Nonlinear Functional Analysis and Applications
Vol. 23, No. 4 (2018), pp. 805-818
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

GENERALIZATIONS OF SOME KKM TYPE RESULTS ON HYPERBOLIC SPACES

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Abstract. In some previous works, it is known that the KKM type results on Hadamard manifolds can be extended to hyperbolic spaces originated from Kirk in 1982 and Reich-Shafrir in 1990. Such results are the KKM theorem, the Fan-Browder fixed point theorem, Nash equilibrium theorem, variational inequalities, etc. based on our theory of abstract convex spaces. In the present article, we show that our method can be applied some recent works on Hadamard manifolds. Historical remarks are added on the study of the KKM type results on Hadamard manifolds and hyperbolic spaces.

1. Introduction

In 1982, Kirk [3] extended the Krasnoselskii iteration scheme for approximation of fixed points of nonexpansive mappings in Banach spaces to a wider class of spaces including convex metric spaces of 'hyperbolic' type or hyperbolic spaces. In 1990, Reich and Shafrir [16] introduced hyperbolic spaces as a class of metric spaces containing all normed vector spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls.

Since we began to study the KKM theory in 1992, we have been studied the foundations of the theory on various types of the KKM spaces, and then finally

[0]Received May 13, 2018. Revised September 26, 2018.
[0]2010 Mathematics Subject Classification: 47H04, 47H10, 49J27, 49J35, 54H25, 90C47, 91B50.
[0]Keywords: Abstract convex space, KKM theorem, (partial) KKM space, hyperbolic space, Hadamard manifold, Fan-Browder type fixed point theorem, Nash equilibrium theorem, variational inequalities.

we established the theory on abstract convex spaces in 2010 [13]. While we were studying the KKM theory, in 2008, we found that any hyperbolic spaces are G-convex spaces [9] and also particular cases of c-spaces [10-13]. Actually, in 2010 [12, 13], we indicated but not concretely that most of key results in the KKM theory can be applied to hyperbolic spaces.

Since then, a number of authors have studied some KKM theoretic results on Hadamard manifolds. For example, Németh [6] introduced and studied variational inequalities on Hadamard manifolds, and Zhou and Huang [19, 20] introduced a KKM type theorem on Hadamard manifolds with some applications to a mixed variational inequality and a Fan-Browder fixed point theorem. Moreover, in 2012, Colao, Lopez, Marino, and Martin-Marquez [1] developed an equilibrium theory in Hadamard manifolds. Further, Yang and Pu [17] proved a Fan-Browder type fixed point theorem on Hadamard manifolds with strongly geodesic convexity. It is clear that such results are closely related to the KKM theory on hyperbolic spaces.

In our previous work [14], we showed that three of key results of Colao et al. [1] can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of ours in [12, 13]. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds.

Moreover, in 2015, Lee [5] showed that the main results of [17, 19] also can be derived from the corresponding ones on abstract convex spaces. This is not the end of story. Very recently in 2018, W. Kim [2] obtained some obsolete results in the KKM theory on Hadamard manifolds.

Our aim in this paper is to show that works of W. Kim and others are still can be generalized and simplified in the frame of our abstract convex spaces following the spirit of Park [14] and Lee [5].

Section 2 devotes to review some preliminary facts on our abstract convex spaces as in [12, 13]. In Section 3, we are concerned with definitions and examples of hyperbolic spaces and we show that any of such spaces are KKM spaces, which means that most results in [12, 13] are applicable to them. Section 4 deals with KKM type theorems on hyperbolic spaces. Some remarks are added for such KKM type theorems for Hadamard manifolds due to other authors.

In Section 5, we give the Fan-Browder type fixed point theorems for hyperbolic spaces generalizing other authors' results on Hadamard manifolds. Section 6 deals on the Nash type equilibrium theorems due to W. Kim [2]. We generalize them to the ones on hyperbolic spaces. Finally, in Section 7, we recall some history of the study of KKM type results on Hadamard manifolds and hyperbolic spaces.

2. Abstract convex spaces

We follow our previous works [12, 13] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Our KKM theory concerns with the study of partial KKM spaces and their applications.

