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CONTRIBUTIONS OF ANDRZEJ GRANAS TO THE KKM THEORY

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Abstract. A partial KKM space is an abstract convex space satisfying an abstract form of the KKM theorem and a KKM space is the one also satisfying the 'open' version of the form. The KKM theory concerns with the study of partial KKM spaces nowadays. In this article, our aim is to recall the contributions of A. Granas to the theory. We introduce the contents of most of works of Granas and his coauthors on the KKM theory and give some comments to compare them with current results in the theory.

1. Introduction

In 1929, the celebrated Knaster-Kuratowski-Mazurkiewitz (simply, KKM) theorem and a Polish mathematician Andrzej Granas were born. At first the KKM theorem was immediately applied to a simple proof of the Brouwer fixed point theorem and, later, to the Sion minimax theorem in 1958. Since then the theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. The KKM theory, first called so by the author in 1992 [29], is the study on applications of numerous equivalent formulations or generalizations of the KKM theorem; see [33, 46, 51] and the references therein.

Granas studied and worked in Poland, Russia, France, Canada, and works now in Poland. He published many articles in the fields of functional analysis such as non-linear equations, convex analysis, ordinary differential equations, and, in the fields of topology such as topology of function spaces and fixed point theory. Many of his works on convex analysis belong to the KKM theory nowadays.

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At the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Ky Fan in 1961-84. F. Browder in 1968 also added some applications.

In 1978-95, Granas contributed to the development of the KKM theory. In fact, he organized several conferences related to the theory, systematically studied the theory on topological vector spaces, and published or guided many works by himself or together with his students and coworkers; for example, Dugundji [1,3,34], Lassonde [9,22,26], Horvath [8,11,18,21,25], Ben-El-Mechaiekh [4-6,16,19,23,27,30,31], Deguire [4-6,14,16,23,27], Liu [7,10,12, 15,17,28], and others; see the references in the end of the present article. Recall that contributions of the Montreal School led by Granas on convex analysis were valuable and very important.

Under Granas' influence, a number of intersection theorems and their applications to various equilibrium problems followed. Then, the KKM theory had been extended to *convex spaces* by Lassonde in 1983 [9], and to *c-spaces* or *H-spaces* by Horvath in 1984-93 [8,11,18,21] and others. Since 1993, the theory was further extended to *generalized convex spaces* or *G-convex spaces* in a sequence of works of the present author and others; see [33,51]. Those basic results of the KKM theory have many applications to various equilibrium problems in nonlinear analysis and many other fields. Moreover many modifications or imitations of G-convex spaces were unified the concept of ϕ_A-*spaces* in the sense of the author; see [51].

In 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [35-51]. A *partial KKM space* is an abstract convex space satisfying an abstract form of the KKM theorem and a *KKM space* is the one also satisfying the 'open-valued' version of the form. Now the KKM theory becomes the study of mainly partial KKM spaces.

In this article, we introduce the contents of most of works of Granas and his coauthors on the KKM theory and give some comments to compare them with current results in the theory.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the collection of all nonempty finite subsets of a set D. Recall the following in [46, 51] and the references therein.

Definition. Let E be a topological space, D a nonempty set, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Now we have the following well-known diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

The following is one of the most general KKM type theorems in [49,50]:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K = Z$;*

(ii) *$\bigcap \{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(iii) *for each* $N \in \langle D \rangle$, *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap \{G(y) \mid y \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap \{G(y) \mid y \in D\} \neq \emptyset$.

Our KKM theory mainly concerns with the study of partial KKM spaces and their applications.

Recall that [46] contains incorrect statements such as (V), (VI), Theorem 4, (XVI), and (XVII). These can be easily corrected.

3. Works of Granas on the KKM theory

In this section, we recall the works of Granas and his coauthors on the KKM theory, in the chronological order. We also add some comments to compare them with current results in the theory. Moreover, in order to give plenty of information, certain comments are borrowed from *Mathematical Reviews* and the author deeply appreciates such reviewer's efforts.

(1) Dugundji-Granas, 1978 – Pisa [1]

From the text: Let E be a vector space, $X \subset E$ an arbitrary subset. A function $G : X \to 2^E$ is called a KKM map if $\mathrm{conv}\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$ for each finite subset $\{x_1, \cdots, x_n\} \subset X$. Further if each $G(x)$ is finitely closed, then the family $\{G(x) \mid x \in X\}$ has the finite intersection property. This theorem is a modification of Ky Fan's generalization [Math. Ann. 142 (1960/61) 305–310] of the KKM theorem. A new proof of the Hartman-Stampacchia theorem on variational inequalities [Acta Math. 115 (1966) 271–310] is given as an application.

