Journal of Nonlinear and Convex Analysis
Volume 19, Number 9, 2018, 1493–1501

ON THE MINIMAX INEQUALITY OF BRÉZIS-NIRENBERG-STAMPACCHIA

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Abstract. Since the celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem appeared in 1929, a large number of its generalizations and modifications followed. Based on a lemma which generalizes the KKM theorem, Brézis-Nirenberg-Stampacchia (simply BNS) obtained a slightly more general result than the 1961 KKM Lemma of Ky Fan. Then they obtained a generalization of the 1972 minimax inequality of Ky Fan and some of its applications. In the present article, we show that one of our previous KKM type theorems for abstract convex spaces can be applied to generalize the KKM type lemma and the minimax inequality due to BNS. Using our results, we can correct certain results of BNS.

1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz (simply, KKM) first considered the closed-valued multimap $F : V \multimap \Delta_n$, where V is the set of vertices of the standard n-simplex Δ_n, satisfying

$$\operatorname{co} A \subset F(A) \quad \text{for each } A \subset V.$$

This kind of multimaps are called later the KKM maps. The KKM (intersection) theorem [4] says that $\bigcap_{v\in V} F(v) \neq \emptyset$. The recently developed KKM theory on abstract convex spaces is the study of applications of various equivalent formulations or generalization of the KKM theorem.

Since 1961 Ky Fan obtained an infinite dimensional version of the KKM theorem in arbitrary topological vector spaces and showed that his KKM lemma provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Consequently, at the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84. Then, the KKM theory was extended to convex spaces by Lassonde in 1983, and to c-spaces (or H-spaces) by Horvath in 1983-93 and others. Moreover, since 1993 the theory is extended to generalized convex (G-convex) spaces in a sequence of articles of the present author and others.

While G-convex spaces were investigated by a large number of authors, the concept has been challenged by several authors who aimed to obtain more general concepts. In fact, a number of modifications or imitations of G-convex spaces followed. It is known in 2007-2009 that all of such spaces belong to the class of ϕ_A-spaces due to ourselves. Furthermore, since 2006, all of the above mentioned classes of spaces are unified to that of abstract convex spaces, and the KKM theory tends to the

1991 *Mathematics Subject Classification.* 47H10, 49J53, 54C60, 54H25, 90A14, 90C76, 91A13, 91A10.

Key words and phrases. Abstract convex space, KKM theorem, KKM space, Ky Fan's minimax inequality.

research of such new spaces. Consequently, a large number of generalizations and modifications of the KKM theorem followed. See [6]-[12] and the references therein.

On the other hand, based on a lemma which generalizes a finite dimensional result of KKM, Brézis, Nirenberg and Stampacchia (simply BNS) in 1972 [1] obtained a slightly more general result than the 1961 KKM Lemma of Ky Fan. Then they obtained a generalization of the 1972 minimax inequality of Ky Fan and some applications.

In the present article, we show that one of our previous KKM type theorems for abstract convex spaces in 2012 [10] can be applied to generalize the KKM type lemma and the minimax inequality due to BNS. Moreover, using our results, we can correct most results of BNS.

Section 2 is a preliminary on our abstract convex spaces. In Section 3, we show that one of our previous KKM theorems in [9] implies generalizations of the main results of BNS [1]. Section 4 deals with the main results of BNS [1] in order to compare them with our corresponding generalizations. In Section 5, we give some applications of our BNS type inequality to generalize the applications given in BNS [1]. In these last two sections, we made some corrections or improvements of results in BNS [1].

2. Abstract convex spaces

For the concepts on our abstract convex spaces, KKM spaces and the KKM maps, we follow [6] and [10] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent works, we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the (partial) KKM principle. See [6], [10] and the references therein.

Definition 2.4. A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept extends the one due to Lassonde for $X = D$.

We obtained the following diagram for abstract convex spaces $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Note that each implication in the above diagram is proper; that is, its converse does not hold.

Consider the following related four conditions for a multimap $G : D \multimap Z$ to a topological space Z:

(1) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(2) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued* [5]).

(3) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(4) G is closed-valued.

