

## Variational Relations in Abstract Convex Spaces

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Academic Editor: [Erdal Karapinar](#)

**Abstract:** Luc [1] initiated the study of variational relations, which is a unifying approach to various models of equilibrium theory and variational inclusions. There, a simple condition was established for the existence of solutions of variational relations and was applied to a number of variational problems. In the present article, certain results in [1] are generalized by reflecting recent development of the KKM theory on abstract convex spaces.

**Keywords:** Variational relations, Equilibria, Variational inequalities, Abstract convex space, (Partial) KKM space, Fixed point.

**MSC:** 47J20, 49J40, 49J53.

### 1 Introduction

The KKM theory is originated from the celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem in 1929. In 1961-1984, Ky Fan investigated various results in the theory on Hausdorff topological vector spaces. His results were elaborated and extended by many authors for various types of more general spaces. Since 2006, such results have been unified and abstracted by our KKM theory on abstract convex spaces. For the history of such research, see [3].

Almost independently to such progress, D. T. Luc [1] in 2008 began to work on the variational relations in order to present a unifying approach to study various models of equilibrium theory and variational

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Received June 09, 2018; accepted June 09, 2018.

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inclusions. Since then a score of authors have published quite large numbers of related works on Hausdorff topological vector spaces. However, such models also appeared in various types of abstract convex spaces. Moreover, we found that the most works on variational relations do not reflect recent development of the KKM theory of abstract convex spaces.

In the present article, we follow the pioneering work of Luc [1] and show that some of his results can be extended to our abstract convex spaces. Therefore, this article is supplementary to [1].

This article is organized as follows: In Section 2, definitions, some basic facts, and some of typical examples of abstract convex spaces are introduced. Section 3 deals with Luc's condition (Theorem 3.1) linking the existence of solutions to the variational relation problem (VR) and the intersection property of a certain multimap. Sections 4 and 5 are concerned with sufficient conditions for existence of solutions of a broad class of models, respectively, in which conditions based on intersection theorems and fixed point theorems are derived.

## 2 Abstract convex spaces

Recall the following in [2] with some later modifications and the references therein.

**Definition 2.1.** Let  $E$  be a topological space,  $D$  a nonempty set,  $\langle D \rangle$  the set of all nonempty finite subsets of  $D$ , and  $\Gamma : \langle D \rangle \multimap E$  a multimap with nonempty values  $\Gamma_N := \Gamma(N)$  for  $N \in \langle D \rangle$ . The triple  $(E, D; \Gamma)$  is called an *abstract convex space* whenever the  $\Gamma$ -convex hull of any  $D' \subset D$  is denoted and defined by

$$\text{co}_\Gamma D' := \bigcup \{ \Gamma_N : N \in \langle D' \rangle \} \subset E.$$

A subset  $X$  of  $E$  is called a  $\Gamma$ -convex subset of  $(E, D; \Gamma)$  relative to some  $D' \subset D$  if for any  $N \in \langle D' \rangle$ , we have  $\Gamma_N \subset X$ , that is,  $\text{co}_\Gamma D' \subset X$ .

In case  $E = D$ , let  $(E; \Gamma) := (E, E; \Gamma)$ .

**Definition 2.2.** Let  $(E, D; \Gamma)$  be an abstract convex space and  $Z$  be a topological space. For a multimap  $F : E \multimap Z$  with nonempty values, if a multimap  $G : D \multimap Z$  satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y \in N} G(y) \quad \text{for all } N \in \langle D \rangle,$$

then  $G$  is called a *KKM map* with respect to  $F$ . A *KKM map*  $G : D \multimap E$  is a KKM map with respect to the identity map  $1_E$  of  $E$ .

**Definition 2.3.** The *partial KKM principle* for an abstract convex space  $(E, D; \Gamma)$  is the statement that, for any closed-valued KKM map  $G : D \multimap E$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

**Example 2.4.** Let  $X$  be a subset of a vector space and  $D$  a nonempty subset of  $X$ . We call  $(X, D)$  a *convex space* if  $\text{co} D \subset X$  and  $X$  has a topology that induces the Euclidean topology on the convex hulls of any  $N \in \langle D \rangle$ . Note that  $(X, D)$  can be represented by  $(X, D; \Gamma)$  where  $\Gamma : \langle D \rangle \rightarrow X$  is the convex hull operator.