Comments: This is the origin of the name of *KKM map* and the concept of *convex spaces* in the sense of Lassonde [9]. The theorem given in [1] is simply tells that $(E, X; \Gamma)$ with the finite topology of E and the convex hull operation Γ is a partial KKM space. Recall that Ky Fan maintained to study convex subsets of Hausdorff topological vector spaces.

Nowadays the KKM theory is mainly concerned with abstract convex spaces and KKM maps with respect to some multimaps. Recall that some peoples complained against our use of triples for abstract convex spaces. Note that here appears a triple $(E, X; \Gamma)$ and such triple appears frequently in the present article.

(2) Granas, 1981 – Birkhäuser [2]

From Introduction: There was a special reason for giving a general survey of the theory of KKM-maps at the *Scottish Book* Conference. The subject, which evolved from some of the research carried out in Lwów and Warsaw during 1929-1939, is closely related to some of the problems in the Scottish Book and, in particular, to Problem 54 (J. Schauder).

In the brief paper, we emphasize those general principles of the theory of KKM-maps which provide the foundation for many of the modern existential results in diverse areas of mathematics. We give a number of applications of these principles illustrating the nature and flavor of the techniques involved. For simplicity, we restrict ourselves to shorter forms of somewhat more general theorems. Of necessity many noteworthy and even important contributions must be omitted or mentioned only briefly. We discuss the status of Problem 54 in the last section.

Comments: This is the first article surveying the early results and applications of the KKM theory. In this survey, Section 1 begins with examples of KKM maps, the principle of KKM maps given in [1], and simple applications to the Mazur-Schauder theorem in 1936 and Ky Fan's fixed point theorem in 1969. Moreover, the Tychnoff fixed point theorem in 1935 and its extension due to Fan in 1969 are deduced by the KKM principle. In Section 2, a Fan-Browder type fixed point theorem and the Fan minimax inequality are obtained, and a fundamental existence theorem in potential theory is mentioned. Section 3 devotes to get some basic facts in the variational inequalities by using KKM maps. In fact, the Hartman-Stampacchia theorem in 1966 and, for Hilbert spaces, its application to Browder-Goehde-Kirk's fixed point theorem for nonexpansive maps and Nikodym's best approximation theorem are deduced. Section 4 devotes to applicability of KKM maps to game theory. In fact, Fan's coincidence theorem is applied to the von Neumann-Sion minimax theorem and Fan's generalized intersection theorem is used to deduce the Nash equilibrium theorem. Finally, Section 5 is concerned with bibliographical and historical comments. Here, Granas states that the Schauder conjecture (Problem 54 in the *Scottish Book*) was an inspiration for numerous later investigations both in fixed point theory and in nonlinear functional analysis. Recall that the more-than 80 years old conjecture seems to be not completely resolved yet.

Note that most results mentioned in [2] are extended to partial KKM spaces in [46]; see also [51].

(3) Dugundji-Granas, 1982 – PWN [3]

From Preface: The aim of this monograph is to give a systematic and unified account of the topics in fixed point theory that lie on the border-line of topology and functional analysis, emphasizing more recent topological developments related to the Leray-Schauder theory.

In this volume, using for the most part geometric methods, our study centers around formulating those general principles of the theory, that provides the foundation for many of the modern results in diverse areas of mathematics. $\cdots$.

The "Miscellaneous results and examples" given in the form of exercises at the end of the paragraphs form an integral part of of the book; they describe extensions and related developments of the theory and indicate further applications not treated in the text. $\cdots$.

Algebraic developments of the theory, including the fixed point index for arbitrary ANR's, will be treated in Volume II.

Comments: This Volume I is mainly concerned with Brouwer's and Schauder's theorems and the Lefschetz theorem for polyhedra. There are three chapters. Chapter I is on elementary fixed point theory with Paragraph 1 on the Banach contraction principle and Paragraph 2 on further results and applications.

Chapter II with three paragraphs is devoted to Borsuk's theorem and the "topological transversality", i.e., essential maps. Paragraph 3 is on theorems of Brouwer and Borsuk, Paragraph 4 on compact maps in normed linear spaces, and Paragraph 5 on further results and applications.

