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GENERALIZATIONS OF THE RICCERI TYPE INTERSECTION THEOREM

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Abstract. In May 2017, Ricceri sent me a theorem and asked me whether it is well known or not. I replied that I did not know. Later he informed me that Kindler in 1993 proved already even a more general result. Ricceri's result is on an upper semicontinuous set-valued map with non-empty closed convex values. Since then we found that Ricceri's theorem can be extended to a map with closed acyclic values, and moreover, to the class $\mathfrak{B}$ of the 'better' admissible multimaps introduced by ourselves in 1996. Finally, we added some remarks.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptiness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or other solutions of various equilibrium problems. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929, which concerns with certain types of multimaps called the KKM maps later.

On May 7, 2017, Ricceri wrote to me:

"I'm writing you to know whether the attached result is already well known."

Theorem R. *Let E be a topological vector space; S a locally convex topological vector space; X a nonempty convex subset of E; Y a non-empty compact*

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convex subset of S; F an upper semicontinuous (u.s.c.) set-valued map from X into Y, with non-empty closed convex values, such that, for each $y \in Y$, the set

$$\{x \in X : y \notin F(x)\}$$

is convex. Then, one has

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

Here S must be Hausdorff.

On May 19, 2017, we replied as follows:

"I know several whole intersection properties beginning from the KKM theorem. But yours seems to be new one."

An immediate reply from Ricceri on May 19, 2017 is as follows:

"In the meanwhile, I was acquainted with the paper "Topological intersection theorems" (PAMS, 1993) by Jürgen Kindler where even a more general result was proved."

The following is from Kindler's paper:

Let Y be a nonvoid set and $\{C_x : x \in X\}$ a family of nonvoid subsets of Y (X a nonvoid index set). Let $C_y^* := \{x \in X : y \notin C_x\},\ y \in Y$, denote the system of conjugate sets and set $\langle X \rangle := \{A \subset X : A \text{ finite nonvoid}\}$.

Theorem K. *The following are equivalent:*

(a) $\bigcap\{C_x : x \in X\} \neq \emptyset$.

(b) *There exist topologies on X and Y such that*

(0) *Y is compact,*

(i) *every set $C_x,\ x \in X$, is closed,*

(ii) *for every closed $F \subset Y$ the subset $\bigcap\{C_y^* : y \in F\}$ is open,*

(iii) *every subset $\bigcap\{C_x : x \in A\},\ A \in \langle X \rangle$, is connected or empty,*

(iv) *every subset $\bigcap\{C_y^* : y \in B\},\ B \subset Y$, is connected or empty.*

In the present paper, we give generalizations of Ricceri's Theorem R to a u.s.c. map with closed acyclic values, and moreover, to the class $\mathfrak{B}$ of the 'better' admissible multimaps introduced by ourselves in 1997. Finally, we add a metatheorem which concerns with the basic ideas of our present works, and some remarks.

2. Preliminaries

We need the following from our previous work in 1994:

A *convex space* X in the sense of Lassonde is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls will be called *polytopes.* See Park et al. [14]:

Lemma P. *Let X be a topological space, Y a convex space, and S, T : $X \multimap Y$ multimaps satisfying*

(1) *for each $x \in X$, $\operatorname{co} S(x) \subset T(x)$; and*

(2) *for each $y \in Y$, $S^-(y)$ is open in X; or $X = \bigcup\{\operatorname{Int} S^-(y) : y \in Y\}$.*

Then, for any nonempty compact subset K of X, there exists a continuous function $f : K \to Y$ such that

(3) *$f(x) \in T(x)$ for each $x \in K$;*

(4) *$f(K)$ is contained in a polytope of Y; and*

(5) *for any compact subset L of X containing K, there exists a continuous extension $\tilde{f} : L \to Y$ of f such that $\tilde{f}(x) \in T(x)$ for each $x \in L$ and $\tilde{f}(L)$ is contained in a polytope of Y.*

