Nonlinear Analysis Forum **22**(2), pp. 7–15, 2017
Available electronically at http://www.na-forum.org

GENERALIZATIONS OF KHAMSI'S KKM AND FIXED POINT THEOREMS ON HYPERCONVEX METRIC SPACES

Sehie Park

Reprinted from the
Nonlinear Analysis Forum
Vol. 22(2), Sep. 2017

Nonlinear Analysis Forum **22**(2), pp. 7–15, 2017
Available electronically at http://www.na-forum.org

GENERALIZATIONS OF KHAMSI'S KKM AND FIXED POINT THEOREMS ON HYPERCONVEX METRIC SPACES

Sehie Park

The National Academy of Sciences, Republic of Korea
Seoul 06579, KOREA; and
Department of Mathematical Sciences
Seoul National University
Seoul 08826, KOREA
E-mail : park35@snu.ac.kr, sehiepark@gmail.com

Abstract. In 1996, Khamsi established the KKM theorem for hyperconvex metric spaces and applied it to obtain a Schauder type fixed point theorem. This line of study has been followed by a large number of authors. In this article, we show that the KKM theorem, best approximation theorem, and the Schauder type fixed point theorem for hyperconvex metric spaces due to Khamsi can be extended to partial KKM metric spaces.

1. Introduction

In 1996, Khamsi [7] introduced the class of the Knaster-Kuratowski-Mazurkiewicz type maps (simply KKM maps) on metric spaces, established the KKM theorem for hyperconvex metric spaces, and applied it to obtain a Schauder type fixed point theorem. This is the origin of the KKM theory of hyperconvex metric spaces and related topics in more than fifty subsequent articles appeared in the literature; see [20] and the references therein.

Since 2006, we established the KKM theory of abstract convex spaces and the multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ related to generalizations of KKM maps; see [10-18]. Later, it is known that most of the results in the KKM theory of hyperconvex metric spaces are simple consequences of much more general results on KKM spaces due to ourselves; see [20].

In this article, we show that the KKM theorem, best approximation theorem, and the Schauder type fixed point theorem for hyperconvex metric spaces in [7] can be extended to partial KKM metric spaces.

Section 2 is a preliminary on our abstract convex space theory. In Section 3, we are concerned with KKM maps in metric spaces. In fact, we show that every hyperconvex metric space is a KKM space and hence, can be

Received & Accepted: Apr. 2017, Online Published: Sep. 2017.
2010 Mathematics Subject Classification: Primary 47H09, Secondary 47H10.
Key words and phrases: abstract convex space, KKM map, fixed point, admissible set.

applicable to the theory of KKM spaces developed by ourselves in [16] with some corrections in [19]. Section 4 devotes to generalize the KKM theorem, the best approximation theorem, and the Schauder type fixed point theorem for hyperconvex metric spaces in [7] to corresponding ones in partial KKM metric spaces. Finally, in Section 5, we add further comments to improve the contents of Khamsi [7] and Yuan [21].

2. Abstract convex spaces

For the basic concepts on our abstract convex spaces and partial KKM spaces, we follow [16] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Remark 2.2. Recently the present author was informed that abstract convex spaces were named Γ-convex spaces by Zafarani [23] earlier in 2004.

Definition 2.3. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Now we have the following well-known diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Note that each implication in the above diagram is proper; that is, its converse does not hold.

3. KKM maps in metric spaces

Let A be a nonempty bounded subset of a metric space (M, d). Then we define the following as in Khamsi [7]:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap\{B \subset M : B$ is a closed ball in M such that $A \subset B\}$.

(ii) $\mathcal{A}(M) := \{A \subset M : A = \mathrm{BI}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

(iii) A is called *subadmissible*, if for each $N \in \langle A \rangle$, $\mathrm{BI}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

For a point $x \in M$ and $t > 0$, let

$$B(x,t) := \{y \in M : d(x,y) \le t\}.$$

We introduce new definitions:

Definition 3.1. An abstract convex space $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, $D \subset M$ is nonempty, and $\Gamma : \langle D \rangle \to \mathcal{A}(M)$ is a map such that $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle D \rangle$. A map $G : D \multimap M$ is a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

A Γ-convex subset of $(M, D; \Gamma)$ is subadmissible and conversely.

Example 3.2. For the following examples, we can make metric spaces $(M, D; \Gamma)$.