Finally, Chapter III is on homology and fixed points, and consists of Paragraph 6 on simplicial homology and of Paragraph 7 on the Lefshetz-Hopf theorem and Brouwer degree.

In this monograph, materials related to the KKM theory appear in the first four sections and the last section (entitled "Miscellaneous results and examples") of Paragraph 5. "The first four sections contain some extensions of a very versatile technique introduced by Ky Fan based on the KKM theorem." The materials given in this part and the last section are almost same to the previous survey of Granas in 1981 [2] with some extra results of Lebesgue and Lassonde. Note that [2] was evidently written later than this monograph [3].

As we noted for [2], most results related to the KKM theory in this monograph are extended to partial KKM spaces in [45]; see [51] also.

(4) Ben-El-Mechaiekh - Deguire - Granas, 1982 – Paris [4]

Abstract: In this Note we first extend the well-known fixed point theorem of Ky Fan to a new class of set-valued maps. Then, using the terms of convex analysis, we establish an alternative, which has several useful consequences, in particular in the theory of variational inequalities.

From Text: Using a generalization of a Ky Fan fixed point theorem for multivalued mappings, the following alternative is obtained:

Theorem 2. *Let X be a convex compact subset of a topological vector space and $f, g : X \times X \to \mathbb{R}$ two real functions such that:*
(i) $g(x, y) \leq f(x, y)$ *for all* $(x, y) \in X \times X$*;*
(ii) $x \to f(x, y)$ *is quasiconcave on X for all* $y \in X$*;*

(iii) *$y \to g(x,y)$ is lower semicontinuous on X for all $x \in X$.*
Then for every $\lambda \in \mathbb{R}$ one of the following statements holds:
(1) *there exists $y_0 \in X$ such that $g(x,y_0) \le \lambda$ for all $x \in X$, or*
(2) *there exists $x_0 \in X$ such that $f(x_0,x_0) > \lambda$.*

As examples of applications, some known results on variational problems and two function version (Corollary 3) of the Ky Fan minimax inequality are given.

Comments: The fixed point theorem is extended to the Fan-Browder fixed point property (V) of partial KKM spaces in [46] and Theorem 2 to the analytic alternative (X) in [46]. Moreover, Corollary 3 is extended to the minimax inequality (XI) in [46]. In fact, those properties (V), (X), (XI) and many others characterize (partial) KKM spaces.

(5) Ben-El-Mechaiekh - Deguire - Granas, (I), (II) 1982 – Paris [5,6]

Abstract: [5] In this and the following Notes we discuss numerous generalizations of some fixed point and coincidence results which appeared in works of Ky Fan and F. Browder.

[6] This note deals with fixed point and coincidence theorems for setvalued maps of type φ and φ^*, introduced in [4]. These new classes of multivalued maps generalize those of type F and F* treated in our previous Note [5]. The results presented here have numerous applications.

From Text: Let X be a convex subset of a Hausdorff topological vector space, and Y a Hausdorff topological space. A multimap $A : X \multimap Y$ is called

(1) an F-map ($A \in F(X,Y)$) if Ax is open for every $x \in X$ and $A^{-1}y = \{x : y \in Ax\}$ is nonempty and convex for every $y \in Y$; and
(2) an F^*-map ($A \in F^*(X,Y)$) if $A^{-1} \in F(X,Y)$.

In [5], the authors consider some properties of F- and F^*-maps and prove the following main results.

If $A \in F(X,Y)$ and the multimap $\varphi : X \multimap Y$ is a compact composition of a finite number of acyclic maps then there exists a coincidence $x_0 \in X$: $A(x_0) \cap \varphi(x_0) \neq \emptyset$.

If both X and Y are convex, then every $A \in F(X,Y)$ has a coincidence with every compact $B \in F^*(X,Y)$.

As consequences, some coincidence and fixed point theorems of Ky Fan and F. Browder and some minimax relations are given.

The classes of mappings $F(X,Y)$ and $F^*(X,Y)$ considered in [5] are extended in [6] in the following way.

A multimap $A : X \multimap Y$ is called a φ-mapping ($A \in \varphi(X,Y)$) if $A^{-1}y$ is nonempty and convex for every $y \in Y$ and there exists a multivalued selection $\tilde{A}$ of A such that $\tilde{A}$ is surjective and $\tilde{A}(U)$ is open for every $U \subset X$.

A multimap A is called a φ^*-map if $A^{-1} \in \varphi(Y, X)$.