The following was given by Krasa and Yannelis [3, Fact I] in 1994 also:

Corollary KY. *Let Δ be a compact metric space. Assume that $\varphi : \Delta \multimap \mathbb{R}^n$ is non-empty, convex valued and, for each $y \in \mathbb{R}^n$, $\varphi^-(y)$ is open in Δ. Then φ has a continuous selection, i.e., there exists a continuous function $f : \Delta \to \mathbb{R}^n$ with $f(x) \in \varphi(x)$ for all $x \in \Delta$.*

Note that, in Corollary KY, Δ can be replaced by any compact topological space and $\mathbb{R}^n$ by any convex space in the sense of Lassonde, without affecting its conclusion.

From now on, t.v.s. means a topological vector space.

3. Generalizations to acyclic maps

A topological space is *acyclic* if all its reduced Čech homology groups over rationals vanish. A multimap is called *acyclic* if it is u.s.c. with compact acyclic values.

The following is given in [4, 15]:

Lemma 3.1. *Let X be a nonempty convex subset of a locally convex Hausdorff t.v.s. E and $F : X \multimap X$ be a composite of acyclic maps. If F is compact, then it has a fixed point.*

Theorem 3.2. *Let X be a convex space; S a locally convex Hausdorff t.v.s.; Y a non-empty compact convex subset of S; Φ a u.s.c. multimap from X into Y, with non-empty closed acyclic values, such that, for each $y \in Y$, the set*

$$\{x \in X : y \notin \Phi(x)\}$$

is convex. Then, one has

$$\bigcap_{x\in X} \Phi(x) \neq \emptyset.$$

Proof. Let the dual $\Phi_* : Y \multimap X$ of Φ be defined by $\Phi_*(y) := \{x \in X : y \notin \Phi(x)\},\ y \in Y$. Then $\Phi_*(y)$ is convex and

$$\Phi_*^-(x) = \{y \in Y : x \in \Phi_*(y)\} = \{y \in Y : y \notin \Phi(x)\} = \{\Phi(x)\}^c,\ x \in Y$$

is open.

If $\Phi_*(y)$ is empty for some $y \in Y$, then $y \in \Phi(x), \forall x \in X$. So we have done.

If $\Phi_*(y)$ is nonempty for all $y \in Y$, suppose $\bigcap_{x\in X} \Phi(x) = \emptyset$. Then

$$\bigcup_{x\in X} \Phi_*^-(x) = \bigcup_{x\in X} \{\Phi(x)\}^c = Y$$

and we have an open cover of Y. Now, by Lemma P, Φ_* has a continuous selection $f : Y \to X$. Then $\Phi \circ f : Y \multimap Y$ has a fixed point $y_0 \in Y$ by Lemma 2.1, that is, $y_0 \in \Phi(x)$ and

$$x = f(y_0) \subset \Phi_*(y_0) = \{x \in X : y_0 \notin \Phi(x)\}.$$

This is a contradiction. Hence, $\bigcap_{x\in X} \Phi(x) \neq \emptyset$. □

In case $E = S$ and $X = Y$, Theorem 3.2 reduces to the following:

Corollary 3.3. *Let E be a locally convex Hausdorff t.v.s.; X a non-empty compact convex subset of E; Φ a u.s.c.multimap from X into X, with non-empty closed acyclic values, such that, for each $y \in X$, the set*

$$\{x \in X : y \notin \Phi(x)\}$$

is convex. Then, one has

$$\bigcap_{x\in X} \Phi(x) \neq \emptyset$$

and Φ has a fixed point.

Proof. By Theorem 3.2, we have an element $x_0 \in X$ such that

$$x_0 \in \bigcap_{x\in X} \Phi(x) \neq \emptyset,$$

that is, $x_0 \in \Phi(x)$ for all $x \in X$. Hence $x_0 \in \Phi(x_0)$. □

Remark. It might be interesting to check whether Kindler's Theorem K works for our Theorem 3.2. In fact, by putting

$$C_x := \Phi(x) \text{ and } C_y^* := \{x \in X : y \notin \Phi(x)\},$$

we have the following from Theorem 2.2:

(0) Y is compact;

(i) C_x is closed;

(iv) C_y^* is convex and, hence, connected.