For typical examples of KKM spaces, see [13] and the references therein.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned} &\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Convex space} \Longrightarrow \text{H-space} \\ &\Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ &\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.} \end{aligned}$$

Recall that, in 2010 [12], we derived generalized forms of the Ky Fan minimax inequality, the von Neumann-Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for partial KKM spaces. Consequently, our results in [12] unify and generalize most of previously known particular cases of the same nature.

Moreover, in [13], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [13] unifies and enlarges previously known several proper examples of such statements for particular types of partial KKM spaces.

3. Hyperbolic spaces

After the pioneering work of Kirk [3] appeared in 1982, Reich and Shafrir [16] in 1990 introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces:

Definition 3.1. ([16]) Let (X, ρ) be a metric space and $\mathbb{R}$ the real line. We say that a map $c : \mathbb{R} \to X$ is a *metric embedding* of $\mathbb{R}$ into X if

$$\rho(c(s), c(t)) = |s - t|$$

for all real s and t. The image of a metric embedding is called a *metric line*. The image of a real interval $[a, b]:= \{t \in \mathbb{R} \mid a \leq t \leq b\}$ under such a map is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines, such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment denoted by $[x, y]$ joining x and y. For each $0 \leq t \leq 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \quad \text{and} \quad \rho(z, y) = (1 - t)\rho(x, y).$$

This point z is denoted by $(1 - t)x \oplus ty$.

We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \leq \frac{1}{2}\rho(y, z)$$

for all x, y and z in X.

Example 3.2. ([16]) The following are examples of hyperbolic spaces:

(1) All normed vector spaces.
(2) All Hadamard manifolds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of constant curvature.
(3) The Hilbert ball equipped with the hyperbolic metric.
(4) Arbitrary product of hyperbolic spaces.

Definition 3.3. ([16]) A subset C of a hyperbolic space X is said to be *convex* if, for each pair of points x and y in C, the metric segment $[x, y]$ is also contained in C. The *closed convex hull* of a subset D of X is the intersection of all closed convex subsets of X which contains D.

In the class of Hadamard manifolds, the authors [6, 17, 19] defined more special concepts of convexity as follows:

Definition 3.4. A set K of an Hadamard manifold M is said to be *geodesic convex* if for any $p, q \in K$, the geodesic joining p to q is contained in K, that is, for any $p, q \in K$, $\exp_p(t \exp_p^{-1} q) \in K$ for all $t \in [0, 1]$.

A set K of an Hadamard manifold M is said to be *strongly geodesic convex* if for any given $o \in M$ and for any $p, q \in K$, $\exp_o((1-t)\exp_o^{-1} p + t \exp_o^{-1} q) \in K$ for all $t \in [0, 1]$.

Since any geodesic is a metric segment and the point o can be given by p, a strongly geodesic convex set in an Hadamard manifold is a geodesic convex set and hence a convex set in a hyperbolic space. See also [4].

In our previous works, we noted that any hyperbolic spaces are G-convex spaces [8] and also particular cases of c-spaces [10-13]. This can be strengthened as follows:

Definition 3.5. The *convex hull* $\operatorname{co} D$ of a subset D of a hyperbolic space X is the intersection of all convex subsets of X which contains D.

Lemma 3.6. ([14]) *Any convex subset Y of a hyperbolic space $X = (X, \rho, M)$ can be made into a c-space $(X; \Gamma)$ and hence a KKM space.*

In view of Lemma 3.6, all results in [12, 13] hold for any convex subset of a hyperbolic spaces.

4. The KKM theorem on hyperbolic spaces

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems in [15] for abstract convex spaces:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K = Z$;*

(ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Since any hyperbolic space is a partial KKM space, Theorem C is applicable to hyperbolic spaces.

