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NEW EXAMPLES OF KKM SPACES

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Abstract. In this article, we introduce some particular types of KKM maps or relatively new KKM spaces. They are hyperconvex metric spaces, E-convex spaces of Youness, Bayoumi's p-convex spaces, R-KKM maps of Sankar Raj and Somasundaran, Kim's R-E-KKM maps, and KKM spaces introduced by Chaipunya and Kummam. We also respond some remarks given by Kulpa and Szymanski on the definition of our abstract convex spaces.

1. Introduction

Since we began to study the KKM theory in 1992, we introduced several types of abstract convex spaces such as G-convex spaces, ϕ_A-spaces, (partial) KKM spaces, and of multimap classes like acyclic maps, the admissible $\mathfrak{A}_c^\kappa$-maps, the better admissible $\mathfrak{B}$-maps, and the KKM-classes $\mathfrak{KC}$, $\mathfrak{KO}$ of multimaps. Recently we found some particular types of KKM maps or relatively new KKM spaces appeared in the literature.

In 1993, Horvath [6] showed that the hyperconvex metric spaces due to Aronszajn and Panitchpakdi [1] is a c-space, which is in turn a KKM space in the sense of ourselves. Moreover, Khamsi in 1996 [8] and Yuan in 1999 [25] showed that every hyperconvex metric space with the finitely metric topology is a KKM space in the sense of ourselves.

In 1999, a class of sets and a class of functions called E-convex sets and E-convex functions were introduced by Youness [24] by relaxing the definitions of convex sets and convex functions, resp. This kind of generalized convexity is based on the effect of an operator E on the sets and domain of definition of the functions. The optimality results for E-convex programming problems are established in [24]. Consecutively, many works on E-convex spaces have appeared; see MathSciNet of A.M.S.

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In 2003, Bayoumi [2] introduced p-convex topological vector spaces, which is known to be KKM spaces later by ourselves. Moreover, Gholizadeh et al. [7] in 2013 extended Park's several fixed point theorems in locally convex spaces to the corresponding ones in locally p-convex spaces; more details can be seen in Park [21].

In 2012, using a KKM type intersection theorem, Sanka Raj and Somasundaram [23] attempted to prove a extended version of Fan-Browder multivalued fixed point theorem, in a normed linear space setting, by providing an existence of a best proximity point.

In a more recent work [9], Kim introduced the E-KKM map using the E-convexity and proved some Fan-Browder type fixed point theorems and an equilibrium existence theorem in E-convex setting. Moreover, in another work [10], he introduced an R-E-KKM map as a common generalization of E-KKM map and R-KKM map in E-convex setting. And then he proved an R-E-KKM theorem and the best proximity theorem in a normed linear space.

Very recently Chaipunya and Kummam [4] in 2015 considered the KKM maps defined for a nonself map and the correlated nonself KKM lemma in Hadamard manifolds. The nonself version of the Browder's fixed point theorem as well as the solvability of a generalized equilibrium problem are studied, as applications of their KKM lemma. Their outputs improved the results of Sanka Raj and Somasundaram [23]. In the present article we show that their proximal pairs can be regarded as a new type of KKM spaces.

Our main aim in the present article is to show that all of the main results in the above mentioned works can be included in our previous KKM theory of abstract convex spaces [15]. Moreover, some of them can be counter-examples of certain remarks on abstract convex spaces given by Kulpa and Szymanski [12] in 2014.

In Section 2, we give only a small portion of basic concepts in our KKM theory of abstract convex spaces as a preliminary. Sections 3-9 are devoted to introduce the works of Horvath, Khamsi and Yuan, Youness, Bayoumi, Sankar Raj and Somasundaran, Kim, and Chaipunya and Kummam, resp. All of them are sources of new KKM spaces. Finally, in Section 10, we state our responses to some remarks given by Kulpa and Szymanski [12] and others as a continuation of Park [20].

In this article Theorems A - D are our own results.

2. Abstract convex spaces

Let $\langle D \rangle$ denote the set of nonempty finite subsets of a set D.

In order to unify various types of convexity spaces appeared in the KKM theory, we introduced the following in 2006 [13]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A : A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Remark. Recently the present author was informed that abstract convex spaces were named Γ-convex spaces by Zafarani [26] more early in 2004.

Later, we add to assume E is a topological space in an abstract convex space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$]. See [14].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent works, we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Exmaple. We gave known examples of (partial) KKM spaces in [15] and the references therein. The following are some of them.