However, we have no evidence to show the following:

(ii) $C_y^* := \{x \in X : y \notin \Phi(x)\}$ is open; and

(iii) every subset $\bigcap\{C_x : x \in A\},\ A \in \langle X \rangle$, is connected or empty.

Therefore, our Theorem 3.2 must be independent from Theorem K.

4. Generalizations to better admissible class $\mathfrak{B}$

Let X and Y be topological spaces. In the following, a *polytope* is a homeomorphic image of a simplex. The following due to the author is well-known:

Definition. An *admissible class* $\mathfrak{A}_c^\kappa(X,Y)$ of maps $T : X \multimap Y$ is the one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K,Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(3) for each polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

Example. Examples of the multimap class $\mathfrak{A}$ are the classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$, the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$, the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, the Simons maps $\mathbb{K}_c$, σ-selectionable maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, the Fan-Browder maps (codomains are convex sets), locally selectionable maps having convex values, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park et al., and approximable maps $\mathbb{A}_c^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of the multimap class $\mathfrak{A}_c^\kappa$.

For the literature, see Park [5, 8, 9], Park and H. Kim [12-14] and the references therein.

Later, we introduced the following class of multimaps, where Δ_n is the standard n-simplex:

Definition. For topological spaces X and Y, (*the better admissible class*) $\mathfrak{B}$ of multimaps $F : X \multimap Y$ is defined as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ has the property that, for any natural $n \in \mathbb{N}$ and for any two continuous functions $\phi : \Delta_n \to X$, $p : F\phi(\Delta_n) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset X \overset{F}{\multimap} F\phi(\Delta_n) \xrightarrow{p} \Delta_n$$

has a fixed point.

Proposition 4.1. *For any topological spaces X and Y, we have $\mathfrak{A}_c^\kappa(X,Y) \subset \mathfrak{B}(X,Y)$.*

Example. 1. A u.s.c. multimap having compact values with trivial shape (that is, contractible in arbitrary neighborhood) belongs to $\mathfrak{B}$; see Ben-El-Mechaiekh [1].

2. The class $\mathfrak{B}$ properly contains $\mathfrak{A}_c^\kappa$; see [6]. The connectivity maps due to Nash and Girolo are examples of $\mathfrak{B}$-maps not belonging $\mathfrak{A}_c^\kappa$; see [7].

3. An important subclass of $\mathfrak{B}$ is the Fan-Browder maps having convex values and nonempty open fibers.

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every 0-neighborhood $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

In 1998, we obtained the following [7, Theorem 10.1]:

Theorem 4.2. *Let E be a Hausdorff t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

In [9], it was shown that Theorem 4.2 subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

Let X be a subset of a t.v.s. E. A compact subset K of X is said to be *Klee approximable into* X if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in X.

Example. We give some examples of Klee approximable sets:

(1) If a subset X of E is admissible (in the sense of Klee), then every compact subset K of X is Klee approximable into E.

(2) Any polytope in a subset X of a t.v.s. is Klee approximable into X.

(3) Any compact subset K of a convex subset X in a locally convex t.v.s. is Klee approximable into X.

(4) Any compact subset K of a convex and locally convex subset X of a t.v.s. is Klee approximable into X.

(5) Any compact subset K of an admissible convex subset X of a t.v.s. is Klee approximable into X.

(6) Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.

Note that (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

In 2004 [10], Theorem 4.2 is generalized as follows:

Theorem 4.3. *Let X be a subset of a Hausdorff t.v.s. E and $F \in \mathfrak{B}(X, X)$ a compact closed multimap. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

The following extends a particular case of Lemma 3.1:

Corollary 4.4. *Let X be a subset of a Hausdorff t.v.s. E and $F : X \to X$ a compact acyclic map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

Now the following is the main result of this section.

