *Let X and Y be convex sets in topological vector spaces and let $W : Y \to 2^X$ be a set-valued map with nonempty values. Each of the following conditions is sufficient for W to have the KKM property:*

(i) *The closure of the image of every convex subset of Y under W is convex;*

(ii) *W has convex values and open fibers;*

(iii) *W is upper semicontinuous and has convex, compact values.*

Recall that the map W satisfying (ii) is called a Fan-Browder map, and the one satisfying (iii) a Kakutani map.

In our language in Section 2, Proposition 4.3 is to give examples of $F \in \mathcal{KC}(Y, X)$. Note that we already introduced a large number of $\mathcal{KC}$-maps; see the references of Sections 2. Recently, we have collected a large number of examples of $\mathcal{KC}$-maps in [12].

In order to establish existence of solutions to a mixed variational relation problem, Balaj and Luc [2] considered the following auxiliary variational inclusion problem.

Let X and Y be topological spaces and let $Q_1,\ Q_2,\ U : X \to 2^Y$ and $F : Y \to 2^X$ be multimaps.

(VI) Find $\bar{x} \in X$ such that
(i) $\bar{x} \in F \circ Q_1(\bar{x})$
(ii) $Q_2(\bar{x}) \subset U(\bar{x})$.

Balaj and Luc [2, Theorem 5] were able to establish sufficient conditions for existence of solutions to (VI). Here their theorem can be generalized by assuming $(Y; \Gamma)$ is an abstract convex space and F is a $\mathcal{KC}$-map.

Theorem 4.4. *Assume that X is a topological space, $(Y; \Gamma)$ is a partial KKM space, and that the following conditions hold:*

(i) *$Fix(F \circ Q_1)$ is a compact set;*

(ii) *Q_2 has nonempty values and open fibers, and $X \setminus Q_2^-(y)$ is compact for at least one $y \in Y$;*

(iii) $\mathrm{co}_\Gamma(Q_2(x)) \subset Q_1(x)$, *for each $x \in X$;*

(iv) *U^- is KKM with respect to F and its values are closed in X;*

(v) *F belongs to $\mathcal{KC}(Y, X)$.*

Then (VI) *has solutions.*

PROOF. Consider the map $P : Y \to 2^X$ defined by

$$P(y) = (X \setminus Q_2^-(y)) \cup (Fix(F \circ Q_1) \cap U^-(y)), \tag{1}$$

where $Fix(F \circ Q_1)$ denotes the set of all fixed points of the map $F \circ Q_1$ on X and $Q_2^-(y)$ is the fiber of Q_2 on y, that is $Q_2^-(y) = \{x \in X : y \in Q_2(x)\}$.

We show that P is a KKM map with respect to F. Let $N = \{y_1, \dots, y_n\}$ be a finite subset of Y and $x \in F(\Gamma_N)$. If $x \in Fix(F \circ Q_1)$, then since U^- is KKM with respect to F, one has

$$x \in Fix(F \circ Q_1) \cap U^-(N) = \bigcup_{i=1}^{n}(Fix(F \circ Q_1) \cap U^-(y_i)) \subset P(N).$$

If $x \in (X \setminus Fix(F \circ Q_1)) \cap Q_2^-(y_i)$ for all $y_i \in N$, then $y_i \in Q_2(x)$ and by (iii), $\Gamma_N \subset \mathrm{co}_\Gamma(Q_2(x)) \subset Q_1(x)$. Thus, $x \in F(\Gamma_N) \subset F(Q_1(x))$; a contradiction. Hence P is a KKM map with respect to F. In view of (v), the family $\{P(y) : y \in Y\}$ has the finite intersection property. Since P has closed values and $P(y)$ is compact for at least one $y \in Y$, by (v), there exists $\bar{x} \in \bigcap_{y \in Y} P(y)$. If $\bar{x} \notin Fix(F \circ Q_1)$ it follows that $\bar{x} \in X \setminus Q_2^-(y)$ for all $y \in Y$, which implies the contradiction $Q_2(x) = \emptyset$ (see (ii)). Hence $\bar{x} \in Fix(F \circ Q_1)$. For each $y \in Q_2(x)$, i.e. $\bar{x} \notin X \setminus Q_2^-(y)$, since $\bar{x} \in P(y)$, we have $\bar{x} \in U^-(y)$, that is, $y \in U(\bar{x})$. Thus $Q_2(\bar{x}) \subset U(\bar{x})$. □