Definition. A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

If $D = X$, we denote $(X;\Gamma)$ instead of $(X, X;\Gamma)$, which is called a c-space by Horvath [6]

Definition. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X, D;\Gamma)$ becomes an abstract convex space.

Now we have the following diagram for triples $(E, D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Note that, in the above diagram, every implication is proper; that is, its converse does not hold.

The following well-known KKM theorem is due to ourselves; see [15,17,18]:

Theorem A. *Let $(E, D;\Gamma)$ be a partial KKM space [resp. KKM space] and $G : D \multimap E$ a KKM map such that*

(1) *G is closed-valued [resp. open-valued].*

Then the family $\{G(z) : z \in D\}$ has the finite intersection property.

Moreover, suppose that

(2) *$\bigcap_{z\in N} \overline{G(z)}$ is compact for some $N \in \langle D\rangle$.*

Then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

This theorem is an immediate routine consequence of the definition of partial KKM spaces, and implies the Fan matching property, some geometric property, the Fan-Browder fixed point property, some minimax inequality, several variational inequalities, von Neumann type minimax theorems, Nash equilibrium theorems, and many others. See [15] with some corrections in [17] and the references therein.

3. Hyperconvex metric spaces — Horvath 1993

In a metric space (M, d), for a point $x \in M$ and $t > 0$, let

$$B(x, t) := \{y \in M : d(x, y) \leq t\}.$$

The following is due to Aronszajn and Panitchpakdi [1]:

Definition. A metric space (H, d) is said to be *hyperconvex* if $\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of closed balls in H for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

It is known that a normed vector space X is not hyperconvex in general, and the spaces $(\mathbb{R}^n, ||\cdot||_\infty)$, l^∞, L^∞ and $\mathbb{R}$-trees are hyperconvex.

The following is due to Horvath [6]:

Lemma. *Any hyperconvex metric space H is a c-space $(H;\Gamma)$, where $\Gamma(A)$ is the intersection of all closed balls in H containing $A \in \langle H\rangle$.*

From Lemma and our KKM theory, we have the following:

Theorem B. *Every hyperconvex metric space is a KKM metric space, that is, a metric space satisfying the KKM principle.*

From this, hyperconvex metric spaces satisfy all of the results in the KKM theory appeared in [15] with some corrections in [17].

4. Hyperconvex metric spaces — Khamsi 1996, Yuan 1997

Let A be a nonempty bounded subset of a metric space (M, d). Then we define the following as in Khamsi [8]:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap\{B \subset M : B$ is a closed ball in M such that $A \subset B\}$.

(ii) $\mathcal{A}(M) := \{A \subset M : A = \mathrm{BI}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.

(iii) A is called *subadmissible*, if for each $N \in \langle A\rangle$, $\mathrm{BI}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

We introduce new definitions:

Definition. An abstract convex space $(M, D; \Gamma)$ is called simply a *metric space* if (M, d) is a metric space, $D \subset M$ is nonempty, and $\Gamma : \langle D\rangle \to \mathcal{A}(M)$ is a map such that $\Gamma_A := \mathrm{BI}(A) \in \mathcal{A}(M)$ for each $A \in \langle D\rangle$. A map $G : D \multimap M$ is a KKM map if $\Gamma_A \subset G(A)$ for each $A \in \langle D\rangle$.

A Γ-convex subset of $(M, D; \Gamma)$ is subadmissible and conversely.

The following is due to Khamsi [8] and Yuan [25]:

Definition. Let (M, d) be a metric space. A subset $S \subset M$ is said to be *finitely metrically closed* [resp. *finitely metrically open*] if for each $F \in \mathcal{A}(M)$, the set $F \cap S = \mathrm{BI}(F) \cap S$ is closed [resp. open]. Note that $\mathrm{BI}(F)$ is always defined and belongs to $\mathcal{A}(M)$. Thus if S is closed [resp. open] in M, it is obviously finitely metrically closed [resp. open].

The following generalization of Theorem B is due to Khamsi [8] and Yuan [25]:

Theorem 1 (KKM-Map Principle)**.** *Let H be a hyperconvex metric space, X an arbitrary subset of H, and $G : X \multimap H$ a KKM map such that each $G(x)$ is finitely metrically closed* [*resp. finitely metrically open*]. *Then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

This shows that any hyperconvex metric space having *finitely metric topology* is a KKM space. Hence such space satisfies all results in [15].

