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From Simplices to Abstract Convex Spaces: A brief history of the KKM theory

Sehie Park[a]

[a] *The National Academy of Sciences, Republic of Korea, Seoul 06579; and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea.*

Abstract

We review briefly the history of the KKM theory from the original KKM theorem on simplices in 1929 to the birth of the new partial KKM spaces by the following steps.

(1) We recall some early equivalent formulations of the Brouwer fixed point theorem and the KKM theorem.

(2) We summarize Fan's foundational works on the KKM theory from 1960s to 1980s.

(3) We note that, in 1983-2005, basic results in the theory were extended to convex spaces by Lassonde, to H-spaces by Horvath, and to G-convex spaces due to Park.

(4) In 2006, we introduced the concept of abstract convex spaces $(E, D; \Gamma)$ on which we can construct the KKM theory. Moreover, abstract convex spaces satisfying an abstract form of the KKM theorem were called *partial KKM spaces*. Now the KKM theory becomes the study of such spaces.

(5) Various properties hold for partial KKM spaces and many new types of such spaces are introduced. We state a metatheorem for common properties or applications of such spaces.

(6) Finally, we introduce the partial KKM space versions of the von Neumann minimax theorem, the von Neumann intersection lemma, the Nash equilibrium theorem, and the Himmelberg fixed point theorem.

Keywords: KM space, partial KKM principle, minimax theorem, equilibrium theorem, fixed point theorem.
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1. Introduction

One of the earliest equivalent formulations of the Brouwer fixed point theorem (1910) is a theorem of Knaster, Kuratowski, and Mazurkiewicz (1929) (simply, the KKM theorem), which concerned with a particular type of multimaps called KKM maps later.

Email addresses: `sehiepark@gmail.com` (Sehie Park), `park35@snu.ac.kr` (Sehie Park)

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The KKM theory, first called by the author in 1992 [1], is the study of applications of various equivalent formulations or generalizations of the KKM theorem. At the beginning, the basic theorems in the theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84. A number of intersection theorems and their applications to various equilibrium problems followed. Then, the KKM theory has been extended to convex spaces by Lassonde in 1983, and to c-spaces (or H-spaces) by Horvath in 1984-93 and others. Since 1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. Those basic theorems have many applications to various equilibrium problems in nonlinear analysis and many other fields.

In 2006-09, we proposed new concepts of abstract convex spaces and KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [3-20]. Now the KKM theory becomes mainly the study of partial KKM spaces satisfying an abstract form of the KKM theorem.

In the present article, we state a brief history of the KKM theory from the original KKM theorem on simplices to the study of the partial KKM spaces. In fact, we found various properties of partial KKM spaces and many new types of such spaces. We state a metatheorem for common properties or applications of such spaces.

Finally, in order to show the usefulness of our theory, we introduce new partial KKM space versions of the von Neumann minimax theorem, the von Neumann intersection lemma, the Nash equilibrium theorem, and the Himmelberg fixed point theorem.

All references given by the form (year) can be found in [19].

2. Early works related to the KKM theory — From 1920s to 1980s

In 1910, the Brouwer fixed point theorem appeared:

Theorem. (Brouwer 1910) *A continuous map from an n-simplex to itself has a fixed point.*

The 'closed' version of the following is the origin of the KKM theory; see [2].

Theorem. (KKM 1929) *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is,* $\operatorname{co} A \subset G(A)$ *for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z\in D} G(z) \neq \emptyset$.*

This is first applied to a direct proof of the Brouwer fixed point theorem by KKM (1929), and then to a von Neumann type minimax theorem for arbitrary topological vector spaces by Sion (1958). Later it was known that the KKM theorem also holds for open-valued KKM map.