Remark 4.5. The compactness of the set $Fix(F \circ Q_1)$ (condition (i) of the above theorem) is assured in each of the following situations:

(i) X is compact, Q_1 is upper semicontinuous with compact values and F is closed;

(ii) Y is compact, one of the maps F and Q_1^- is closed and the other is upper semicontinuous with compact values.

PROOF. (i) Let $\{x_t\}$ be a net in $Fix(F \circ Q_1)$ converging to a point x. Then, there exists a net $\{y_t\}$ in Y such that $y_t \in Q_1(x_t)$ and $x_t \in F(y_t)$, for all t. Since Q_1 is upper semicontinuous with compact values, there exist $y \in Q_1(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y. F is closed, and so $x \in F(y) \subset F(Q_1(x))$. Thus, $Fix(F \circ Q_1)$ is a closed subset of the compact X, hence it is compact too.

(ii) It is easy to see that the fixed point set of the map $F \circ Q_1$ coincides with the range of the map $F \cap Q_1^-$. Under the given conditions the map $F \cap Q_1^-$ is upper semicontinuous with compact values (see [2]). Since Y is compact, $(F \cap Q_1^-)(Y)$ is a compact set. □

When Y is not an abstract convex space, using the concept of F-KKM(b) maps, we may also establish existence of solutions to (VI).

Theorem 4.6. *Assume that* $(X; \Gamma)$ *is a partial KKM space,* Y *is a topological space, and that the following conditions hold:*

(i) $Fix(F \circ Q_1)$ *is compact;*

(ii) Q_2 *has nonempty values and open fibers, and* $X \setminus Q_2^-(y)$ *is compact for at least one* $y \in Y$*;*

(iii) $\mathrm{co}_\Gamma(F \circ Q_2(x)) \subseteq FQ_1(x)$ *for every* $x \in X$*;*

(iv) U^- *is F-KKM(b) and has closed values.*

Then (VI) *has solutions.*

Proof. We wish to show first that the map P defined by (1) is F-KKM(b). Let $\{y_1, \dots, y_n\}$ be a finite set in Y. By (iv) there are $x_i \in F(y_i)$ such that for each subset of indices $I \subset \{1, \dots, n\}$,

$$\mathrm{co}_\Gamma\{x_i : i \in I\} \subset \bigcup_{i \in I} U^-(y_i). \tag{2}$$

Let x be a point from the Γ-convex hull of $\{x_i : i \in I\}$. We prove that (2) implies

$$x \in \bigcup_{i \in I} P(y_i). \tag{3}$$

By (1) and (2) it follows that (3) holds when $x \in Fix(F \circ Q_1)$. If $x \in X \setminus Fix(F \circ Q_1) \cap Q_2^-(y_i)$, for all $i \in \{1, \dots, n\}$, then $y_i \in Q_2(x)$ and by (iii), $x \in \mathrm{co}_\Gamma\{x_i : i \in I\} \subset \mathrm{co}_\Gamma(F \circ Q_2(x)) \subset FQ_1(x)$; a contradiction. Thus P is F-KKM(b) and generalized KKM in the sense of Section 3 as well. Consequently the family $\{P(y) : y \in Y\}$ has the finite intersection property. Moreover, it follows from the hypothesis that P has closed values and at least one compact value. Hence the family $\{P(y) : y \in Y\}$ has some point $\bar{x}$ in common. Using the argument of proof of Theorem 4.4 we conclude that $\bar{x}$ is a solution of (VI). □

Moreover, conditions that assure the closedness of the values $P(y)$ can be weakened to the so-called intersectional closedness recently developed by Luc as follows:

$$\bigcap_{y \in Y} \mathrm{cl}\, P(y) = \mathrm{cl}(\bigcap_{y \in Y} P(y)).$$

Similarly, other results in [2] also can be improved or extended by reflecting recent development of the KKM theory.