If  $X = D$  is convex, then  $X = (X, X)$  becomes a convex space in the sense of Lassonde.

**Example 2.5.** A triple  $(X, D; \Gamma)$  is called an *H-space* if  $X$  is a topological space,  $D$  a nonempty subset of  $X$ , and  $\Gamma = \{\Gamma_N\}$  a family of contractible (or, more generally,  $\omega$ -connected) subsets of  $X$  indexed by  $N \in \langle D \rangle$  such that  $\Gamma_M \subset \Gamma_N$  whenever  $M \subset N \in \langle D \rangle$ .

If  $D = X$ , we denote  $(X; \Gamma)$  instead of  $(X, X; \Gamma)$ , which is called a *c-space* by Horvath or an *H-space* by Bardaro and Ceppitelli.

**Example 2.6.** A *generalized convex space* or a *G-convex space*  $(X, D; \Gamma)$  consists of a topological space  $X$ , a nonempty set  $D$ , and a map  $\Gamma : \langle D \rangle \rightarrow X$  such that for each  $N \in \langle D \rangle$  with the cardinality  $|N| = n + 1$ , there exists a continuous function  $\phi_N : \Delta_n \rightarrow \Gamma(N)$  such that  $J \in \langle N \rangle$  implies  $\phi_N(\Delta_J) \subset \Gamma(J)$ .

Here,  $\Delta_n = \text{co}\{e_i\}_{i=0}^n$  is the standard  $n$ -simplex, and  $\Delta_J$  the face of  $\Delta_n$  corresponding to  $J \in \langle A \rangle$ ; that is, if  $N = \{a_0, a_1, \dots, a_n\}$  and  $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset N$ , then  $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$ .

There are a lot of examples of G-convex spaces.

**Example 2.7.** A *space having a family*  $\{\phi_A\}_{A \in \langle D \rangle}$  or simply a  $\phi_A$ -*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle}) \text{ or } (X, D; \phi_A)$$

consists of a topological space  $X$ , a nonempty set  $D$ , and a family of continuous functions  $\phi_A : \Delta_n \rightarrow X$  (that is, singular  $n$ -simplices) for  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ .

For a  $\phi_A$ -space  $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ , a subset  $C$  of  $X$  is said to be  $\phi_A$ -*convex* with respect to a subset  $D' \subset D$  if for each  $B \in \langle D' \rangle$ , we have  $\text{Im } \phi_B := \phi_B(\Delta_{|B|-1}) \subset C$ .

By putting  $\Gamma_A := \phi_A(\Delta_n)$ , any  $\phi_A$ -space becomes a KKM space.

Now the following diagram for triples  $(E, D; \Gamma)$  is well-known:

$$\begin{aligned} \text{Simplex} &\implies \text{Convex subset of a t.v.s.} \implies \text{Lassonde type convex space} \\ &\implies \text{H-space} \implies \text{G-convex space} \implies \phi_A\text{-space} \implies \text{KKM space} \\ &\implies \text{Partial KKM space} \implies \text{Abstract convex space.} \end{aligned}$$

### 3 Variational relation problem

According to Luc [1], we assume:  $A$ ,  $B$ , and  $Y$  are nonempty sets,  $S_1 : A \multimap A$ ,  $S_2 : A \multimap B$  and  $T : A \times B \multimap Y$  are multimaps with nonempty values. Let  $R(a, b, y)$  be a relation linking  $a \in A$ ,  $b \in B$  and  $y \in Y$ . We consider the following problem, denoted **(VR)**:

Find  $\bar{a} \in A$  such that:

- (1)  $\bar{a}$  is a fixed point of  $S_1$ , that is  $\bar{a} \in S_1(\bar{a})$ ;
- (2)  $R(\bar{a}, b, y)$  holds for every  $b \in S_2(\bar{a})$  and  $y \in T(\bar{a}, b)$ .

This problem is called a *variational relation problem* in which the multimaps  $S_1, S_2, T$  are constraints and  $R$  is a variational relation. The relation  $R$  is often determined by equalities and inequalities of real functions or by inclusions and intersections of set-valued maps. Typical instances of variational relation problems are the following as shown by Luc [1]:

- (i) Optimization Problem
- (ii) Equilibrium Problem
- (iii) Variational Inclusion Problem
- (iv) Differential Inclusion

To study the variational relation problem (VR), Luc [1] defined a multimap  $P : B \multimap A$  by

$$P(b) = [A \setminus S_2^-(b)] \cup \{a \in A : a \in S_1(a), R(a, b, y) \text{ holds } \forall y \in T(a, b)\}.$$

The following main theorem of [1] expresses the existence of solutions of (VR) by an intersection relation:

**Theorem 3.1.** ([1]) *A point  $\bar{a} \in A$  is a solution of the variational relation problem (VR) if and only if it belongs to the set  $\bigcap_{b \in B} P(b)$ .*

The following corollary is useful in establishing sufficient conditions for the existence of solutions via fixed point theorems.