1. Any normed vector space.
2. Any metrizable topological vector space.
3. Any hyperconvex metric spaces; see below.

Remark 3.3. 1. Let X be a nonempty subadmissible subset of a metric space (M, d), D a nonempty subset of X such that $\mathrm{BI}(D) \subset X$. In [3, Theorem 4], it is claimed that $(X, D; \Gamma)$ is a partial KKM space with an incorrect proof.

2. A variant of the KKM map is given in [4].

Definition 3.4. A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [6] or an H-space by Bardaro and Ceppitelli [2].

Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces, and these are examples of compact c-spaces without having the fixed point property.

Now we recall some notions and basic facts about hyperconvex metric spaces.

The following is due to Aronszajn and Panitchpakdi [1]:

Definition 3.5. A metric space (H, d) is said to be *hyperconvex* if $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

It is known that a normed vector space X is not hyperconvex in general, and the spaces $(\mathbb{R}^n, ||\cdot||_\infty)$, l^∞, L^∞ and $\mathbb{R}$-trees are hyperconvex.

Since each admissible subset of a hyperconvex metric space is hyperconvex and hence contractible, the following is due to Horvath [6]:

Lemma 3.6. *Any hyperconvex metric space H is a c-space* $(H; \Gamma)$, *where* $\Gamma_A = BI(A)$ *for each* $A \in \langle H \rangle$.

From Lemma 3.6 and our KKM theory, we have the following:

Lemma 3.7. *Every hyperconvex metric space is a KKM metric space, that is, a metric space satisfying the KKM principle.*

From this, hyperconvex metric spaces satisfy all of the results in the KKM theory appeared in [16] with some corrections in [19].

4. KKM theorems and fixed point theorems

In this section, we generalize the KKM theorem, best approximation theorem, and the Schauder type fixed point theorem for hyperconvex metric spaces in [7] to corresponding ones for partial KKM metric spaces. In each proof, we follow faithfully the original one given by Khamsi [7].

From the partial KKM principle we have a whole intersection property of the Fan type as follows [16, 17, 18]:

Theorem 4.1. *Let* $(E, D; \Gamma)$ *be a partial KKM space* [*resp.* *KKM space*] *and* $G : D \multimap E$ *a KKM map such that*

(1) G *is closed-valued* [*resp. open-valued*].

Then the family $\{G(z) : z \in D\}$ *has the finite intersection property.*

Moreover, suppose that

(2) $\bigcap_{z\in N} \overline{G(z)}$ *is compact for some* $N \in \langle D \rangle$.

Then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

This theorem is an immediate routine consequence of the definition of partial KKM spaces, and implies the Fan matching property, some geometric property, the Fan-Browder fixed point property, some minimax inequality, several variational inequalities, von Neumann type minimax theorems, Nash equilibrium theorems, and many others. See [16] with some corrections in [19] and the references therein.

In order to prove the analogue of Ky Fan's best approximation theorem [5] for partial KKM metric spaces, we need the following direct application of Theorem 4.1.

Theorem 4.2. *Let $(M, d) = (M, X; \Gamma)$ be a partial KKM metric space where $X \subset M$ is a compact subadmissible subset. Let $f : X \to M$ be continuous. Then there exists $y_0 \in X$ such that*

$$d(y_0, f(y_0)) = \min_{x \in X} d(x, f(y_0)).$$

Proof. Consider the map $G : X \multimap M$ defined by

$$G(x) := \{y \in M : d(y, f(y)) \leq d(x, f(y))\}.$$

Since f is continuous, $G(x)$ is closed for any $x \in X$. We claim that G is a KKM map on the partial KKM metric space. Indeed, assume not. Then there exist $A = \{x_1, \ldots, x_n\} \in \langle X \rangle$ and $y \in \mathrm{BI}(A)$ such that $y \notin G(A)$. This clearly implies

$$d(x_i, f(y)) < d(y, f(y)) \quad \text{for } i = 1, \ldots, n.$$

Let $\varepsilon > 0$ such that $d(x_i, f(y)) \leq d(y, f(y)) - \varepsilon$ for each i. Hence $x_i \in B(f(y), d(y, f(y)) - \varepsilon)$ for each i. Therefore, we have $\mathrm{BI}(A) \subset B(f(y), d(y, f(y)) - \varepsilon)$, which implies $y \in B(f(y), d(y, f(y)) - \varepsilon)$. Clearly this gets us our contradiction which completes the proof of our claim. By the compactness of X, we deduce that $G(x)$ is compact for any $x \in X$. Therefore, there exists $y_0 \in \bigcap_{x \in X} G(x)$. This clearly implies $d(y_0, f(y_0)) \leq d(x, f(y_0))$ for any $x \in X$ which implies $d(y_0, f(y_0)) = \min_{x \in X} d(x, f(y_0))$ and the proof is complete.