Theorem 4.5. *Let X be a convex space; Y a compact Klee approximable subset of a Hausdorff t.v.s.; and $\Phi \in \mathfrak{B}(X,Y)$ a multimap with closed graph such that, for each $y \in Y$, the set*

$$\{x \in X : y \notin \Phi(x)\}$$

is convex. Then, one has

$$\bigcap_{x \in X} \Phi(x) \neq \emptyset.$$

Proof. Since $\Phi \in \mathfrak{B}(X,Y)$ is closed, it is closed-valued. Let $\Phi_* : Y \multimap X$ be defined by $\Phi_*(y) := \{x \in X : y \notin \Phi(x)\}$ for $y \in Y$. Then $\Phi_*(y)$ is convex and $\Phi_*^-(x) = \{y \in Y : x \in \Phi_*(y)\} = \{y \in Y : y \notin \Phi(x)\} = \{\Phi(x)\}^c$ is open for each $x \in Y$.

If $\Phi_*(y)$ is empty for some $y \in Y$, then $y \in \Phi(x)$ for all $x \in X$. So we have done.

If $\Phi_*(y)$ is nonempty for all $y \in Y$, suppose $\bigcap_{x\in X} \Phi(x) = \emptyset$. Then $\bigcup_{x\in X} \Phi_*^-(x) = \bigcup_{x\in X}\{\Phi(x)\}^c = Y$ and we have an open cover of Y. Since Y is compact, by Lemma P, Φ_* has a continuous selection $f : Y \to X$ such that $f(Y)$ is contained in a polytope in X. Moreover, since $\Phi \in \mathfrak{B}(X,Y)$, for any natural $n \in \mathbb{N}$ and for any two continuous functions $\phi : \Delta_n \to Y$, $p : \Phi \circ f(\phi(\Delta_n))) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset Y \xrightarrow{f} f(\phi(\Delta_n)) \overset{\Phi}{\multimap} \Phi \circ f(\phi(\Delta_n)) \xrightarrow{p} \Delta_n$$

has a fixed point; that is, the composition

$$\Delta_n \xrightarrow{f\phi} f\phi(\Delta_n) \subset X \overset{\Phi}{\multimap} \Phi \circ f(\phi(\Delta_n)) \xrightarrow{p} \Delta_n$$

has a fixed point.

Then $\Phi \circ f \in \mathfrak{B}(Y,Y)$ has a fixed point $y_0 \in Y$ by Theorem 4.3, that is, $y_0 \in \Phi(x)$ and

$$x = f(y_0) \subset \Phi_*(y_0) = \{x \in X : y_0 \notin \Phi(x)\}.$$

This is a contradiction. Hence, $\bigcap_{x\in X} \Phi(x) \neq \emptyset$. □

As for Corollary 3.3, in case $E = S$ and $X = Y$, Theorem 4.5 reduces to the following:

Corollary 4.6. *Let X be a compact convex Klee approximable subset of a Hausdorff t.v.s.; and $\Phi \in \mathfrak{B}(X,X)$ a multimap with closed graph such that, for each $y \in X$, the set*

$$\{x \in X : y \notin \Phi(x)\}$$

is convex. Then, one has

$$\bigcap_{x\in X}\Phi(x)\neq\emptyset$$

and Φ has a fixed point.

Theorem 4.5 is a far-reaching generalization of Ricceri's theorem and our Theorem 3.2. However, for its usefulness, we have to wait and see Ricceri's coming work.