The authors consider some properties of φ- and φ^*-maps in general analogous to the properties of F- and F^*-maps and give some applications to nonlinear alternatives in convex analysis.

Comments: Recently, from our KKM theorem C in [49,50], we deduced the following coincidence theorem of the Fan-Browder type.

Theorem D. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $S : D \multimap Z$, $T : E \multimap Z$ maps. Suppose that*

(1) *for each $z \in F(E)$, we have $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$\bigcap_{y\in M} \overline{Z \setminus S(y)} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{Z \setminus S(y)} \subset K.$$

(α) *If S is transfer open-valued and $\overline{F(E)} \cap K \subset S(D)$, then there exist $\bar{x} \in E$ and $\bar{z} \in \overline{F(E)} \cap K$ such that $\bar{z} \in F(\bar{x}) \cap T(\bar{x})$.*

(β) *if S is unionly open-valued and $Z = S(D)$, then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.*

Note that Theorem D generalizes and unifies the coincidence theorems in [5,6] and many previous results as in [50].

(6) Granas-Liu, 1983 – BIMAS [7]

Abstract: A well-known result of Ky Fan concerning systems of inequalities (in 1957) is generalized to situations involving three families of functions. The result is applied to formulate a generalization of Kneser's minimax theorem (in 1952).

Comments: Let Y be a nonempty convex compact subset of a linear topological space and $\mathcal{F}, \mathcal{G}, \mathcal{H} \subset \mathbb{R}^Y$ be three families. The authors of [7] prove, using the KKM theorem,

Theorem 1. *If $\mathcal{F} \leq \mathcal{G} \leq \mathcal{H}$, each $f \in \mathcal{F}$ is lower semicontinuous, each $g \in \mathcal{G}$ is convex, and the family $\mathcal{H}$ is "concave" (in a sense made precise), then given $\lambda \in \mathbb{R}$ either*

(a) *there is $h \in \mathcal{H}$ such that $h(y) > \lambda$ for all $y \in Y$ or*

(b) *there is $y \in Y$ such that $f(y_0) \leq \lambda$ for all $f \in \mathcal{F}$ simultaneously.*

(Here $F \leq G$ means that, for all $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that $f \leq g$). They then deduce from their result a three-function generalization of Kneser's minimax theorem in 1952.

It is possible to establish similar results using the Hahn-Banach theorem rather than the KKM theorem. In fact, two years later, S. Simons [13] gave a remark on [7] as follows:

Summary: We generalize a recent result of Granas and Liu about systems of inequalities involving three families of functions. Our proof differs from theirs in that it uses the Hahn-Banach theorem rather than the KKM theorem.

Comments: It would be very interesting to study some equivalency or close relationship between the Hahn-Banach theorem and the KKM theorem. Actually the Hahn-Banach theorem follows from the KKM theorem; see [24].

(7) Granas-Liu, 1984 – Paris [10]

Summary: In this note we announce certain new minimax theorems. The proofs of these results involve fixed point and coincidence theorems for set-valued maps (in particular, those of type Φ and Φ^*). The results obtained have a number of useful consequences.

Comments: Some important results of Fan, Fan-Glicksberg, and von Neumann are extended on topological vector spaces. However, such extensions are generalized to abstract convex spaces; see [48, 51].

(8) Granas-Liu, 1985 – Paris [12]

Summary: In this note we present some theorems in the minimax theory which hold under rather weak convexity assumptions and generalize several results known in the convex case. The proof is based on the von Neumann minimax theorem in $\mathbb{R}^n$.

Comments: Theorem 1 is a four function version of a minimax theorem extending a Nikaido-von Neumann minimax theorem; and Theorem 2 is a three function version extending a Fan-Nikaido-Kneser minimax theorem.

(9) Granas-Liu, 1986 – JMPA [15]

From Text: In this paper we discuss some new general coincidence results. Then using the new terms of convex analysis we translate the geometrical formulations into analytical results. As a consequence, several new general minimax theorems are established and a number of noteworthy applications follow. The central geometrical result of this paper is the following whose proof is based on the Lefschetz-type fixed point theorem for composites of acyclic maps:

(4.1) **Theorem.** *Let X be a convex subset of a vector space with finite topology and Y be a topological space. Let G, $S : X \multimap Y$ such that:*
(1) *G has an open-valued surjective selection and convex pre-images;*
(2) *S is compact and has nonempty acyclic images and demi-closed graph.*
Then G and S have a coincidence $x_0 \in X$: $G(x_0) \cap S(x_0) \neq \emptyset$.