By putting $E = Z$ and $F=\mathrm{id}_E$, we immediately have the following form of the KKM theorem in the setting of hyperbolic spaces:

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, where E is a hyperbolic space, $D \subset E$, and Γ is the convex hull operation, and $G : D \multimap E$ a map such that*

(1) *G is a closed-valued KKM map; and*

(2) *there exists a nonempty compact subset K of E such that either*

(i) *$K = E$;*

(ii) *$\bigcap\{G(y) \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a closed compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, and*

$$L_N \cap \bigcap_{y\in D'} G(y) \subset K.$$

Then we have

$$K \cap \bigcap_{y \in D} G(y) \neq \emptyset.$$

The following is a simple observation:

Theorem 4.2. *Let M be a hyperbolic space and $K \subset M$ a convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is closed. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*

Moreover, if there exists $x_0 \in K$ such that $G(x_0)$ is compact, then

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

Proof. By Lemma 3.6, a convex subset $K \subset M$ is a KKM space. Hence, by the definition itself, the conclusion follows. □

In [19, Theorem 3.1], the first half of Theorem 4.2 for Hadamard manifolds with a KKM map $G : K \multimap M$ and compact K was proved. They could deduce only the finite intersection property of map-values of G. Here, the compactness is redundant.

In [19, Theorem 3.2], Theorem 4.2 for Hadamard manifolds with a KKM map $G : K \multimap M$ and compact K was proved as follows:

Corollary 4.3. *Let M be a Hadamard manifold and $K \subset M$ a geodesic convex subset. Let $G : K \multimap M$ be a closed-valued KKM mapping and there exists at least one $x_0 \in K$ such that $G(x_0)$ is compact in M, then*

$$\bigcap_{x \in K} G(x) \neq \emptyset.$$

This is the basis of Zhou and Huang [19, 20].

Colao et al. [1, Lemma 3.1] provided Corollary 4.3 with almost two page proof.

For open-valued KKM map, we have the following:

Theorem 4.4. *Let M be a hyperbolic space and $K \subset M$ a convex subset. Let $G : K \multimap K$ be a KKM map such that, for each $x \in K$, $G(x)$ is open. Then $\{G(x) \mid x \in K\}$ has the finite intersection property.*

Proof. By Lemma 3.6, K is a KKM space. Hence, by the definition itself, the conclusion follows. □

Kim [2, Theorem 3.7] is the following:

Corollary 4.5. *Let X be a nonempty geodesic convex subset of a Hadamard manifold M, and $T : X \multimap M$ be a geodesic KKM map such that for each*

$x \in X$, $T(x)$ is an open subset of M. Then the family of sets $\{T(x) \mid x \in X\}$ has the finite intersection property.

5. Fan-Browder type fixed point theorem on hyperbolic spaces

We begin with the following correct form of [13, Theorem 1(V)], which is a basis of the Fan-Browder type fixed point theorem:

Theorem 5.1. *An abstract convex space $(E, D; \Gamma)$ is a KKM space if and only if the following Fan-Browder fixed point property holds:*
Let $S : E \multimap D$, $T : E \multimap E$ be multimaps satisfying
(1) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
(2) *$S^-(z) := \{x \in E \mid z \in S(x)\}$ is open [resp. closed]; and*
(3) *$E = \bigcup_{z \in M} S^-(z)$ for some $M \in \langle D \rangle$.*
Then T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$.

In fact, $x_0 \in \mathrm{co}_\Gamma S(x_0) \subset T(x_0)$ in this theorem.

From this theorem we immediately obtain the following Fan-Browder type fixed point theorem:

Theorem 5.2. *Let X be a nonempty compact convex subset of a hyperbolic space M, and $T : X \multimap X$ be a multimap such that*
(i) *For any $x \in X$, $T(x)$ is nonempty and convex in X;*
(ii) *For any $y \in X$, $T^-(y)$ is open in X.*
Then there exists $x_0 \in X$ such that $x_0 \in T(x_0)$.

Recall that Zhou and Huang [19, Theorem 4.2] obtained Theorem 5.2 for a geodesic convex set X of a Hadamard manifold M by using the KKM principle on a Hadamard manifold.