5. A metatheorem

The essences of Theorems R, 3.2, and 4.3 can be stated by the following metatheorem:

Theorem 5.1. *Let X, Y, $\Phi : X \multimap Y$, and $\Phi_* : Y \multimap X$ satisfy the following:*

(a) *$\Phi_*(y)=\emptyset$ for some $y\in Y$; or*

(b) *if $\{\Phi_*^-(x)\}_{x\in X}$ covers Y, then Φ_* has a selection $f : Y\to X$ such that $\Phi_*\circ f : Y\multimap Y$ has a fixed point.*

Then, one has

$$\bigcap_{x\in X}\Phi(x)\neq\emptyset.$$

Proof. If $\Phi_*(y)$ is empty for some $y\in Y$, then $y\in\Phi(x)$ for all $x\in X$. So we have done.

If $\Phi_*(y)$ is nonempty for all $y\in Y$, suppose that $\bigcap_{x\in X}\Phi(x)=\emptyset$. Then $\bigcup_{x\in X}\Phi_*^-(x)=\bigcup_{x\in X}\{\Phi(x)\}^c=Y$ and, by (b), Φ_* has a selection $f : Y\to X$ such that $\Phi_*\circ f$ has a fixed point $y_0\in Y$. Let $x_0=f(y_0)$. Then $y_0\in\Phi(x_0)$ and

$$x_0=f(y_0)\subset\Phi_*(y_0)=\{x\in X : y_0\notin\Phi(x)\}.$$

This is a contradiction. Hence, $\bigcap_{x\in X}\Phi(x)\neq\emptyset$. □

Note that Theorem 5.1 includes all of Theorems R, 3.2, and 4.3.

We note that condition (b) seems to be essential; that is, $\bigcap_{x\in X}\Phi(x)=\emptyset$ implies that $\Phi_*\circ f : Y\multimap Y$ has no fixed point.

Example. Under the situation of Corollary KY, let

$$X\equiv\mathbb{R}^n,\ Y\equiv\Delta,\ \text{and}\ \Phi_*\equiv\varphi:\Delta\multimap\mathbb{R}^n.$$

We apply Theorem 5.1. Since $\Phi_*(y)=\varphi(y)\neq\emptyset$ for all $y\in Y$, the case (a) does not hold. So we consider only case (b).

(1) $\{\Phi_*^-(x)\}_{x\in X}=\{\varphi^-(x)\}_{x\in X}$ covers Y. — In fact, for each $y\in\Delta$, we know $\varphi(y)\neq\emptyset$ and hence, there exists $x\in\varphi(y)\subset\mathbb{R}^n$ or $y\in\varphi^-(x)$.

(2) $\varphi\equiv\Phi_*$ has a continuous selection $f:\Delta\to\mathbb{R}^n$ by Corollary KY.

(3) $\Phi:\mathbb{R}^n\multimap\Delta$ holds $\Phi\equiv(\varphi^c)^-$. — In fact, for each $x\in\mathbb{R}^n$, we have

$$(\varphi^c)^-(x)=\{y\in\Delta : x\in\varphi^c(y)\}=\{y\in\Delta : x\notin\varphi(y)\}=\varphi_*(x),$$

and $\Phi_*(y)=\varphi(y)$ for each $y\in\Delta$.

(4) Suppose $\varphi \circ f : \Delta \multimap \Delta$ has a fixed point. Then by Theorem 4.1, we have

$$\bigcap_{x \in X} \Phi(x) = \bigcap_{x \in \mathbb{R}^n} \varphi_*(x) \neq \emptyset.$$

Hence there exists $y \in \varphi_*(x)$ such that $x \notin \varphi(y)$ or $x \in \mathbb{R}^n \setminus \varphi(y)$ for all $x \in \mathbb{R}^n$. Therefore $\mathbb{R}^n \subset \mathbb{R}^n \setminus \varphi(y)$ and hence $\varphi(y) = \emptyset$, a contradiction.

(5) Therefore $\varphi \circ f : \Delta \multimap \Delta$ can not have a fixed point.

6. Epilogue

Later I sent Sections 1-4 of this paper to Ricceri and asked him whether he had any follow-ups of his theorem. He replied that he was trying to get something and recommended one of his new papers [16] which is concerned with his new results.

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