Relatively early equivalent formulations of the Brouwer fixed point theorem are as follows; see [2]:

1883 Poincaré's theorem
1904 Bohl's non-retraction theorem
1912 Brouwer's fixed point theorem
1928 Sperner's combinatorial lemma
1929 The Knaster-Kuratowski-Mazurkiewicz theorem
1930 Caccioppoli's fixed point theorem
1930 Schauder's fixed point theorem
1935 Tychonoff's fixed point theorem
1937 von Neumann's intersection lemma
1940 Intermediate value theorem of Bolzano-Poincaré-Miranda
1941 Kakutani's fixed point theorem
1950 Bohnenblust-Karlin's fixed point theorem
1950 Hukuhara's fixed point theorem
1951 Klee's intersection theorem
1951 The Nash equilibrium theorem.
1952 Fan-Glicksberg's fixed point theorem
1954 Steinhaus' chessboard theorem

1955 Main theorem of mathematical economics on Walras equilibria of Gale (1955), Nikaido (1956), and Debreu (1959)
1958 Sion's minimax equality
1960 Kuhn's cubic Sperner lemma
1961 Fan's KKM lemma
1961 Fan's geometric or section property of convex sets
1964 Debrunner-Flor's variational equality
1966 Fan's theorem on sets with convex sections
1966 Hartman-Stampacchia's variational inequality
1967 Browder's variational inequality
1967 Scarf's intersection theorem
1968 Fan-Browder's fixed point theorem
1969 Fan's best approximation theorems
1972 Fan's minimax inequality
1972 Himmelberg's fixed point theorem
1973 Shapley's generalization of the KKM theorem
1976 Tuy's generalization of the Walras excess demand theorem
1981 Gwinner's extension of the Walras theorem to infinite dimensions.
1983 Yannelis-Prabhakar's existence of maximal elements in mathematical economics.
1984 Fan's matching theorems
1997 Horvath-Lassonde's intersection theorem
1998 Greco-Moschen's KKM type theorem

It is known that many generalizations of the results in the above list are also equivalent to the Brouwer theorem. We introduce some articles on this matter.

Horvath and Lassonde (1997) introduced some intersection theorems of KKM-type, Klee-type, and Helly-type which are also equivalent to the Brouwer theorem.

Park and Jeong (2001) collected equivalent statements closely related to Euclidean spaces, n-simplices or n-balls. Among them are the Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

Recently, Idzik et al. (2014) collected three sets of equivalents of the Brouwer fixed point theorem. The first set covers some classic results connected to surjectivity property of continuous functions under proper assumptions on their boundary behavior. The second covers some results related to the Himmelberg fixed point theorem. The third loop involves equivalence of the existence of economic equilibrium and the Brouwer theorem.

Many of the above theorems are applied to lots of fields in mathematical sciences. One of them is the minimax theory originated from the von Neumann minimax theorem (1928).

3. Fan's works on the KKM theory — From 1960s to 1980s

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone on the history of the KKM theory was erected by Fan (1961). He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma. (Fan 1961) *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *convex hull of any finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem. Fan also obtained the geometric or section property of convex sets, which is equivalent to the preceding Lemma. Fan applied this property to give a simple proof (1961) of the Tychonoff theorem and to prove two results (1963) generalizing the Pontrjagin-Iohvidov-Kreĭn theorem on existence of invariant subspaces of certain linear operators. Also, Fan (1964) applied his KKMF theorem to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff theorem. The main results of Fan (1964) were extended by Ma (1969), who obtained a generalization of the Nash theorem for infinite case.

Moreover, "a theorem concerning sets with convex sections" was applied to prove the following results in Fan (1966):

An intersection theorem (which generalizes the von Neumann lemma (1937)).
An analytic formulation (which generalizes the equilibrium theorem of Nash (1951) and the minimax theorem of Sion (1958)).
A theorem on systems of convex inequalities of Fan (1957).
Extremum problems for matrices.
A theorem of Hardy-Littlewood-Pólya concerning doubly stochastic matrices.
A fixed point theorem generalizing Tychonoff (1935) and Iohvidov (1964).
Extensions of monotone sets.
Invariant vector subspaces.
An analogue of Helly's intersection theorem for convex sets.

On the other hand, Browder (1968) obtained an equivalent result to Fan's geometric lemma (1961) in the convenient form of a fixed point theorem which is known as the Fan-Browder fixed point theorem. Later this is also known to be equivalent to the Brouwer theorem. Browder (1968) applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding (1976) and Yannelis and Frabhakar (1983), to the existence of maximal elements in mathematical economics.

Fan (1969) deduced best approximation theorems from his geometric lemma and applied them to generalizations of the Brouwer theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps.

Moreover, Fan (1972) established a minimax inequality from the KKMF theorem and applied it to the following:

A variational inequality (extending Hartman-Stampacchia (1966) and Browder (1967)).
A geometric formulation of the inequality (equivalent to the Fan-Browder fixed point theorem).
Separation properties of u.d.c. multimaps, coincidence and fixed point theorems.
Properties of sets with convex sections (Fan 1966).
A fundamental existence theorem in potential theory.