5 Luc, Sarabi, and Soubeyran 2010 [6]

In [6], two main existence conditions for solutions of variational relation problems are established without convexity. The first one is based on a finite solvability property and the second one on generalized KKM map. These conditions unify and strengthen several existing results in the literature on the topic. A model of satisficing process by rejection is considered which gives an economic interpretation of the introduced concepts.

In Section 4 of [6], the authors establish existence conditions for variational relation problems that share certain properties of the so-called KKM maps.

We will use generalized KKM maps in the sense of Section 3. Now we assume that A and B are nonempty subsets of a abstract convex space $(X; \Gamma)$.

Definition 5.1. The relation R is said to be *generalized KKM* if for every finite subset $\{b_1, \dots, b_m\}$ of B there exists a corresponding subset $\{a_1, \dots, a_m\}$ of A such that $\mathrm{co}_\Gamma\{a_1, \dots, a_m\} \subset A$, for any subset $I \subseteq \{1, \dots, m\}$ and any $\bar{a} \in \mathrm{co}_\Gamma\{a_j : j \in I\}$, one can find some index $i \in I$ such that $R(\bar{a}, b_i, y)$ holds for all $y \in T(\bar{a}, b_i)$.

As for multimaps, KKM relations in [1, 13] are generalized KKM, but the converse is not true in general.

Here is the main result of this section on existence of solutions of (VR) when the relation R is generalized KKM.

Theorem 5.2. *The following conditions are sufficient for* (VR)*to have a solution:*
(i) *A is a nonempty compact set;*
(ii) *The multimap $P(.)$ is intersectionally closed on B;*
(iii) *$S_1(a) = A$ for every $a \in A$;*
(iv) *The relation R is generalized KKM.*

PROOF. We first prove that P is generalized KKM. Consider a finite subset $\{b_1, \ldots, b_m\}$ of B. Using (iv), we can find a corresponding subset $\{a_1, \ldots, a_m\}$ of A such that for any subset $I \subset \{1, \ldots, m\}$ and any $\bar{a} \in \mathrm{co}_\Gamma\{a_j : j \in I\}$, one can find some index $j \in I$ such that $R(\bar{a}, b_j, y)$ holds for all $y \in T(\bar{a}, b_j)$. This yields $\bar{a} \in P(b_j)$ which shows that P is generalized KKM. Since P is generalized KKM, for each $b \in B$ there is some $a \in A$ such that $a \in P(b)$. In particular $P(b)$ is nonempty for each $b \in B$. Now consider the multimap $b \mapsto \mathrm{cl}(P(b))$. It is a generalized KKM map too. Similarly to Lemma 4.1 of [5] the family $\{\mathrm{cl}(P(b)) : b \in B\}$ has the finite intersection property. By the abstract convex space version of the 1961 KKM-Fan lemma, that family has a common point, and so does the family $\{P(b) : b \in B\}$ in view of (ii). By Theorem 2.1 of [5] problem (VR) has a solution. □

The above theorem generalizes Theorem 3.1 of [5] in three aspects; for details, see [6].

6 Lin 2012 [4]

L.-J. Lin [4] studied the existence theorems of solutions for variational relation problems. From the existence theorems, he studied equivalent forms of generalized Fan-Browder fixed point theorem, existence theorems of solutions for Stampacchia vector equilibrium problems, and generalized Stampacchia vector equilibrium problems. His results contains many original results and have many applications in Nonlinear Analysis.