As an application of Lemma 3.6 and Theorem 5.1 with $X = E = D$ for open-valued case, we have the following Fan-Browder type fixed point theorem:

Theorem 5.3. *Let X be a nonempty convex subset of a hyperbolic space M, and S, $T : X \multimap X$ be two multimaps such that*
(1) *for each $y \in X$, $T^-(y)$ is (possibly empty) open in X;*
(2) *for each $x \in X$, $T(x)$ is nonempty and $S(x)$ is convex, and $T(x) \subset S(x)$;*
(3) *there exists an $x_0 \in X$ such that $X \setminus T^-(x_0)$ is compact.*
Then S has a fixed point $\bar{x} \in X$, that is, $\bar{x} \in S(\bar{x})$.

Proof. For each $x \in X$, by (2), $y \in T(x)$ or $x \in T^-(y)$ for some $y \in X$. Since $X \setminus T^-(x_0)$ is compact by (3), X is covered by a finite number of open $T^-(y)'$s. Since $T(x) \subset S(x)$ by (2), by Theorem 5.1, there exists an $\bar{x} \in X$ such that $\bar{x} \in \mathrm{co}\, T(\bar{x}) \subset S(\bar{x})$. □

For geodesic convex sets in Hadamard manifolds, the above theorem reduces to W. Kim [2, Theorem 3.1] with a quite lengthy proof.

As an application of Theorem 5.1, we can prove the second Fan-Browder type fixed point theorem for convex sets in a hyperbolic space as follows:

Theorem 5.4. *Let X be a convex subset of a hyperbolic space M and $S, T : X \multimap X$ be two multimaps such that*

(1) *for each $x \in X$, $S(x)$ is convex, and $T(x) \subset S(x)$;*

(2) *for each $y \in X$, $T^{-}(y)$ is a closed subset of X;*

(3) *there exists a finite subset $\{x_1, \ldots, x_n\}$ of X such that $X \subset \bigcup_{i=1}^{n} T^{-}(x_i)$.*

Then S has a fixed point $\bar{x} \in X$, i.e., $\bar{x} \in S(\bar{x})$.

For geodesic convex sets in Hadamard manifolds, the above theorem reduces to W. Kim [2, Theorem 3.8].

6. The Nash equilibrium for an 1-person game

As an application of Theorem 5.1, we have the following:

Theorem 6.1. *Let X be a nonempty compact convex subset of a hyperbolic space M. Suppose that $f : X \times X \to \mathbb{R}$ is a function, and $T : X \multimap X$ is a multimap such that*

(1) *the set $\{(x, y) \in X \mid f(x, x) > f(x, y)\}$ is open;*

(2) *T has open graph in $X \times X$, and $T(x)$ is nonempty for each $x \in X$;*

(3) *$\{y \in X \mid f(x, x) > f(x, y)\} \cap T(x)$ is a convex subset of X for each $x \in X$.*

Then there is an $x_0 \in X$ such that

$$f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in T(x_0).$$

Furthermore, if $(x_0, x_0) > f(x_0, y)$ for each $y \notin T(x_0)$, then $x_0 \in X$ satisfies

$$x_0 \in T(x_0) \text{ and } f(x_0, x_0) \leq f(x_0, y) \quad \text{for each } y \in T(x_0).$$

Proof. First, we define a multimap $S : X \to X$ by $S(x) := \{y \in X \mid f(x, x) > f(x, y)\} \cap T(x)$ for each $x \in X$. By (3), each $S(x)$ is a convex subset of X. For each $y \in X$, we have

$$S^{-}(y) = \{x \in X \mid y \in S(x)\} = \{x \in X \mid f(x, x) > f(x, y)\} \cap T^{-}(y).$$

By (2), T^{-} has also open graph in $X \times X$ so that $S^{-}(y)$ is open for each $y \in X$. Since X is compact, and each $S^{-}(y)$ is open in X, the set $X \setminus S^{-}(y)$ is compact. Therefore, if $S(x)$ is nonempty for each $x \in X$, then S satisfies all the assumptions of Theorem 5.1 in case of $S = T$ so that there exists a fixed point $\check{y} \in X$ of S, i.e., $\check{y} \in S(\check{y})$. This implies that $f(\check{y}, \check{y}) > f(\check{y}, \check{y})$, a contradiction. Therefore, $S(x_0)$ should be empty for some $x_0 \in X$. Since $T(x_0)$ is nonempty, we can obtain the conclusion

$$f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in T(x_0).$$

Furthermore, by the assumption, if $x_0 \notin T(x_0)$, then $f(x_0, x_0) > f(x_0, x_0)$, a contradiction. Therefore, we obtain that $x_0 \in T(x_0)$ which completes the proof. □

The above proof is followed the corresponding one of [2, Theorem 3.3].