In [5], Luc et al. noted that (1) $\Longleftarrow$ (2) $\Longleftarrow$ (3) $\Longleftarrow$ (4), and gave examples of multimaps satisfying (2) but not (3). According to Luc, the concept of intersectionally closed sets are originated from Rockafellar in 1970.

The following KKM type theorem is due to ourselves; see [10].

Theorem 2.5. *Let $(E, D; \Gamma)$ be a partial KKM space [resp. KKM space] and $G : D \multimap E$ a KKM map such that*

(1) *G is closed-valued [resp. open-valued].*

Then the family $\{G(z) \mid z \in D\}$ has the finite intersection property.

Moreover, suppose that

(2) *there exists a nonempty compact subset K of E such that one of the following holds:*
   (i) *$K = E$;*
   (ii) *$K = \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D \rangle$; or*
   (iii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $L_N \cap \bigcap_{z \in D'} \overline{G(z)} \subset K$.*

Then $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

Furthermore,

(α) *if G is transfer closed-valued, then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$;*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Cases (i) and (ii) are immediate routine consequences of the definition of partial KKM spaces. Theorem 2.5 implies the Fan matching property, some geometric property, the Fan-Browder fixed point property, some minimax inequality, several variational inequalities, von Neumann type minimax theorems, Nash equilibrium theorems, and many others. See [5] with some corrections in [10] and the references therein.

3. Generalization of the Brézis-Nirenberg-Stampacchia inequality

In this section, we show that our KKM theorem 2.5 implies generalizations of the main results of BNS [1].

Theorem 1 of [1] can be generalized as follows:

Theorem 3.1. *Let $(E, C; \Gamma)$ be an abstract convex space such that C is a Γ-convex subset of E. Let f be a real valued function defined on $C \times C$ such that*

(a) *$f(x,x) \leq 0$ for all $x \in C$.*
(b) *For every fixed $x \in C$, the set $\{y \in C \mid f(x,y) > 0\}$ is Γ-convex.*
(c) $\overline{\bigcap_{y\in C}\{x \in C \mid f(x,y) \leq 0\}} = \bigcap_{y\in C} \overline{\{x \in C \mid f(x,y) \leq 0\}}$.
(d) *There exist a closed compact subset L of E and $y_0 \in L \cap C$ such that $f(x, y_0) > 0$ for $x \in C$, $x \notin L$.*

Conclusion — *There exists $x_0 \in L \cap C$ such that*

$$f(x_0, y) \leq 0 \quad for\ all \quad y \in C.$$

Thus in particular $\inf_{x\in C} \sup_{y\in C} f(x,y) \leq 0$.

Note that (c) holds when $x \mapsto f(x,y)$ is l.s.c. for each $y \in C$.

The following generalizes Lemma 1 of [1]:

Lemma 3.2. *Let $(E, X; \Gamma)$ be a partial KKM space, X an arbitrary subset of E, and $F : X \multimap E$ a multimap satisfying*

(p) *$\overline{F(x_0)} = K$ is compact for some $x_0 \in X$.*
(q) *$\Gamma_A \subset F(A)$ for each $A \in \langle X \rangle$.*
(r) *(intersectional closedness) $\overline{\bigcap_{x\in X} F(x)} = \bigcap_{x\in X} \overline{F(x)}$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

Proof of Lemma 3.2. Note that $\overline{F} : X \multimap E$ is a closed-valued KKM map by (q). Then $\{\overline{F}(x) \mid x \in X\}$ has the finite intersection property by the first part of Theorem 2.5. Moreover, $K = \overline{F(x_0)}$ is compact. Hence, by Theorem 2.5(ii), we have $\bigcap\{\overline{F(x)} \mid x \in X\} \neq \emptyset$. Moreover, by (r), F is intersectionally closed-valued. Therefore, by Theorem 2.5 for the case (β), we have $\bigcap_{x\in X} F(x) \neq \emptyset$. This completes our proof. □

The coercivity condition (p) satisfies (ii) of Theorem 2.5. Instead we can adopt other coercivity condition (i) or (iii) and obtain another variants of Lemma 3.2.