Note that Theorem 4.2 reduces to [7, Lemma 4] when $(M, d) = (M; \Gamma)$ is a hyperconvex metric space. We followed Khamsi's proof faithfully.

Corollary 4.3. *Let (M, d) be a metrizable topological vector space and X be a compact ball. Let $f : X \to M$ be continuous. Then there exists $y_0 \in X$ such that*

$$d(y_0, f(y_0)) = \min_{x \in X} d(x, f(y_0)).$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of F.)

Remark 4.4. 1. Note that, in Corollary 4.3, $(M, X; \Gamma)$ is a metric space with $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle X \rangle$, and that $\mathrm{co}(A) \subset \mathrm{BI}(A)$ where co denotes the convex hull. Since $(M, X; \mathrm{co})$ is a partial KKM metric space, so is $(M, X; \Gamma)$.

2. It is not known yet Corollary 4.3 holds when X is a compact convex subset of M.

3. Ky Fan [5] deduced the following best approximation theorem from his geometric lemma. Here $(E, X; \Gamma)$ is an H-space with $\Gamma_A := \mathrm{co}(A)$ for each $A \in \langle X \rangle$.

Theorem 4.5 ([5])**.** *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$||y_0 - f(y_0)|| = \min_{x \in X} ||x - f(y_0)||.$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of F.)

Fan [5] also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. These are known as best approximation theorems and applied to obtain generalizations of the Brouwer fixed point theorem and some non-separation theorems on upper demicontinuous multimaps.

Note that Corollary 4.3 is a consequence of Theorem 4.5 when M is a normed vector space, and another generalization of the Brouwer fixed point theorem.

We now are ready to state a Fan type fixed point theorem [7] in partial KKM metric spaces.

Theorem 4.6. *Let $(M, X; \Gamma)$ be a partial KKM metric space where $X \subset M$ is a compact subadmissible subset. Let $f : X \to M$ be continuous and such that, for every $x \in X$ with $x \neq f(x)$, there exists an $\alpha \in (0, 1)$ such that*

$$(*) \qquad X \cap B(x, \alpha d(x, f(x)) \cap B(f(x), (1-\alpha)d(x, f(x)) \neq \emptyset.$$

Then f has a fixed point, i.e., $f(y) = y$ for some $y \in X$.

Note that the condition $(*)$ means that a metric convex combination of x and $f(x)$ belongs to X. In particular, it is satisfied if $f : X \to X$.

Proof. By the previous Theorem 4.2, there exists $y_0 \in X$ such that

$$d(y_0, f(y_0)) = \inf_{x \in X} d(x, f(y_0)).$$

We claim that such an element y_0 is a fixed point of f. Indeed, assume not, i.e., $y_0 \neq f(y_0)$. Then our assumption on X implies the existence of an $\alpha \in (0, 1)$ such that

$$X \cap B(y_0, \alpha d(y_0, f(y_0)) \cap B(f(y_0), (1-\alpha)d(y_0, f(y_0)) \neq \emptyset.$$

Let x be an element of the nonempty left-hand side. Then $d(x, f(y_0)) = (1-\alpha)d(y_0, f(y_0)$. Since $d(y_0, f(y_0)) \leq d(x, f(y_0))$, we clearly get a contradiction. This completes our proof.

Corollary 4.7. *Any continuous self-map of a compact subadmissible subset X of a partial KKM metric space has a fixed point.*

Remark 4.8. 1. Note that Theorem 4.6 generalizes [7, Theorem 6] where $(M, d) = (M; \Gamma)$ is a hyperconvex metric space and X is admissible. We followed Khamsi's proof faithfully.

2. Khamsi noted that his Theorem 6 can be easily extended to a class of subsets other than admissible ones, i.e., we may relax this assumption by a kind of metric convexity. Note that by using the extension properties of hyperconvex metric spaces, one can easily derive this conclusion via the Schauder theorem.

5. Comments on Khamsi's KKM theorem

The following is due to Khamsi [7] and Yuan [21]:

Definition 5.1. Let (M, d) be a metric space. A subset $S \subset M$ is said to be *finitely metrically closed* [resp. *finitely metrically open*] if for each $F \in \mathcal{A}(M)$, the set $F \cap S = \mathrm{BI}(F) \cap S$ is closed [resp. open]. Note that $\mathrm{BI}(F)$ is always defined and belongs to $\mathcal{A}(M)$. Thus if S is closed [resp. open] in M, it is obviously finitely metrically closed [resp. open].