**Corollary 3.2.** ([1]) *A point  $\bar{a} \in A$  is a solution of (VR) if and only if the set  $B \setminus P^-(\bar{a})$  is empty. In particular, if  $A = B$ , then (VR) has a solution under the following conditions:*

- (i) *The map  $a \mapsto A \setminus P^-(a)$ ,  $a \in A$ , has a fixed point whenever it has nonempty values.*
- (ii) *For each  $a \in A$ ,  $S_2(a) \subset S_1(a)$ .*
- (iii) *For each fixed point  $a$  of  $S_1$ , the relation  $R(a, a, y)$  holds for all  $y \in T(a, a)$ .*

## 4 Criteria Based on Intersections

In this section, we derive two sufficient conditions for the existence of solutions of (VR) as in Section 3 of Luc [1]:

**Definition 4.1.** ([1]) We say that the problem (VR) is *finitely solvable* if, for every finite subset  $N \in \langle B \rangle$ , there is some  $a_0 \in A$  such that, for each  $b \in N$ , either  $b \notin S_2(a_0)$  or  $a_0 \in S_1(a_0)$  and  $R(a_0, b, y)$  holds for all  $y \in T(a_0, b)$ .

**Proposition 4.2.** ([1]) *Assume that  $A$  is a compact set. Then, the variational relation problem (VR) has a solution if and only if it is finitely solvable.*

From now on, we assume that  $A = B$ , a nonempty subset of a partial KKM space  $(X; \Gamma)$ , and that  $(Y; \Lambda)$  is another partial KKM space.

**Definition 4.3.** ([1]) We say that the relation  $R$  is  $T$ -KKM (or KKM for short) if, for every finite subset  $N = \{a_1, \dots, a_k\}$  of  $A$  and for every  $a \in \Gamma_N$ , one can find some index  $i$  such that  $R(a, a_i, y)$  holds for all  $y \in T(a, a_i)$ .

This definition is an adaptation of the KKM maps to variational relations. We recall that a multimap  $G : A \multimap A$  is said to be KKM if for every finite subset  $N = \{a_1, \dots, a_k\}$  of  $A$ , we have  $\Gamma_N \subset G(N) = \bigcup_{i=1}^n G(a_i)$ .

The following intersection theorem of the KKM-Fan type for our abstract convex space theory will be needed: If  $A$  is a nonempty compact and  $\Gamma$ -convex subset of a partial KKM space and if  $G : A \multimap A$  is a KKM map with nonempty closed values, then  $\bigcap_{a \in A} G(a) \neq \emptyset$ .

**Theorem 4.4.** *The following conditions are sufficient for (VR) to have a solution:*

- (i)  $(A; \Gamma)$  is a compact partial KKM space.
- (ii) The map  $P$  has closed values.
- (iii) For every  $a \in A$ , the  $\Gamma$ -convex hull of  $S_2(a)$  is contained in  $S_1(a)$ .
- (iv) The relation  $R$  is KKM.

PROOF. Consider the map  $P$  on  $A$ . We start with proving that for each  $a \in A$ , the set  $P(a)$  is nonempty. Indeed, if not, say  $P(a_0)$  is empty for some  $a_0 \in A$ . By the definition of  $P$ , every  $S_2(a)$  contains  $a_0$ . In particular,  $a_0 \in S_2(a_0) \subset S_1(a_0)$ . Since  $R$  is KKM, we deduce  $a_0 \in P(a_0)$ , a contradiction. We show next that  $P$  is KKM. To this purpose, let  $N = \{a_1, \dots, a_k\} \in \langle A \rangle$  and let  $a \in \Gamma_N$ . If  $a$  belongs to the set  $A \setminus S_2^-(a_i)$  for some  $i$ , then we are done because  $a$  belongs to  $P(a_i)$  as well. If not, in view of (iii),  $a$  belongs to the  $\text{co}_\Gamma S_2(a)$ , and hence  $a \in S_1(a)$ . As  $R$  is KKM, there is some index  $i$  such that  $R(a, a_i, y)$  holds for all  $y \in T(a, a_i)$ . This implies  $a \in P(a_i)$ , and  $P$  is KKM. It remains to apply our KKM-Fan theorem and Theorem 3.1 to conclude.  $\square$

When  $A$  is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector spaces, Theorem 4.4 reduces to Luc [1, Theorem 3.1].