Proof of Theorem 3.1. For each $y \in C$ let

$$F(y) := \{x \in C \mid f(x,y) \leq 0\}.$$

We show the following:

(p) $\Longleftarrow$ (d): Since

$$x \in C \setminus L \Longrightarrow f(x, y_0) > 0 \Longrightarrow x \notin F(y_0)$$

or

$$x \in F(y_0) \Longrightarrow x \notin C \setminus L \Longrightarrow x \in L,$$

we have $F(y_0) \subset L$. Since L is closed, we have $\overline{F(y_0)} \subset \overline{L} = L$. Since L is compact, so is $\overline{F(y_0)}$.

(r) $\Longleftarrow$ (c): Clear.

(q) $\Longleftarrow$ (a) and (b): Suppose that (q) does not hold. Then for some choice of $A := \{y_1, y_2, \ldots, y_n\} \in \langle C \rangle$ with $\Gamma_A \not\subset F(A)$, and we have an $x_0 \in \Gamma_A$ such that

$$x_0 \notin F(A) = \bigcup_{i=1}^{n} F(y_i) \Longrightarrow x_0 \notin F(y_i) \Longrightarrow f(x_0, y_i) > 0 \ \forall i.$$

By (b) the set $\{y \in C \mid f(x_0, y) > 0\}$ is Γ-convex and contains A. Hence

$$x_0 \in \Gamma_A \subset \{y \in C \mid f(x_0, y) > 0\} \Longrightarrow f(x_0, x_0) > 0$$

which contradicts (a).

The conclusion of Theorem 3.1 — which is equivalent to $\bigcap_{y \in C} F(y) \neq \emptyset$ — follows. This completes our proof of Theorem 3.1. □

Instead of the closedness of L in (d), we may assume Hausdorffness of E in Theorem 3.1.

Moreover, note that Theorem 3.1(d) is based on the coercivity condition (ii) in Theorem 2.5, but we can obtain another variant of Theorem 3.1 by adopting other coercivity condition (i) or (iii).

4. Main results of Brézis-Nirenberg-Stampacchia

In order to compare our new results, Theorem 3.1 and Lemma 3.2, with the corresponding ones of BNS [1], we state them as follows:

Theorem 4.1 ([1]). *Let C be a convex subset of a Hausdorff topological vector space E and let f be a real valued function defined on $C \times C$ such that*

(1) *$f(x, x) \leq 0$ for all $x \in C$.*
(2) *For every fixed $x \in C$, the set $\{y \in C \mid f(x, y) > 0\}$ is convex.*
(3) *For every fixed $y \in C$, $f(x, y)$ is an l.s.c. function of x on the intersection of C with any finite dimensional subspace of E.*
(4) *Whenever $x, y \in C$ and x_α is a filter on C converging to x, then $f(x_\alpha, (1-t)x + ty) \leq 0$ for every $t \in [0, 1]$ implies $f(x, y) \leq 0$.*
(5) *There exist a compact subset L of E and $y_0 \in L \cap C$ such that $f(x, y_0) > 0$ for $x \in C$, $x \notin L$.*

Conclusion — *There exists $x_0 \in L \cap C$ such that*

$$f(x_0, y) \leq 0 \quad \textit{for all} \quad y \in C.$$

Thus in particular $\inf_{x \in C} \sup_{y \in C} f(x, y) \leq 0$.

In (3), the lower semicontinuity is stated using the finitely generated topology of E.

Early in 1996, Chowdhury and Tan [2] noted that "if the compact set L is a subset of C, C is not required to be closed in E in Theorem 4.1. Note also that in Theorem 4.1, the set C was not assumed to be closed in E. However, this is false in general as is observed by" an example in a lecture note by Tan.

Note that the following counter-example of Theorem 4.1:

Example. Let $C := (0,1)$ in $E := \mathbb{R}$, the real line, and $f(x,y) = y - x$. Then the requirements (1)-(4) are clearly satisfied. For (5), take $L := [0.5, 1]$ and $y_0 := 0.5$. Then the conclusion of Theorem 4.1 does not hold. Moreover, Theorem 4.1(3) does not imply Theorem 3.1(c).