In view of Theorem 1, we have the following due to Khamsi [8, Theorem 4]:

Theorem 2. *Let H be a hyperconvex metric space and $X \subset H$ an arbitrary subset. Let $G : X \multimap H$ be a KKM map such that $G(x)$ is closed for any $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then we have*

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

Note that H can have the finitely metric topology. Therefore it is natural, but not practical, to assume in the KKM theory that *every hyperconvex metric space has the finitely metric topology.*

From now on the numbers attached to Definitions or Theorems are given in their original source.

5. E-convex spaces of Youness — 1999

In [24], a class of sets and a class of functions called E-convex sets and E-convex functions are introduced as follows:

Definition 2.1 ([24])**.** A set $M \subset \mathbb{R}^n$ is said to be *E-convex* iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that $(1 - \lambda)E(x) + \lambda E(y) \in M$, for each $x, y \in M$ and $0 < \lambda < 1$.

There is an example of an E-convex set, which is not convex [24].

An E-convex set is an example of an abstract convex space and hence can be applied our KKM theory.

In fact, let D be a nonempty subset of M. Then $(M, D; \Gamma)$ is an abstract convex space, where $\Gamma : \langle D \rangle \to M$ is defined by

$$\Gamma\{x_0, \ldots, x_n\} = \operatorname{co} E\{x_0, \ldots, x_n\} = \{\Sigma_{i=0}^n \lambda_i E(x_i) : 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^n \lambda_i = 1\}$$

for each $A := \{x_0, x_1, \ldots, x_n\} \in \langle D \rangle$.

Hence, every E-convex set $M \subset \mathbb{R}^n$ is an abstract convex space $(M, D; \Gamma)$ with any nonempty $D \subset M$. Note that Γ is convex-valued and that there is an obvious continuous map $\phi_A : \Delta_n \to \Gamma_A$. Therefore, $(M, D; \Gamma)$ is an example of H-spaces and ϕ_A-spaces, and hence KKM spaces. Therefore, it satisfies so many results in the KKM theory as shown in [15] and many other related articles.

More generally, we have the following way of making new abstract convex spaces from old:

Definition. Let $(X, D; \Gamma)$ be an abstract convex space, Y be a nonempty set, and $E : Y \multimap D$ be a map. Then $(X, Y; \Gamma^E)$ is called an abstract E-convex space whenever

$$\Gamma^E(A) := \Gamma(E(A)) \text{ for each } A \in \langle Y \rangle.$$

A multimap $T : Y \multimap X$ is called an *E-KKM map* if for any set $A \in \langle Y \rangle$, we have

$$\Gamma(E(A)) \subset T(A).$$

Note that these concepts reduce to those of an abstract convex space and a usual KKM map whenever $E = 1_D$, the identity map on D.

Theorem C. *As in the above Definition, if* $(X, D; \Gamma)$ *is a KKM space, so is* $(X, Y; \Gamma^E)$.

Proof. Let $T : Y \multimap X$ be an E-KKM map having closed [resp. open] values. We have to show $\bigcap_{y \in A} T(y) \neq \emptyset$ for each $A \in \langle Y \rangle$. Let $T' : E(A) \multimap X$ such that $T'(E(y)) := T(y)$ for each $y \in A$. Then

$$\Gamma(E(J)) = \Gamma^E(J) \subset T(J) = T'(E(J))$$

for each $J \subset A$. Hence $T' : E(A) \multimap X$ is a KKM map on the KKM space $(X, E(A); \Gamma)$. Therefore $\bigcap_{z \in E(A)} T'(z) \neq \emptyset$ and hence $\bigcap_{y \in A} T(y) \neq \emptyset$. This completes our proof.

Note that Theorem C can be used to deduce a large number of examples of new KKM spaces.

6. Bayoumi's KKM spaces – 2003

Let $0 < p \leq 1$. Recall the definitions introduced by Bayoumi [2, 7]:

Definition (*p-convex set*)**.** A set A in a vector space V is said to be *p-convex* if, for any $x, y \in A$, $s, t \geq 0$, we have

$$(1-t)^{1/p}x + t^{1/p}y \in A, \quad \text{whenever } 0 \leq t \leq 1.$$

Definition (*p-convex hull*)**.** If X is a topological vector space and $A \subset X$, the closed p-convex hull of A denoted by $\overline{C}_p(A)$ is the smallest closed p-convex set containing A.

Definition (*p-convex combination*)**.** Let A be p-convex and $x_1, \cdots, x_n \in A$, and $t_i \geq 0, \sum_1^n t_i^p = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$. If $\sum_1^n |t_i|^p \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition (*locally p-convex space*)**.** A topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way.