From MR 867668 (88d:54062): The authors consider some coincidence theorems for set-valued maps of the following types: compact convex-valued and acyclic-valued closed maps and open-valued with convex pre-images (or vice versa).

Two analytic versions of Theorem (4.1) are formulated as nonlinear alternatives. Then it is shown that these results permit one to give a unified treatment to minimax inequalities and minimax theorems with a controlling function. One of the corollaries is applied to systems of inequalities and a generalization of the Ky Fan-Nikaido-Kneser theorem is obtained. Among other applications the coincidence theorem for inward and outward maps (in the sense of Ky Fan) and two minimax results in topological vector spaces with sufficiently many linear functionals should be mentioned. The final section of the work contains some new results related to the Walras excess demand theorem, and one of them contains as a special case the well-known theorem of Gale-Nikaido-Debreu. [Reviewed by Valeri Obukhovskiĭ]

(10) Deguire-Granas, 1986 – SM [14]

Abstract: In this note, using the Principle of the KKM-maps, we establish an alternative, formulated in terms of convex analysis, which has several useful consequences in the theory of variational inequalities and contains as a special case the well-known theorems of Sion, von Neumann, and Ky Fan-Nash.

Comments: The well-known theorems of Sion, von Neumann, and Ky Fan-Nash are all extended within the frame of abstract convex spaces in [46].

(11) Ben-El-Mechaiekh - Deguire - Granas, 1987 – Paris [16]

Abstract: In this Note we present some geometrical results which extend numerous fixed points and coincidence theorems, in particular those of our previous Notes [6].

From Text: Let X be a Hausdorff topological space and Y be a convex subset of a topological vector space. A multimap $A : X \multimap Y$ belongs to the class $M^*(X, Y)$ if for every compact $K \subset X$ there exist a finite subset $\{y_1, \cdots, y_k\} \subset Y$ and a continuous selection $s : K \to Y$ of A such that $s(K) \subset \mathrm{co}\{y_1, \cdots, y_k\}$. The class M^* contains the classes of maps F^* and φ^* investigated in Parts I and II [5,6] and also finite compositions of maps from these classes.

Some properties of the class M^* are described, in particular, analogues of fixed point and coincidence theorems from Parts I and II are given. It is noted also that analytical interpretations of these statements contain a number of known minimax and variational relations.

(12) Granas-Liu 1987 – NCA [17]

Abstract: In numerous extensions of the von Neumann's Minimax Theorem, one of the main purposes was to eliminate as much as possible the underlying convexity structure from the original hypothesis. In this note we discuss some new general minimax results without convexity. In the first part, we extend the well-known version of of the von Neumann Principle due to Ky Fan [Proc. Nat. Acad. Sci., U.S.A. 39 (1953), 42–47]. In the second part (as an application) we give a further generalization of the Ky Fan-Nikaido-Neser Theorem concerning systems of inequalities. Some of the main results presented here were announced without proof in [12]. We remark that the proof of the main results is based on the von Neumann Minimax Theorem in $\mathbb{R}^n$.

(13) Granas 1990 – Montreal [20]

From the preface: In Part I we recall the general principle of KKM maps. We formulate their analytic version, from which we derive several applications in minimax and fixed-point theory. Part II is devoted to the exposition of some coincidence principles for mutimaps, their analytic versions in the form of nonlinear alternatives, and several other applications to minimax theory, variational inequalities and game theory. Part III contains several more specialized results along with various applications.

Comments: This is a rich source of earlier works in the KKM theory with topological methods in convex analysis.

(14) Granas-Lassonde, 1991 – SM [24]

Abstract: In this note we present a new elementary approach in the theory of minimax inequalities. The proof of the main result (called the geometric principle) uses only some simple properties of convex functions. The geometric principle (which is equivalent to the well-known lemma of Klee (1951)) is shown to have numerous applications in different areas of mathematics.

Comments: This paper concerns with many known applications of the convex-valued KKM maps. The main result is as follows:

Theorem 1. (Geometric Principle) *Let D be a a nonempty subset of a t.v.s. E and let $G : D \multimap E$ be a KKM map having convex closed values; then the family $\{G(x) \mid x \in D\}$ has the finite intersection property.*

Note that the proof of Ky Fan's 1961 KKM lemma contains the geometric principle without assuming convexity of $G(x)$ for all $x \in D$ based on the original KKM theorem in 1929. However, the authors' aim in this paper is to give an elementary proof without using the KKM theorem. Since the authors' $(E, D; \mathrm{co})$ is a particular partial KKM space, it satisfies a large number of equivalent results in [46]. Some of them appear in this paper in their particular forms.