In view of this example we should assume that *"C is closed"* in Theorem 4.1.

Proof of Theorem 4.1 from Theorem 3.1. We may assume E has the finitely generated topology. Then E becomes a convex space. Note that the requirements (1), (2), and (5) imply (a), (b), and (d) of Theorem 3.1, respectively. Moreover, we see the following.

(3) $\Longrightarrow$ (c): For any finite dimensional subspace H of E and any $y \in C$,

$$\{x \in C \mid f(x,y) \leq 0\} \cap F = \{x \in C \cap F \mid f(x,y) \leq 0\}$$

is closed, and hence $\{x \in C \mid f(x,y) \leq 0\}$ is finitely closed. Therefore, (c) holds.

Now the conclusion follows from Theorem 3.1. □

Note that *the requirement (4) in Theorem 4.1 are redundant* and C *should be closed* in the above proof. Moreover, *Hausdorffness of E can be replaced by the closedness of L.*

As we noted at the end of Section 3, the requirement (d) is based on the coercivity condition (ii) in Theorem 2.5. Similarly, we can obtain another variant of Theorem 4.1 by adopting other coercivity condition (i) or (iii).

The paper [1] is based on their Lemma 1 (the following Lemma 4.2) which generalizes a finite dimensional result of KKM [3] and is slightly more general than the 1961 KKM Lemma of Ky Fan.

Lemma 4.2 ([1])**.** *Let X be an arbitrary set in a Hausdorff topological vector space E, and $F : X \multimap E$ a map satisfying*

(6) $\overline{F(x_0)} = L$ *is compact for some* $x_0 \in X$.
(7) $\mathrm{co}A \subset F(A)$ *for each* $A \in \langle X \rangle$.
(8) *For every $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace is closed.*
(9) *For every convex subset D of E we have*

$$\overline{\bigcap_{x \in X \cap D} F(x)} \cap D = \bigcap_{x \in X} F(x) \cap D.$$

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Other results in [1] are consequences of Lemma 4.2. In Lemma 4.2, the closure operation on E is given with respect to its original topology, and (8) is for extra finitely generated topology with respect to the family of finite dimensional subspaces. However, *we proved Lemma 4.2 without assuming Hausdorffness of E and (8)*. Moreover we can replace (9) by its following particular case for $D = E$:

(r′) (transfer closedness) $\overline{\bigcap_{x\in X} F(x)} = \bigcap_{x\in X} F(x)$.

Here we add the following proof of Theorem 4.1 with the closed C:

Proof of Theorem 4.1 from Lemma 4.2. Let C be a *closed* convex subset and, for each $y \in C$, let

$$F(y) := \{x \in C \mid f(x, y) \le 0\}.$$

Requirements (6) and (8) follow clearly from (5) and (3), respectively. As in BNS [1], (1) and (2) imply (7). The condition (4) may be needed in this proof in order to show (9). To see this, let D be any convex subset of E. Let $y \in \overline{\bigcup_{x\in C\cap D} F(x)} \cap D$. Then $y \in D$ and there exists a filter $(y_\alpha)_{\alpha\in\Lambda}$ in $\bigcup_{x\in C\cap D} F(x)$ such that $y_\alpha \to y$. Thus

$$f(x, y_\alpha) \le 0 \text{ for all } \alpha \in \Lambda \text{ and } x \in C \cap D \qquad (*)$$

Since C is closed (this is an essential condition) and $y \in D$, we have $y \in C \cap D$. Since $x, y \in C \cap D$ and $C \cap D$ is convex,

$$tx + (1-t)y \in C \cap D \quad \text{for all } t \in (0, 1].$$

It follows from $(*)$ that

$$f(tx + (1-t)y, y_\alpha) \le 0 \text{ for all } \alpha \text{ and } x \in C \cap D.$$

By (4), $f(x, y) \le 0$ for all $x \in C \cap D$, so $y \in F(x)$ for all $x \in C \cap D$ and $y \in \bigcup_{x\in C\cap D} F(x) \cap D$. Thus

$$\overline{\bigcap_{x\in X\cap D} F(x)} \cap D = \bigcap_{x\in X\cap D} F(x) \cap D.$$

By Lemma 4.2, the conclusion of Theorem 4.1 — which is equivalent to $\bigcap_{y\in C} F(y) \ne \emptyset$ — follows. □

5. Applications

BNS [1] applied their results to four problems. In this section, we give some comments to applications in the BNS paper [1].