In order to develop sufficient conditions for (ii) and (iv), it is recalled some definitions of continuity of multimaps. Let  $G$  be a multimap between two topological spaces  $X$  and  $Z$ . It is *closed* (resp. *open*) if its graph is a closed (resp. open) set in  $X \times Z$ ; it is *upper semicontinuous* if for  $x \in X$  and an open set  $V \subset Z$  containing  $G(x)$ , there is some open neighborhood  $U \subset X$  of  $x$  such that  $G(U) \subset V$ ; and it is *lower semicontinuous* if for  $x \in X$  and an open set  $V \subset Z$  with  $V \cap G(x) \neq \emptyset$ , there is some open neighborhood  $U \subset X$  of  $x$  such that  $G(x') \cap V \neq \emptyset$  when  $x' \in U$ .

**Definition 4.5.** Let  $b \in A$  be given. We say that the relation  $R(., b, .)$  is *closed* in the first and the third variables if, for every net  $\{(a_\alpha, y_\alpha)\}$  converging to some  $(a, y)$ , and if  $R(a_\alpha, b, y_\alpha)$  holds for all  $\alpha$ , the relation  $R(a, b, y)$  holds too.

We set

$$Z := \{a \in A : a \in S_1(a)\},$$

$$P_R(b) := \{x \in A : R(x, b, y) \text{ holds for all } y \in T(x, b)\}.$$

It is clear that  $P(b)$  is the union of the sets  $A \setminus S_2^{-1}(b)$  and  $Z \cap P_R(b)$ . Therefore,  $P(b)$  is closed if these two latter sets are closed. Moreover, the set  $Z$  of all fixed points of  $S_1$  on  $A$  is closed if the map  $S_1$  is closed. The converse is evidently not always true.

**Lemma 4.6.** ([1]) *Let  $b \in A$ . Assume that:*

- (i) *The set  $A$  and the set  $Z$  of all fixed points of  $S_1$  are closed.*
- (ii) *The inverse value  $S_2^{-1}(b)$  is open in  $A$ .*
- (iii)  *$T(\cdot, b)$  is lower semicontinuous in the first variable.*
- (iv)  *$R(\cdot, b, \cdot)$  is closed in the first and the third variables.*

*Then, the set  $P(b)$  is closed.*

**Corollary 4.7.** *The following conditions are sufficient for (VR) to have a solution:*

- (i)  *$(A; \Gamma)$  is a compact partial KKM space.*
- (ii) *The set of all fixed points of  $S_1$  is closed.*
- (iii) *The map  $S_2$  has open inverse values and, for every  $b \in A$ , the  $\Gamma$ -convex hull of  $S_2(b)$  is contained in  $S_1(b)$ .*
- (iv) *For every given  $b \in A$  fixed,  $T(\cdot, b)$  is lower semicontinuous in the first variable.*
- (v) *The relation  $R$  is KKM and, for every given  $b \in A$ ,  $R(\cdot, b, \cdot)$  is closed in the first and the third variables.*

PROOF. Apply Lemma 4.6 and Theorem 4.1.  $\square$

When  $A$  is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector spaces, Corollary 4.7 reduces to Luc [1, Corollary 3.1].

The concept of KKM relations can be found in the majority of papers on variational inequalities in one or another form. Luc [1] mentioned some of them as follows:

- (i) Diagonally Quasiconvex Maps.
- (ii) Properly Quasimonotone Maps.
- (iii) Quasiconvex Inclusions.

## 5 Criteria Based on Fixed Points

The criteria that we are going to establish in this section are based on Corollary 3.2, in which fixed point theorems are involved. As before, it is assumed that  $A = B$  is a nonempty subset of a partial KKM space  $(X; \Gamma)$  and that  $(Y; \Lambda)$  is another partial KKM space. Consider the map  $Q : A \rightarrow A$  defined by

$$Q(a) = \{x \in A : R(a, x, y) \text{ does not hold for some } y \in T(a, x)\}.$$

It can be seen that

$$A \setminus P^-(a) = \begin{cases} S_2(a), & \text{if } a \notin S_1(a); \\ S_2(a) \cap Q(a), & \text{else.} \end{cases}$$

The next result gives a relationship between  $R$ ,  $P_R$  and  $Q$ .