Using these concepts, in [7], new definitions of almost p-convex sets and the p-convexly almost fixed point property are introduced as generalizations of almost convex sets (due to Himmelberg) and the almost fixed point property, resp.

Now we have a new KKM space:

Theorem D. *Suppose that* X *is a subset of a topological vector space* E *and* D *is a nonempty subset of* X *such that* $C_p(D) \subset X$*. Let* $\Gamma_N := C_p(N)$ *for each* $N \in \langle D \rangle$*. Then* $(X, D; \Gamma)$ *is a* ϕ_A*-space.*

Proof. Since $C_p(D) \subset X$, Γ_N is well-defined. For each $N = \{x_0, x_1, \cdots, x_n\} \subset D$, define $\phi_N : \Delta_n \to \Gamma_N$ by

$$\sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} (t_i)^{\frac{1}{p}} x_i.$$

Then clearly $(X, D; \Gamma)$ is a ϕ_A-space.

7. Sankar Raj - Somasundaram's KKM spaces — 2012

Recently, there have appeared a number of authors who adopted the concept of the so-called *generalized R-KKM maps* which were used to rewrite known results in the KKM theory. In our previous work [16], we showed that those maps are simply KKM maps on G-convex spaces and should be destroyed in order to preserve the elegancy of the G-convex space theory. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

In this section, we introduce another example called R-KKM spaces.

Sankar Raj and Somasundaram [23] considered two nonempty subsets A, B of a normed linear space X, introduced an R-KKM map $T : A \to 2^B$, and discussed some sufficient conditions for which the set $\bigcap\{T(x) : x \in A\}$ is nonempty. Using this nonempty intersection theorem, they prove an Fan-Browder type fixed point theorem by providing an existence of a best proximity point.

Let A, B be nonempty subsets of a normed linear space X. Then,

$\text{Prox}(A,\ B) = \{(x, y) \in A \times B : \|x - y\| = dist(A, B)\}$,
$A_0 = \{x \in A : \|x - y\| = dist(A,\ B)$ for some $y \in B\}$
$B_0 = \{y \in B : \|x - y\| = dist(A,\ B)$ for some $x \in A\}$.

Definition 1 ([23])**.** Let A, B be nonempty subsets of a metric space X. Then the pair (A, B) is said to be a *proximal pair* if, for each $(x, y) \in A \times B$, there exists $(\tilde{x}, \tilde{y}) \in A \times B$ such that $\|x - \tilde{y}\| = \|\tilde{x} - y\| = dist(A, B)$. Note that a pair (A, B) is a proximal pair if and only if $A = A_0$ and $B = B_0$.

Now the authors define the notion of R-KKM mappings.

Definition 2 ([23])**.** Let $(A,\ B)$ be a nonempty proximal pair of a normed linear space X. A multivalued mapping $T : A \multimap B$ is said to be a *R-KKM map* if, for any $\{x_1, \ldots, x_n\} \subset A$, there exists $y_1, \ldots, y_n \in B$ with $\|x_i - y_i\| = dist(A,\ B)$, for all $i = 1, \ldots, n$, such that $\text{co}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^n T(x_i)$.

Since $\text{co}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^n T(x_i) \subset B$, we better assume that B is a convex subset of X.

Note that, if $T : A \multimap B$ is an R-KKM map, then $\text{dist}(x,\ T(x)) = dist(A,\ B)$ for any $x \in A$. Also, if $A = B$, then the definition of R-KKM maps reduces to that of KKM maps. A subset C of a normed linear space X is said to be finitely closed if $C \cap L$ is closed for every finite-dimensional subspace L of X.

Theorem 3.1 ([23]). *Let (A, B) be a nonempty proximal pair in a normed linear space X and $T : A \to 2^B$ be an R-KKM map such that $T(x)$ is finitely closed for all $x \in A$. Then the family $\{T(x) : x \in A\}$ has finite intersection property.*

Theorem 3.2 ([23]). *Let (A, B) be a nonempty proximal pair in a normed linear space X and $T : A \to 2^B$ be an R-KKM map. If, for each $x \in A$, $T(x)$ is closed in X and there exists at least one $x_0 \in A$ such that $T(x_0)$ is compact in X, then $\bigcap\{T(x) : x \in A\}$ is nonempty.*

Comments: Let us assume X has the finitely generated topology. Note that if (A, B) is a proximal pair, then the abstract convex space $(B, A; \Gamma)$ with $\Gamma\{x_1, \ldots, x_n\} := \text{co}\{y_1, \ldots, y_n\}$ as in Definition 2 is a partial KKM space by Theorem 3.1.