The proof of the geometric principle is very simple and depends only on the geometric structure induced by convexity. It is shown that numerous applications in different areas of mathematics are consequences of the principle. Such applications are systemically given to systems of inequalities, variational inequalities, minimax equalities, theorems of Markhoff-Kakutani, Mazur-Orliicz and Hahn-Banach, variational problems, maximal monotone operators, and others in convex analysis. Unlike other approaches, the derivation of results is obtained without using arguments such as convexity, the separation theorem, and fixed point theory.

Recall that some people complained against our use of triples for abstract convex spaces. Note that here also appear many triples $(E, D; \Gamma)$.

(15) Granas-Lee-Liu 1992 – CM [28]

From Text: In this note, we present a minimax theorem (Theorem A) formulated by only in language of set theory. This result permits to deduce many well-known minimax theorems by immediate method (of using a lemma in general topology).

Comments: One of the consequences is the Nikaidô-von Neumann minimax theorem (Corollaire 2).

(16) Granas-Lassonde, 1995 – TMNA [36]

From *Introduction* : In a recent paper [24], the authors presented a new geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics. In this note, we complement and elucidate the above approach within the context of complete metric spaces.

More precisely, we concentrate on super-reflexive Banach spaces and show that a large part of the theory of these spaces (and, in particular, Hilbert spaces) can be obtained in a very elementary way, without using weak topology or compactness.

Comments: This paper concerns with many known applications of the convex-valued KKM maps on super-reflexive Banach spaces. The main tool is the following

Theorem 2.2. (Intersection Principle) *Let* $(E, ||\cdot||)$ *be super-reflexive and let* $\{C_i \mid i \in I\}$ *be a family of closed convex sets in* E *with the finite intersection property. If* C_{i_0} *is bounded for some* $i_0 \in I$*, then the intersection* $\bigcap\{C_i \mid i \in I\}$ *is not empty.*

The aim of the paper is to provide simple proofs of several results, stated in super-reflexive Banach spaces, concerning minimization of quasi-convex functions, variational inequalities, game theory, systems of inequalities, and maximal monotone operators. As the authors point out, most of the results are valid for arbitrary reflexive spaces.

As the authors noted at the end of [36], in weak topology, closed bounded subsets of a super-reflexive Banach spaces are compact. Therefore the intersection principle is a variant of the general KKM type theorem in the recently developed abstract convex space theory.

(17) Granas-Dugundji, 2003 – Springer [38]

From MR 1987179 (2004d:58012): This is the most comprehensive, well-written and complete book on fixed point theory to date. The book studies just about every aspect of fixed point theory in terms of classes of operators, spaces, and methodology. Chapter I deals with elementary fixed point theorems. It contains results based on completeness, order, and convexity. Chapter II treats the theorem of Borsuk and topological transversality. The Lyusternik-Shnirelman-Borsuk and Borsuk-Ulam theorems can also be found here, as well as fixed points for compact mappings in normed linear spaces. Homology and fixed points are considered in Chapter III. The Leray-Schauder degree theory and the fixed point index are included in Chapter IV. Here one can find, among other things, a nice introduction to absolute neighborhood retracts (ANRs) and the Leray-Schauder principle and the fixed point index in them. In Chapter V the authors include the Lefschetz-Hopf theory. This is where one looks for singular homology, the Lefschetz theory for maps of ANRs, and the Hopf index theorem. Selected topics comprise the content of Chapter VI, where one finds, among other things, finite-codimensional Cĕch cohomology, Vietoris fractions, and coincidence theory. The book carries an extensive literature on the subject and many examples. Many of the interesting results, given as exercises, constitute an extension of the theory established in the main text. No results from the main text rely upon those of the exercises. I recommend this excellent volume on fixed point theory to anyone interested in this core subject of nonlinear analysis. [Reviewed by A. G. Kartsatos]

Comments: This is an excellent book on fixed point theory for beginners. However, materials related to the KKM theory are almost same to its previous version [3]. Therefore, it does not reflect the current research on the KKM theory.

Final Remark. The following references are given in chronological order. For related works, see the references in the ones listed here.

Competing interests. The author declares that he has no competing interests.

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