Furthermore, Fan (1979, 1984) introduced a KKM theorem with a coercivity (or compactness) condition for noncompact convex sets and, from this, extended many of known results to noncompact cases. We list some main results as follows:

Generalizations of the KKM theorem for noncompact cases.
Geometric formulations.
Fixed point and coincidence theorems.
Generalized minimax inequality (extends Allen's variational inequality (1977)).
A matching theorem for open (closed) covers of convex sets.
The 1978 model of the Sperner lemma.
Another matching theorem for closed covers of convex sets.
A generalization of Shapley's KKM theorem (Shapley 1973).
Results on sets with convex sections.

A new proof of the Brouwer theorem.

In 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows:

Potential theory.
Pontrjagin spaces or Bochner spaces in inner product spaces.
Operator ideals.
Weak compactness of subsets of locally convex topological vector spaces.
Function algebras.
Harmonic analysis.
Variational inequalities.
Free boundary value problems.
Convex analysis.
Mathematical economics.
Game theory.
Mathematical statistics.

We may add the following fields to this list: nonlinear functional analysis, approximation theory, optimization theory, fixed point theory, and some others.

4. Convex spaces, H-spaces, and G-convex spaces — From 1980s to 2005

The concept of convex sets in a topological vector space is extended to convex spaces by Lassonde (1983), and further to c-spaces by Horvath in (1983-91). A number of other authors also extended the concept of convexity for various purposes.

Definition. Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Park (1994). Note that (X, D) can be represented by $(X, D; \Gamma)$ where $\Gamma : \langle D \rangle \to X$ is the convex hull operator. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde (1983).

Lassonde (1983) presented a simple and unified treatment of a large variety of minimax and fixed point problems. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem; the principal topics treated by him may be listed as follows:

Fixed point theory for multimaps.
Minimax equalities.
Extensions of monotone sets.
Variational inequalities.
Special best approximation problems.

There have followed many applications of Lassonde's convex spaces in the KKM theory and fixed point theory.

The KKM theorem was further extended to pseudo-convex spaces, contractible spaces, and spaces with certain contractible subsets or c-spaces by Horvath in (1983-1991). In his works, replacing convexity by contractibility, many of Fan's results in the KKM theory are extended to c-spaces; and a large number of new examples of c-spaces were given. Horvath also added some applications of his results to various types of new spaces. This line of generalizations was followed by Bardaro and Ceppitelli (1988, 1989, 1990) and many others.

Definition. A triple $(X, D; \Gamma)$ is called an H-*space* by Park (1992) if X is a topological space, D a nonempty subset of X, $\langle D \rangle$ the set of nonempty finite subsets of D, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, we denote $(X;\Gamma)$ instead of $(X, X;\Gamma)$, which is called a c-space by Horvath (1991) or an H-space by Bardaro and Ceppitelli (1988).

Any convex space X is an H-space $(X;\Gamma)$ by putting $\Gamma_A = \mathrm{co}\, A$, the convex hull of $A \in \langle D \rangle$. Other examples of H-spaces are any pseudo-convex space (Horvath 1983), any homeomorphic image of a convex space, any contractible space, and so on; see Bardaro and Ceppitelli (1988) and Horvath (1991). Every n-simplex Δ_n is an H-space $(\Delta_n, D;\Gamma)$, where D is the set of vertices and $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle D \rangle$.

A number of other authors also extended the concept of convexity on topological spaces for various purposes.

In the last decade of the 20th century, Park and Kim (1993, 1996-98) unified various general convexities to generalized convex spaces or G-convex spaces. For these spaces, the foundations of the KKM theory with respect to the admissible class of multimaps were established by Park and Kim (1997), and some general fixed point theorems on G-convex spaces were obtained by Kim (1998) and Park (1999).

Definition. A *generalized convex space* or a G-*convex space* $(X, D;\Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \mathrm{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X, \Gamma) = (X, X;\Gamma)$.

There are lots of examples of G-convex spaces; see [2] and references therein.

For a G-convex space $(X, D;\Gamma)$, a map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$. So, the KKM theory was extended to the study of KKM maps on G-convex spaces.