Application 1 (Ky Fan [3]). *Assume C is compact convex in a t.v.s. E, f satisfies (a) and (b), and for any $y \in C$, $f(x, y)$ is an l.s.c. function of x on C. Then there exists an $x_0 \in C$ such that $f(x_0, y) \le 0$ for all $y \in C$.*

This follows from Theorem 3.1 with $L = \emptyset$ and without assuming Hausdorffness of E. Similarly, Theorem 3.1 implies the following:

Application 1′. *Assume C is compact convex in a t.v.s. E, f satisfies (a), (b), and (c). Then there exists an $x_0 \in C$ such that $f(x_0, y) \le 0$ for all $y \in C$.*

Here requirement (c) can be replaced by some particular forms without affecting its conclusion.

Application 2 ([1]). *Let C be a convex subset of a t.v.s. E and let $f(x,y)$ be a real valued function on $C \times C$ satisfying (1), (3), (5) of Theorem 4.1 and*

$(2')$ *For every $x \in C$ and for every $c \geq 0$, the set $\{y \in C \mid f(x,y) \geq c\}$ is closed and convex.*

(10) *For every $x, y \in C$, $f(x,y) \leq 0$ implies $f(y,x) \geq 0$.*

(11) *If $f(x,y_1) > f(x,y_2) \geq 0$, then*

$$f(x, ty_1 + (1-t)y_2) \geq f(x,y) \ for 0 < t \leq 1.$$

Conclusion — *There exists $x_0 \in L \cap C$ such that $f(x_0,y) \leq 0$ for all $y \in C$.*

In order to apply Theorem 4.1, we have to assume *"C is closed."*

Note that BNS proved Application 2 by showing $(2') \Rightarrow (2)$ and (10)&(11) $\Rightarrow$ (4) in their Theorem 4.1. However, we already showed that *the requirement (4) is redundant* in Theorem 4.1 and hence, *so are (10) and (11)* in Application 2.

Application 3 ([1]). *Let C be a convex subset of E and let $f(x,y) = \langle Ax, x-y\rangle + \varphi(x) - \varphi(y)$ where A is a pseudo-monotone mapping from C into E^* (i.e. whenever x_α is a filter converging to x with $\limsup\langle Ax_\alpha, x_\alpha - x\rangle \leq 0$ then $\liminf\langle Ax_\alpha, x_\alpha - y\rangle \geq \langle Ax, x-y\rangle$ for every $y \in C$). Assume that A is continuous on any finite dimensional subspace and that φ is an l.s.c. convex function. If (5) holds, then there exists $x_0 \in L \cap C$ such that*

$$\langle Ax_0, x_0 - y\rangle + \varphi(x_0) - \varphi(y) \leq 0 \quad for\ every \quad y \in C.$$

In order to apply Theorem 4.1, we have to assume *"C is closed."*

Note that assumptions (1), (2), (3) in Theorem 4.1 are clearly satisfied. BNS proved (4) from the pseudo-monotonicity of A. However, *assumption (4) is redundant in Theorem 4.1. Therefore, so is the pseudo-monotonicity of A in Application 3.*

Brézis, Nirenberg, and Stampacchia [1] gave another Application 4 to the von Neumann-Sion type minimax principle which can also be improved by applying Theorem 3.1 instead of their Theorem 4.1. Recall that we had already some abstract forms of that principle in [7, 9].

Moreover, particular cases of our Theorem 2.5 have many known applications; see [6, 7, 8, 9, 10, 11, 12] and the references therein.

Acknowledgement

The author would like to express his gratitude to Professor Mau-Hshiang Shih for his various suggestions to this paper. Especially, the example in Section 4 and a new proof of Theorem 4.1 from Lemma 4.2 are suggested by him.

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