The following Fan-Browder type fixed point theorem is due to the present author [9] in 2016:

Theorem 6.1. ([9]) *Let $(E, D; \Gamma)$ be a partial KKM space* [*resp. a KKM space*], *and $F : E \to 2^D$, $G : E \to 2^E$ multimaps. Suppose that*
(1) *F^- is open-valued* [*resp. closed-valued*]*;*
(2) *for each $x \in E$, $\mathrm{co}_\Gamma F(x) \subset G(x)$;*
(3) *there exists a nonempty subset K of E such that $K \subset F^-(N)$ for some $N \in \langle D \rangle$; and*
(4) *either*
(i) *$E \setminus K \subset F^-(M)$ for some $M \in \langle D \rangle$; or*
(ii) *there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $L_N \setminus K \subset F^-(M)$ for some $M \in \langle D' \rangle$.*
Then there exists $\bar{x} \in E$ such that $\bar{x} \in G(\bar{x})$.

From this, we can deduce the following:

Theorem 6.2. *Let $(X;\Gamma)$ be a partial KKM space, Z be a set, and $H,\ S : X \to 2^Z$. Suppose that*

(i) *for each $y \in X$, $\{x \in X : H(x) \cap S(y) = \emptyset\}$ is open;*
(ii) *for each $x \in X$, the set $\{y \in X : H(x) \cap S(y) = \emptyset\}$ is nonempty and Γ-convex;*
(iii) *there is a compact subset K of X; and*
(iv) *$X \setminus K \subset \bigcup_{y\in M}\{x \in X : H(x) \cap S(y) = \emptyset\}$ for some $M \in \langle X\rangle$;*
Then there exists $\bar{x} \in X$ such that $H(\bar{x}) \cap S(\bar{x}) = \emptyset$.

Proof. Let $A : X \to 2^X$ be defined by

$$A(x) = \{y \in X : H(x) \cap S(y) = \emptyset\}.$$

By (ii), for each $x \in X,\ A(x)$ is a nonempty Γ-convex set. Note that

$$A^-(y) = \{x \in X : H(x) \cap S(y) = \emptyset\}$$

is open in X by (i) and that K is covered by a finite number of $A^-(y)'$s. By (iv), for each $x \in X \setminus K$, there exists $y \in M$ such that $x \in A^-(y)$. That is $X \setminus K = \bigcup_{y\in M} A^-(y)$. Then it follows from Theorem 6.1, there exists $\bar{x} \in X$ such that $\bar{x} \in A(\bar{x})$. Therefore $H(\bar{x}) \cap S(\bar{x}) = \emptyset$. □

Remark 6.3. In Theorem 3.1, if $H(x_1) = \emptyset$ for some $x_1 \in X$, then $H(x_1) \cap S(x_1) = \emptyset$, and Theorem 6.2 holds.

The base of [4] is Theorem 3.1 there, which can be extended to the following consequence of Theorem 6.1:

Theorem 6.4. *Theorem 6.2 holds by assuming the following instead of* (iv)*:*

(v) *there exists a Γ-convex subset L_N of X relative to some $D' \subset X$ such that $N \subset D'$ and $L_N \setminus K \subset \bigcup_{y\in M}\{x \in X : H(x) \cap S(y) = \emptyset\}$ for some $M \in \langle D'\rangle$.*

Proof. As in the proof of Theorem 6.2, Conditions (1)–(3) of Theorem 6.1 are satisfied. Moreover, condition (v) satisfies Theorem 6.1(ii). Hence, the conclusion follows from Theorem 6.1. □

7 Balaj 2013 [1]

Balaj [1] investigates the existence of solutions for three types of variational relation problems which encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field.

Let $X,\ Y$ and Z be nonempty sets. A nonempty subset R of the product $X \times Y \times Z$ determines a relation $R(x, y, z)$ in a natural manner: we say that $R(x, y, z)$ holds if and only if $(x, y, z) \in R$. When Z is a parameter set, then R is called a variational relation.

Now we generalize Balaj's three types as follows:

Assume that $(X;\Gamma)$ is an abstract convex space and Y and Z are two sets, endowed for each problem with an adequate topological and/or algebraic structure. Let $T : X \to 2^Y,\ P : X \to 2^Z$ be two multimaps and $R(x, y, z)$ be a relation linking elements $x \in X,\ y \in Y,\ z \in Z$.

(VRP1a) Find $\bar{x} \in X$ such that $R(\bar{x}, y, z)$ holds for all $y \in T(\bar{x})$ and all $z \in P(\bar{x})$.