The following is due to Khamsi [7] and Yuan [21]:

Theorem 5.2 (KKM-Map Principle)**.** *Let H be a hyperconvex metric space, X an arbitrary subset of H, and $G : X \multimap H$ a KKM map such that each $G(x)$ is finitely metrically closed* [*resp. finitely metrically open*]. *Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

This shows that any hyperconvex metric space having *finitely metric topology* is a KKM space. Hence such space satisfies all results in [16].

In view of Theorem 4.1, we have the following due to Khamsi [7, Theorem 4]:

Theorem 5.3. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Note that H can have the finitely metric topology. Therefore it is natural, but not practical, to assume that *every hyperconvex metric space has the finitely metric topology.* This assumption simplifies the texts of [9, 21, 22].

Khamsi [7] noted that the compactness assumption of $G(x_0)$ in the above Theorem 5.3 may be a stronger one. We can still reach the conclusion if one involves an auxiliary multimap and a suitable topology on H such as the ball topology, for example.

Theorem 5.4. *Let E be a partial KKM space and D an arbitrary nonempty set. Let $G : D \multimap E$ be a KKM map. Assume there is a multimap $K : D \multimap E$ such that $G(x) \subset K(x)$ for every $x \in D$ and*

$$\bigcap_{x \in D} K(x) = \bigcap_{x \in D} G(x).$$

If there is some topology on E such that each $K(x)$ is compact, then

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

The proof is obvious. This reduces to [7, Theorem 5] when E is hyperconvex.

Competing interests. The author declares that he has no competing interests.

References

[1] N. Aronszajn and P. Panitchpakdi, *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439

[2] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484–490.

[3] C.-M. Chen, *KKM property and fixed point theorems in metric spaces*, J. Math. Anal. Appl. **323** (2006), 1231–1237.

[4] C.-M. Chen and T.-H. Chang, *Some results for the family 2–$_gKKM(X,Y)$ and the Φ-mapping in hyperconvex metric spaces*, Nonlinear Anal. **69** (2008), 2533–2540.

[5] Ky Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[6] C. D. Horvath, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[7] M. A. Khamsi, *KKM and Ky Fan theorems in hyperconvex metric spaces*, J. Math. Anal. Appl. **204** (1996), 298–306.

[8] M. A. Khamsi and N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **73** (2010), 3123–3129. doi:10.1016/j.na.2010.06.084.

[9] W. A. Kirk, B. Sims, and G. X.-Z. Yuan, *The Knaster–Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*, Nonlinear Anal. **39** (2000), 611–627.

[10] S. Park, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11** (2006), 67–77.

[11] ———, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. Nat. Inst. Math. Sci. **2**(4) (2007), 35–47.

[12] ———, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008), 1–27.

[13] ———, *New foundations of the KKM theory*, J. Nonlinear Convex. Anal. **9**(3) (2008), 331–350.

[14] ———, *Remarks on the partial KKM principle*, Nonlinear Anal. Forum **14** (2009), 51–62.

[15] ———, *Comments on the KKM theory on hyperconvex metric spaces*, Tamkang J. Math. **41**(1) (2010), 1–14.

[16] ———, *The KKM principle in abstract convex spaces — Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[17] ———, *A genesis of general KKM theorems for abstract convex spaces*, J. Nonlinear Anal. Optim. **2**(1) (2011), 133–146.

[18] ———, *Remarks on certain coercivity in general KKM theorems*, Nonlinear Anal. Forum **16** (2011), 1–10.

[19] ———, *Review of recent studies on the KKM theory*, II, Nonlinear Funct. Anal. Appl. **19**(1) (2014), 143–155

[20] ———, *Evolution of the KKM theory of hyperconvex spaces*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **54**(2) (2015), 1–28.

[21] G. X.-Z. Yuan, *The characterization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications*, J. Math. Anal. Appl. **235** (1999), 315–325.

[22] ———, *KKM Theory and Applications in Nonlinear Analysis*, Pure Appl. Math. **vol. 218**, Marcel Dekker, New York, 1999.

[23] J. Zafarani, *KKM property in topological spaces*, Bull. Soc. Royale Sci. Liège **73**(2-3) (2004), 171–185.