Moreover, since Γ is convex-valued, $(B, A; \Gamma)$ is an H-space and hence a KKM space. Note that Theorems 3.1 and 3.2 follows from our Theorem A.

It is routine to deduce equivalent formulations of Theorem 3.1 or 3.2. Note that the Fan-Browder type fixed point theorem is one of them.

8. Kim's R-E-KKM maps – 2013

In a recent work [9], Kim introduced the E-KKM map using the E-convexity and proved some Fan-Browder type fixed point theorems and an equilibrium existence theorem in E-convex setting.

However, Kim's paper seems to do not reflect current studies on the KKM theory. It is noted that all of the results in [9] are consequences of the corresponding ones for ϕ_A-spaces. For example, the following is the main result in [9].

Theorem 3.1 ([9]). *Let X be a nonempty subset of a Hausdorff topological vector space Y, $E : Y \to Y$ a map, and $T : X \multimap Y$ be an E-KKM map. If $T(x)$ is nonempty closed for each $x \in X$, and $T(x_0)$ is compact for some $x_0 \in X$, then*

$$\bigcap_{x\in X} T(x) \neq \emptyset.$$

This is a very obsolete particular consequence of Theorems A and C. Other results in [9] are routine applications of his Theorem 3.1 and can be improved by reflecting current studies.

Moreover, in a consequent work [10], Kim introduced an R-E-KKM map as a common generalization of E-KKM map and R-KKM map in E-convex setting. And then he proved an R-E-KKM theorem and the best proximity theorem in a normed linear space.

Definition 2.2 ([10]). Let $(A,\ B)$ be a pair of nonempty subsets of a normed linear space X with a map $E : X \to X$. A multimap $T : A \multimap B$ is called an *R-E-KKM map* on A if for any finite subset $\{x_1, \ldots, x_n\} \subset A$, there exists a finite subset $y_1, \ldots, y_n \in B$ with $d(E(x_i), E(y_i)) = dist(A,\ B)$ for all $i = 1, \ldots, n$, such that $\text{co}\{E(y_1), \ldots, E(y_n)\} \subset \bigcup_{i=1}^n T(x_i)$.

Since $\operatorname{co}(E(y_i)) \subset T(x_i) \subset B$, we better assume that B is a convex subset of X. Define $\Gamma : \langle A \rangle \to B$ by

$$\Gamma\{x_0, \ldots, x_n\} = \operatorname{co} E\{y_0, \ldots, y_n\} = \{\Sigma_{i=0}^n \lambda_i E(y_i) : 0 \le \lambda_i \le 1, \Sigma_{i=0}^n \lambda_i = 1\}$$

for each $\{x_0, \ldots, x_n\} \in \langle A \rangle$.

Note that Γ is convex-valued and that there is an obvious continuous map $\phi_A : \Delta_n \to \Gamma\{x_0, \ldots, x_n\}$.

Hence, the abstract convex space $(B, A; \Gamma)$ is an H-space and a ϕ_A-space, and hence a KKM space. This can be seen also by the fact that B is a convex subset of a normed linear space X. Therefore, it satisfies so many results in the KKM theory as shown in [15] and many other related articles.

Kim [10] obtained the following:

Theorem 3.1 ([10])**.** *Let A, B be nonempty subsets of a normed linear space X with a map $E : X \to X$, and $T : A \multimap B$ be an R-E-KKM map on A such that each $T(x)$ is closed in B. Then the family of sets $\{T(x) : x \in A\}$ has the finite intersection property.*

This is a consequence of our Theorem A. Other results in [10] seem to be routine ones.

9. Chaipunya and Kummam's KKM spaces – 2015

In 2015, Chaipunya and Kumam [4] considered the KKM maps defined for a nonself map and the correlated intersection theorems in Hadamard manifolds. They also study some applications of the intersection results. Their outputs improved the results of Sanka Raj and Somasundaram [23].

In [4], the authors occupy the nonself KKM lemma in Hadamard manifolds. The nonself version of the Browder's fixed point theorem as well as the solvability of a generalized equilibrium problem are studied, as applications of their KKM lemma.

The pair (A, B) set up by two given nonempty subsets A and B of a metric space (S, d) is called a *proximal pair* if to each point $(x, y) \in A \times B$, there corresponds a point $(\overline{x}, \overline{y}) \in A \times B$ such that

$$d(x, \overline{y}) = d(\overline{x}, y) = \operatorname{dist}(A, B),$$

where $\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

In addition, if both A and B are convex, we say that (A, B) is a *convex proximal pair*.