(VRP1b) Find $\bar{x} \in X$ such that for each $y \in T(\bar{x})$ there exists $z \in P(\bar{x})$ such that $R(\bar{x}, y, z)$ holds.

(VRP2) Find $\bar{x} \in X$ and $\bar{z} \in P(\bar{x})$ such that $R(\bar{x}, y, \bar{z})$ holds for all $y \in T(\bar{x})$.

These problems encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field. Actually, Balaj [1] listed a few typical examples.

In order to study the solution existence of problems (VRP1a) and (VRP1b), Balaj established the inclusion result [1, Theorem 3.1], which can be generalized as follows:

Theorem 7.1. *Let $(X; \Gamma)$ be a partial KKM space, and Y be a nonempty set. Assume that $T, S : X \to 2^Y$ are two multimaps with nonempty values satisfying:*
(i) *T has open fibers and $X \setminus T^-(y)$ is compact for at least one $y \in Y$;*
(ii) *S has closed fibers;*
(iii) *the set $Z = \{x \in X : x \in (S^-T)(x)\}$ is compact;*
(iv) *S^- is a generalized KKM map.*
Then there exists $\bar{x} \in X$ such that $T(\bar{x}) \subseteq S(\bar{x})$.

Proof. Consider the map $Q : Y \to 2^X$ defined by $Q(y) = (X \setminus T^-(y)) \cup (Z \cap S^-(y))$. We show that Q is a generalized KKM map as in Section 3. If $\{y_0, ..., y_n\}$ is a finite subset of Y , by (iv), there exists a subset $\{x_0, ..., x_n\}$ of X such that for each subset of indices $I \subseteq \{0, ..., n\}$,

$$\mathrm{co}_\Gamma\{x_i : i \in I\} \subset \bigcup_{i \in I} S^-(y_i). \tag{1}$$

Let $x \in \mathrm{co}_\Gamma\{x_i : i \in I\}$. We prove that (1) implies

$$x \in \bigcup_{i \in I} Q(y_i). \tag{2}$$

If $x \in Z$, by (1) one has

$$x \in Z \cap (\bigcup_{i \in I} S^-(y_i)) = \bigcup_{i \in I} (Z \cap S^-(y_i)) \subset \bigcup_{i \in I} Q(y_i).$$

If $x \in X \setminus Z$, we claim that $x \in X \setminus T^-(y_i)$ for some index $i \in I$. Suppose on the contrary that $y_i \in T(x)$ for all $i \in I$. Then $S^-(y_i) \subset S^-(T(x))$. In view of (1), we have

$$x \in \mathrm{co}_\Gamma\{x_i : i \in I\} \subset \bigcup_{i \in I} S^-(y_i) \subset S^-(T(x));$$

a contradiction. Hence, $x \in \bigcup_{i \in I}(X \setminus T^-(y_i)) \subset \bigcup_{i \in I} Q(y_i)$. Since Q has closed values and $Q(y)$ is compact for at least one $y \in Y$, by Corollary 3.2, there exists $\bar{x} \in \bigcap_{y \in Y} Q(y)$. For each $y \in T(\bar{x})$, i.e. $\bar{x} \notin X \setminus T^-(y)$, since $\bar{x} \in Q(y)$, we have $\bar{x} \in S^-(y)$, that is $y \in S(\bar{x})$. Thus $T(\bar{x}) \subset S(\bar{x})$ and this means exactly the conclusion of the theorem. □

Remark 7.2. Let us observe that

$$Z = \{x \in X : \exists y \in Y \text{such that } x \in T^-(y) \cap S^-(y)\} = (T^- \cap S^-)(Y).$$

Hence condition (iii) in Theorem 7.1 means actually the compactness of the range of the map $T^- \cap S^-$.

Actually Theorem 7.1 is due to Balaj [1, Theorem 3.1] for a convex set X in a topological vector space, and we followed his proof. He applied his theorem to study the solution existence of certain variational relation problems (VRP1a) and (VRP1b).

Other results in [1] might be extended to abstract convex spaces.

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