The following Fan type KKM theorem for G-convex spaces due to the author is a simple consequence of the KKM theorem:

Theorem 1. *Let $(X, D;\Gamma)$ be a G-convex space and $G : D \multimap X$ a multimap such that*

(1.1) *G has closed [resp., open] values; and*

(1.2) *G is a KKM map.*

Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)

Further, if

(1.3) *$\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

This is the basis of the extensively developed G-convex space theory. For details, see the works of Park and Kim in (1993-98) and Park in (1993-2006). A large number of authors also contributed to the theory in several hundred publications. Motivated by this, some authors tried to imitate, modify, or generalize the concept of G-convex spaces and also published a large number of papers in vain.

5. Theory of the KKM spaces — From 2006

In order to destroy such unnecessary concepts and to upgrade the KKM theory, we proposed in (2006-09) new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [3-20].

Definition. An *abstract convex space* $(E, D;\Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples. In [5-13], we gave plenty of examples of abstract convex spaces as follows:

1. The triple $(\Delta_n \supset V; \mathrm{co})$ in the original KKM theorem (1929), where V is the set of vertices of Δ_n and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

2. A triple $(X, D; \Gamma)$, where X and D are subsets of a topological vector space E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma (1961) is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

3. A convex space $(X, D; \Gamma)$ of the Lassonde type.

4. An H-space. Horvath found a large number of examples of H-spaces (1984-93).

5. Hyperconvex metric spaces due to Aronszajn and Panitchpakdi (1956) are particular ones of c-spaces.

6. Hyperbolic spaces due to Reich and Shafrir (1990) are also particular ones of c-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.

7. Any topological semilattice $(X, \leq)$ with path-connected interval introduced by Horvath and Llinares (1996).

8. A generalized convex space or a G-convex space. All examples given above are G-convex spaces.

9. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with $|A| = n+1$. The so-called FC-spaces and L-spaces are ϕ_A-spaces. Every ϕ_A-space can be made into a G-convex space; see [7,8,13].

10. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree M, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_M(A)$; see Kirk and Panyanak (2007). Then $(M \supset X; \Gamma)$ is an abstract convex space.

11. According to Horvath (2008), a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

12. A $\mathbb{B}$-space due to Briec and Horvath (2008) is an abstract convex space.

Note that each of these examples has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Example. We give examples of KKM spaces [9-12]:

1. Every G-convex space is a KKM space; see Theorem 1.

2. A connected linearly ordered space $(X, \leq)$ can be made into a KKM space; see [6].

3. The extended long line L^* is a KKM space $(L^* \supset D;\Gamma)$ with the ordinal space $D := [0,\Omega]$. But L^* is not a G-convex space; see [10].

4. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree M, and $\Gamma_A := conv_M(A)$ for each finite $A \in \langle X\rangle$. Then Kirk and Panyanak (2007) showed that $(M, X;\Gamma)$ satisfies the partial KKM principle. Later we found that $(M, X;\Gamma)$ is a KKM space.

5. For Horvath's convex space $(X,\mathcal{C})$ with the weak Van de Vel property [18], the corresponding abstract convex space $(X;\Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X\rangle$.

6. A $\mathbb{B}$-space due to Briec and Horvath (2008) is a KKM space.

In our previous works [9-11,17], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Now we have the following diagram for triples $(E, D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ partial KKM space $\Longrightarrow$ Abstract convex space.

6. A Metatheorem on KKM spaces

In the KKM theory, it is routine to reformulate the (partial) KKM principle to the following equivalent forms [9,11]:

Fan type matching property.
Another intersection property.
Geometric or section properties.
The Fan-Browder type fixed point theorems.
Existence theorems of maximal elements.
Analytic alternatives (a basis of various equilibrium problems).
Fan type minimax inequalities.
Variational inequalities, and others.