**Lemma 5.1.** *The following assertions hold:*

(i) *For  $a \in A$ , the relation  $R(a, a, y)$  holds for all  $y \in T(a, a)$  if and only if  $a$  is not a fixed point of  $Q$ . In particular, if  $R$  is KKM, then  $Q$  has no fixed points.*

(ii) *If  $Q(a)$  is  $\Gamma$ -convex for all  $a \in A$  and if  $Q$  does not have fixed points, then  $R$  is KKM.*

(iii) *For  $a \in A$ , one has  $A \setminus Q^-(a) = P_R(a)$ .*

*Consequently, the map  $Q$  has open inverse values if and only if the map  $P_R$  has closed values.*

PROOF. The first assertion is clear. For the second assertion, suppose to the contrary that  $R$  is not KKM. Then, there exist  $N = \{a_1, \dots, a_k\} \in \langle A \rangle$  and  $a \in \Gamma_N$  such that, for each  $i$ ,  $R(a, a_i, y_i)$  does not hold for some  $y_i \in T(a, a_i)$ . In other words, all  $a_i$ 's belong to  $Q(a)$ . Under the convexity hypothesis,  $a$  is a fixed point of  $Q$ , a contradiction. In the last assertion, the equality is obtained by direct calculation.  $\square$

When  $A$  is a nonempty convex compact subset of  $X$ , and  $X$  and  $Y$  are (not necessarily Hausdorff) topological vector spaces, Lemma 5.1 reduces to Luc [1, Lemma 3.1].

The next result is a consequence of Theorem 4.4 and Lemma 5.1, but we shall give another proof based on the Fan-Browder fixed point theorem in our abstract convex space theory, which states that, if  $(A; \Gamma)$  is a compact partial KKM space and if  $G : A \multimap A$  is a multimap with  $A = \bigcup_{a \in A} \text{int}G^-(a)$ , then there is some  $a \in A$  belonging to the  $\Gamma$ -convex hull of  $G(a)$ .

**Theorem 5.2.** *The problem (VR) has a solution if the following conditions hold:*

(i)  *$(A; \Gamma)$  is a compact partial KKM space.*

(ii) *The set of all fixed points of  $S_1$  on  $A$  is closed.*

(iii) *The map  $S_2$  has  $\Gamma$ -convex values and open inverse values, and  $S_2(a) \subset S_1(a)$  for every  $a \in A$ .*

(iv) *The map  $Q$  has  $\Gamma$ -convex values, open inverse values and no fixed points.*

PROOF. We recall that  $Z$  denotes the set of all fixed points of  $S_1$  on  $A$ . Consider the multimap  $A \setminus P^-$  on  $A$ . If, for some point  $a \in A$ , the set  $A \setminus P^-(a)$  is empty, then  $a$  is a solution of (VR) (Corollary 3.2). Assume that this map has nonempty values. It follows that  $A = \bigcup_{a \in A} (A \setminus P^-)^-(a)$ . Moreover, one has

$$\begin{aligned} [A \setminus P^-]^-(a) &= \{x \in A \setminus E : a \in S_2(x)\} \cup \{x \in E : a \in S_2(x) \cap Q(x)\} \\ &= \{(A \setminus Z) \cup Q^-(a)\} \cap S_2^-(a). \end{aligned}$$

By the hypotheses (ii)-(iv), this set is open in  $A$ . Hence,  $A = \bigcup_{a \in A} \text{int}(A \setminus P^-)^-(a)$ . Apply the Fan-Browder theorem to find a fixed point  $\bar{a} \in A$  of  $A \setminus P^-$ . In particular, this point belongs to  $S_2(\bar{a})$ , hence to  $Z$  as well. By this,  $\bar{a} \in Q(\bar{a})$ , which contradicts (iv). The proof is complete.  $\square$

When  $A$  is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector space, Theorem 5.2 reduces to Luc [1, Theorem 4.1].

In this paper, all proofs are imitations of corresponding ones of Luc, and some of other results of him not mentioned here also can be extended to abstract convex spaces.

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