Kim [2] in his Section 2 recalled some notions and terminologies on the generalized Nash equilibrium for pure strategic games. See also our Section 6 of [14].

As an application of Theorem 6.1, Kim proved an existence of Nash equilibrium for an 1-person game of compact geodesic convex settings in a Hadamard manifold as follows [2, Theorem 3.3]:

Corollary 6.2. *Let $\mathcal{G} = (X; T, f)$ be an 1-person game such that X is a nonempty compact geodesic convex subset of a Hadamard manifold M. Suppose $f : X \times X \to \mathbb{R}$ is a function on $X \times X$, and $T : X \multimap X$ is a multimap such that*

(1) *the set $\{(x, y) \in X \mid f(x, x) > f(x, y)\}$ is open;*

(2) *T has open graph in $X \times X$, and $T(x)$ is nonempty for each $x \in X$;*

(3) *$\{y \in X \mid f(x, x) > f(x, y)\} \cap T(x)$ is a geodesic convex subset of X for each $x \in X$.*

Then there is an $x_0 \in X$ such that

$$f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in T(x_0).$$

Furthermore, if $(x_0, x_0) > f(x_0, y)$ for each $y \notin T(x_0)$, then $x_0 \in X$ is a Nash equilibrium for the game $\mathcal{G}$, that is,

$$x_0 \in T(x_0) \text{ and } f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in T(x_0).$$

In Theorem 6.1, when $T(x) := X$ for each $x \in X$, a variational inequality is obtained as follows:

Theorem 6.3. *Suppose that X is a nonempty compact convex subset of a hyperbolic space M. Suppose $f : X \times X \to \mathbb{R}$ is a continuous function on $X \times X$ such that $\{y \in X \mid f(x, x) > f(x, y)\}$ is a convex subset of X for each $x \in X$. Then there is an $x_0 \in X$ such that*

$$f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in X.$$

The following is Kim [2, Corollary 3.5]:

Corollary 6.4. *Suppose that X is a nonempty compact geodesic convex subset of a Hadamard manifold M. Suppose $f : X \times X \to \mathbb{R}$ is a continuous function on $X \times X$ such that $\{y \in X \mid f(x, x) > f(x, y)\}$ is a geodesic convex subset of X for each $x \in X$. Then there is an $x_0 \in X$ such that*

$$f(x_0, x_0) \leq f(x_0, y) \text{ for each } y \in X.$$

Recall that Theorem 6.3 is comparable to the following correct form of Park [13, Theorem 5(XVI)]:

Theorem 6.5. *For a compact partial KKM space* $(X;\Gamma)$*, the following variational inequality* (XVI) *hold:*
(XVI) *Let* $f, g : X \times X \to \mathbb{R}$ *be functions satisfying*
(1) *for any* $x, y \in X$, $f(y,y) - f(x,y) \leq g(y,y) - g(x,y)$*;*
(2) *for each* $x \in X$, $\{y \in X \mid f(x,y) < f(y,y)\}$ *is open; and*
(3) *for each* $y \in X$, $\{x \in X \mid g(x,y) < g(y,y)\}$ *is* Γ*-convex.*
Then there exists a $y_0 \in X$ *such that*

$$f(x, y_0) \geq f(y_0, y_0) \quad \text{for all } x \in X$$

and

$$\sup_{y \in X} \inf_{x \in X} f(x,y) \geq \inf_{x \in X} f(x,x).$$

7. Historical Remarks

In this section, we introduce the contents of the major references by showing their abstract or our comments in order to clarify the history of the present study.