Any of such statements can be used to characterize KKM spaces or partial KKM spaces. For example, the Fan-Browder type theorem is used for the following:

Theorem 2. A characterization of KKM spaces. *An abstract convex space $(X, D;\Gamma)$ is a KKM space iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*

(2.1) *$S(z)$ is open [resp., closed] for each $z \in D$;*
(2.2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(2.3) *$X = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$,*

T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem 3. The Fan type KKM theorem. *Let $(X, D;\Gamma)$ be a partial KKM space, K a nonempty compact subset of X, and $G : D \multimap X$ a map such that*

(3.1) *$\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ [that is, G is transfer closed-valued];*
(3.2) *$\overline{G}$ is a KKM map; and*
(3.3) *either*
 (i) *$K := \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D\rangle$; or*

(ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K := L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

From this theorem we can deduce many equivalent formulations. In fact, for a compact abstract convex space $(X;\Gamma)$, we deduced 15 theorems from any of the characterizations of a partial KKM space; see [9]. Moreover, we noticed there that, for a compact G-convex space $(X;\Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem or the Brouwer fixed point theorem.

Actually, the following are equivalent formulations of Theorem 3 for partial KKM spaces [10,17]:

Theorems of Sperner and Alexandroff-Pasynkoff
Fan type matching theorem
Tarafdar type intersection theorem
Geometric or section properties
Fan-Browder type fixed point theorems
Maximal element theorems
Analytic alternatives
Fan type minimax inequalities
Variational inequalities
Horvath type fixed point theorem
Browder type coincidence theorem
von Neumann type minimax theorem
Nash type equilibrium theorem
Analytic alternatives (a basis of various equilibrium problems)
Fan type minimax inequalities
Variational inequalities, and others

Further applications of our theory on partial KKM spaces are given as follows [9,10,12]:

Best approximations (under certain restrictions)
von Neumann type intersection theorem
Nash type equilibrium theorem
Himmelberg fixed point theorem for KKM spaces
Weakly KKM maps

Consequently, we have the following as is suggested in [15,20]:

Metatheorem. *For any partial KKM space, all theorems mentioned in this section hold.*

7. New examples of KKM spaces

A large number of examples of G-convex spaces were already given in Section 4. In this section, we introduce important examples of KKM spaces chronologically. Some of them are already given in Section 4 and most of them are due to ourselves. For details of them, see our most recent survey [20].

1956 Hyperconvex metric spaces — Aronszajn and Panitchpakdi
1990 Hyperbolic spaces — Reich and Shafrir
1993 Transfer FS convex map — Tian
1996 Topological semilattices — Horvath and Llinares-Ciscar
1996 Hyperconvex metric spaces — Khamsi
1999 E-convex spaces — Youness
2003 locally p-convex spaces — Bayoumi

2004 Γ-convex spaces — Zafarani
2007 ℝ-tree — Kirk and Panyanak
2007 Connected linearly ordered spaces — Park
2008 Horvath's convex space — Horvath
2008 𝔹-spaces — Briec and Horvath
2008 Extended long line L^* — Park
2012 R-KKM spaces — Sankar Raj and Somasundaram
2015 KKM spaces — Chaipunya and Kumam

Finally we introduce an example of a partial KKM space due to Kulpa and Szymanski (2014) which is not a KKM space.

Example. Define an abstract convex space $([0,1];\Gamma)$ by defining $\Gamma : \langle [0,1] \rangle \multimap [0,1]$ as follows: for $0 < p < 0.5 < q < 1$, we define

$$\Gamma(\{p\}) = \{p\}, \quad \Gamma(\{q\}) = \{q\}, \quad \Gamma(\{p,q\}) = [0,1] \setminus \{0.5\},$$

and define $\Gamma(A) = [0,1]$ for other $A \in \langle [0,1] \rangle$.

Then, $([0,1];\Gamma)$ is a partial KKM space, but not a KKM space.

For such examples, the metatheorem in Section 6 works and hence a large number of results can be obtained for each of such examples. This is why so many works on the KKM theory have appeared in the literature; see [20].

8. The von Neumann type, Nash type, and Himmelberg type theorems

In this section, in order to show the usefulness of our theory, we introduce the abstract convex space versions of the minimax theorem and the intersection lemma due to von Neumann, of equilibrium theorem due to Nash, and the Himmelberg fixed point theorem. Those are only some of historically well-known useful results of the KKM theory.

Recall that, for a family of abstract convex spaces, their cartesian product can be made into an abstract convex space [9-11].