In the future contents, they assume that M is a Hadamard manifold with the geodesic distance d. Given a point $x \in M$ and two nonempty subsets $A, B \subset M$, they write $d(x, A) := \inf_{z \in A} d(x, z)$.

For any nonempty subset $A \subset M$, denoted by $\operatorname{co}(A)$ the geodesically convex hull of A, i.e., the smallest geodesically convex set containing A. Note that the geodesically convex hull of any finite subset is compact.

Definition 3.1 ([4]). Let (A, B) be a proximal pair in a Hadamard manifold M. A nonself map $T : A \multimap B$ is said to be KKM if for each finite subset $D := \{x_1, x_2, \cdots, x_m\} \subset A$, there is a subset $E := \{y_1, y_2, \cdots, y_m\} \subset B$ such that $d(x_i, y_i) = \mathrm{dist}(A, B), \forall i \in \{1, 2, \cdots, m\}$, and

$$\mathrm{co}(\{y_i : i \in I\}) \subset T(\{x_i : i \in I\})$$

for every $\emptyset \neq I \subset \{1, 2, \cdots, m\}$.

Theorem 3.2 ([4]). *Suppose that (A, B) is a proximal pair in a Hadamard manifold M and $T : A \multimap B$ is a KKM map with nonempty closed values. Then, the family $\{T(x) : x \in A\}$ has the finite intersection property.*

Theorem 3.3 ([4]). *Suppose that (A, B) is a proximal pair in a Hadamard manifold M and $T : A \multimap B$ is a KKM map with nonempty closed values. If $T(x_0)$ is compact at some $x_0 \in A$, then the intersection $\bigcap\{T(x) : x \in A\}$ is nonempty.*

Comments: We better assume that B is geodesically convex as we noted twice.

In this paper, an example of a partial KKM space $(B, A; \Gamma)$, where A and B are not comparable, is given. Moreover, a nonself KKM map $T : A \multimap B$ is a generalized KKM in the sense of Chang and Zhang [5].

In 1990, Reich and Shafrir [22] andintroduced hyperbolic spaces in order to try to develop a theory of nonexpansive iterations in more general infinite-dimensional manifolds than normed vector spaces. This class of metric spaces contains all normed vector spaces and Hadamard manifolds, as well as the Hilbert ball and the Cartesian product of Hilbert balls.

In 1992, we began to study the KKM theory and, in 2006, to extend it to abstract convex spaces. Since 2008, we found that any hyperbolic spaces are G-convex spaces and also particular cases of c-spaces. Actually, Park [15] indicated but not concretely that most of key results in the KKM theory can be applied to hyperbolic spaces.

Note that if (A, B) is a proximal pair, then $(B, A; \mathrm{co})$ with $\mathrm{co} : \langle A\rangle \multimap B$ is a KKM space. Hence Theorems 3.2 and 3.3 follow from Theorem A.

10. Questions on definition of abstract convex spaces

Since we introduced the concept of abstract convex spaces in the KKM theory in 2006, some readers raised certain questions or comments on them. In 2015 [19], we clarified such things raised by Ben-El-Mechaiekh [3] and Kulpa - Szymmanski [11]. Moreover, a revised version of [11] appeared as [12] and our response to it was given by [20].

Recall the following two remarks:

(1) Ben-El-Mechaiekh [3]: I cannot see the importance of the auxiliary set D. Perhaps I am not aware of concrete situation where the existence of a set $D \neq E$ on which Γ acts is absolutely needed. In such case, kindly send me references.

(2) Kulpa - Szymmanski [12, Remark 2.6(a)]. Park's general notion of *abstract convex space* $(E, D; \Gamma)$ seem to be a bit superfluous. For if $|D| \leq |E|$, then we may rename the elements of D by elements of a subset of E. If so, then the original abstract convex space $(E, D; \Gamma)$ can be regarded as given in the form $(E \supset D; \Gamma)$. No example of abstract convex space $(E, D; \Gamma)$ with $|D| > |E|$ has ever been considered.

We gave already some sufficient responses to the above remarks in [19] and [20]. However, the KKM spaces introduced by ourselves from the works of Sankar Raj and Somasundaran [23], Kim [9,10] and Chaipunya and Kummam [4] are concrete counter-examples against the claims (1) and (2).

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