(1) Kirk [3] in 1982 : A well-known iteration scheme due to Krasnoselskii for approximation of fixed points of of nonexpansive mappings in Banach spaces is extended to a wider class of spaces. This class includes convex metric spaces of 'hyperbolic' type, and the results apply to the study of holomorphic self-mappings of the unit ball in complex Hilbert space.

(2) In 1990, Reich and Shafrir [16] introduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces. This class of metric spaces contains all normed vector spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls.

(3) Németh [6] in 2003 : The notion of variational inequalities is extended to Hadamard manifolds and related to geodesic convex optimization problems. Existence and uniqueness theorems for variational inequalities on Hadamard manifolds are proved. A convexity property of the solution set of a variational inequality on a Hadamard manifold is presented.

The basic lemma is based on the Brouwer fixed point theorem.

(4) In 2008, while Park was studying the KKM theory, he found that any hyperbolic spaces are G-convex spaces [9] and also particular cases of c-spaces [10-13]. Actually, in 2010 [12, 13], he indicated but not concretely that most of key results in the KKM theory can be applied to hyperbolic spaces.

(5) In Zhou and Huang [19], a new notion of KKM mapping is introduced and a generalized KKM theorem is proved on Hadamard manifolds. As applications, an existence theorem of solution for a generalized mixed variational inequality and a fixed point theorem for a set-valued mapping are obtained on Hadamard manifolds.

(6) In 2012, Colao, Lopez, Marino, and Martin-Marquez [1] developed an equilibrium theory in Hadamard manifolds. They provided an analogous KKM theorem in the setting of Hadamard manifolds which is essential in proving the main result of this paper. The existence of equilibrium points for a bifunction is proved under suitable conditions, and applications to variational inequality, fixed point and Nash equilibrium problems are provided. The convergence of Picard iteration for firmly nonexpansive mappings along with the definition of resolvents for bifunctions in this setting is used to devise an algorithm to approximate equilibrium points.

(7) In 2012, Yang and Pu [17] introduced a generalized Browder-type fixed point theorem on Hadamard manifolds, which can be regarded as a generalization of the one on an Euclidean space. As applications, a maximal element theorem, a section theorem, a Ky Fan-type Minimax Inequality and an existence theorem of Nash equilibrium for non-cooperative games on Hadamard manifolds are established.

(8) In 2012, Yang and Pu [18] proved a Fan-Browder type fixed point theorem with strongly geodesic convexity on Hadamard manifolds. It is clear that such results are closely related to the KKM theory on hyperbolic spaces.

(9) In 2013, using an analogous to KKM lemma [1, 19], Zhou and Huang [20] proved the existence of solutions for the vector variational inequalities on Hadamard manifolds. The results presented in this paper generalize some previous ones from Euclidean spaces to Hadamard manifolds.

(10) In 2013, Park [14] showed that three of key results (the KKM lemma, the Ky Fan type minimax inequality, and Nash equilibrium theorem) on Hadamard manifolds in [1] can be extended to hyperbolic spaces and are particular ones for abstract convex spaces in the sense of [12, 13]. Similarly, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces and Hadamard manifolds.

(11) Kristály, Li, Lopez, and Nicolae [4] : Various results based on some convexity assumptions (involving the exponential map along with affine maps, geodesics and convex hulls) have been recently established on Hadamard manifolds. In this paper we prove that these conditions are mutually equivalent and they hold if and only if the Hadamard manifold is isometric to the Euclidean space. In this way, we show that some results in the literature obtained

on Hadamard manifolds are actually nothing but their well known Euclidean counterparts.

(12) In 2015, Lee [5] showed that main results of Yang and Pu [17] can be obtained from the one in the more general spaces. As applications, he claimed that their maximal element theorems, section theorems, Ky Fan type minimax inequality, and equilibrium theorem about non-cooperative games on Hadamard manifolds are already obtained in some sense.

(13) In 2018, Kim [2] provided two basic Fan-Browder type fixed point theorems for multimaps on geodesic convex sets in Hadamard manifolds. Also, an existence theorem of Nash equilibrium for an 1-person game in Hadamard manifolds is established.

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