The following 2009 versions of the corresponding original ones are given in [9,10,16]:

Theorem 4. Generalized von Neumann-Sion minimax theorem. *Let $(X;\Gamma_1)$ and $(Y;\Gamma_2)$ be compact abstract convex spaces, $(E;\Gamma) := (X \times Y;\Gamma_{X\times Y})$ the product abstract convex space, and $f,g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions satisfying*

(4.1) *$f(x,y) \leq g(x,y)$ for each $(x,y) \in X \times Y$;*

(4.2) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and $g(x,\cdot)$ is quasiconvex on Y; and*

(4.3) *for each $y \in Y$, $f(\cdot,y)$ is quasiconcave and $g(\cdot,y)$ is u.s.c. on X.*

If $(E;\Gamma)$ is a partial KKM space, then we have

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \leq \max_{x\in X} \inf_{y\in Y} g(x,y).$$

Given a cartesian product $X = \prod_{i=1}^n X_i$ of sets, let $X^i = \prod_{j\neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x,y \in X$, we let

$$[x^i, y_i] := (x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n).$$

Theorem 5. Generalized von Neumann-Fan intersection theorem. *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that their product $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ is a partial KKM space and, for each i, let A_i and B_i are subsets of X satisfying the following:*

(5.1) *for each* $y \in X$, $B_i(y) := \{x \in X \mid [x^i, y_i] \in B_i\}$ *is open; and*
(5.2) *for each* $x \in X$, $\emptyset \neq \mathrm{co}_\Gamma B_i(x) \subset A_i(x) := \{y \in X \mid [x^i, y_i] \in A_i\}$.
Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Theorem 6. Generalized Nash-Fan equilibrium theorem. *Let* $\{(X_i;\Gamma_i)\}_{i=1}^n$ *be a finite family of compact abstract convex spaces such that their product* $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ *is a partial KKM space and, for each* i, *let* $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ *be real functions such that*
(6.1) $g_i(x) \leq f_i(x)$ *for each* $x \in X$;
(6.2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is quasiconcave on* X_i;
(6.3) *for each* $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ *is u.s.c. on* X_i; *and*
(6.4) *for each* $x_i \in X_i$, $x^i \mapsto g_i[x^i, x_i]$ *is l.s.c. on* X^i.
Then there exists a point $\hat{x} \in X$ *such that*

$$f_i(\hat{x}) \geq \max_{y_i \in X_i} g_i[\hat{x}^i, y_i] \quad \text{for all } i.$$

Definition. A KKM uniform space $(E, D;\Gamma;\mathcal{U})$, where $\mathcal{U}$ is a basis of a Hausdorff uniformity of E, is called an *L*Γ-*space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Theorem 7. Generalized Himmelberg fixed point theorem. *Let* $(X, D;\Gamma;\mathcal{U})$ *be an* $L\Gamma$*-space and* $T : X \multimap X$ *a compact u.s.c. map with closed* Γ*-convex values. Then* T *has a fixed point* $x_0 \in X$.

Particular forms of Theorem 7 were obtained by Kakutani, Himmelberg, Horvath, and Park; see [10,14,16]. Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Rhee, and others. For the literature, see [2,14,18].

Note that Theorems 1-7 generalize many results given in Section 2 to KKM spaces or partial KKM spaces.

Finally, recall that there are several hundred published works on the KKM theory and we can cover only an essential part of it. For the more historical background for the related fixed point theory, the reader can consult with [2,14,18] and references therein. For more details on the results in this paper, see the references below and the literature therein. Moreover, early fixed point theorems related to the KKM theory were applied to the following problems in the author's works in 1991-2007 (see [2] and MATHSCINET):

> Best approximations, variational inequalities, quasi-variational inequalities, the Leray-Schauder type alternatives, existence of maximal elements, minimax inequalities, the Walras excess demand theorems, generalized equilibrium problems, generalized complementarity problems, condensing maps, openness of multimaps, the Birkhoff-Kellogg type theorems, saddle points in nonconvex sets, acyclic or other versions of the Nash equilibrium theorems, quasi-equilibrium theorems, extensions of monotone sets, eigenvector problems, the KKM theory, and others.

Final Remark. This article is a revised and expanded version of our previous work entitled *"From Simplices to KKM Spaces – A brief history of the KKM theory"*, which was given as a plenary talk at the International Conference on Anatolian Communications in Nonlinear Analysis at Abant Izzet Baysal University, Bolu, Turkey, 3-6 July, 2013. In this occasion we would like to express our heartily gratitude to the hospitality of the organizers. More detailed history and literature on the KKM theory, the readers can consult with our recent work [19].

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