RECENT APPLICATIONS OF SOME ANALYTICAL FIXED POINT THEOREMS

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Abstract. In the last three decades, we introduced several fixed point theorems for multimap classes on various types of abstract convex spaces. Such are the classes of acyclic maps, the Fan-Browder type maps, admissible maps $\mathfrak{A}_c^\kappa$, better admissible maps $\mathfrak{B}$, and the KKM maps $\mathfrak{KC}$ and $\mathfrak{KO}$. In our previous reviews, several hundred papers related to applications of such fixed point theorems were introduced. In the present review, we introduce some *recent* results in analytical fixed point theory based on our previous works. Most of them are not appeared in our previous reviews.

1. Introduction

In 1992 [17], we began to study the newly named KKM theory as the field of applications of equivalents or generalizations of the celebrated Knaster-Kuratowski-Mazurkiewicz theorem in 1929. The key results in the KKM theory were first studied by Ky Fan for topological vector spaces in 1969-1984. Later, it is extended to convex spaces by Lassonde, to H-spaces by Horvath, and to generalized (or G-) convex spaces by ourselves. Since 2006, the theory has became the study of abstract convex spaces in the sense of Park.

One of the main fields of the KKM theory is closely related to analytical fixed point theory (other than metric or topological fixed point theories). Since 1992, we studied the multimap classes of acyclic maps, Fan-Browder maps, admissible maps $\mathfrak{A}_c^\kappa$, better admissible maps $\mathfrak{B}$, and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$. Motivated by such works, there have appeared several hundreds of papers on modifications or generalizations or applications of multimap classes in the frame of the KKM theory. These were reviewed by the author in 2009-2014.

In the present review, we introduce some recent results in analytical fixed point theory based on our previous works. Most of such results are not treated in our previous reviews in 2009-2014.

In Section 2, we introduce the contents of our previous reviews in 2009-2014. Section 3 deals with our previous works, from which some recent results in the analytical fixed point theory have been deduced.

2. Our previous reviews

In this section, we introduce the contents of our review articles in 2009-2014.

Park 2009 — VJM 37 [30]

Abstract: We review applications of our fixed point theorems on compact compositions of acyclic maps. Our applications are mainly on acyclic polyhedra, locally convex topological vector spaces, admissible (in the sense of Klee) convex sets, and

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almost convex or Klee approximable sets in topological vector spaces. Those applications are concerned with general equilibrium problems like as (collective) fixed point theorems, the von Neumann type intersection theorems, the von Neumann type minimax theorems, the Nash type equilibrium theorems, cyclic coincidence theorems, best approximation theorems, (quasi-) variational inequalities, and the Gale-Nikaido-Debreu theorem. Finally, we briefly introduce some related results mainly appeared in other authors' works.

Park 2011 — JNAS 50 [34]

Abstract: Since the Brouwer fixed point theorem appeared in 1910, several hundred generalizations or equivalent formulations have been found. The celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, KKM theorem) in 1929 is one of them and the origin of the KKM theory. Now this theory is well established for the new category of abstract convex spaces. Based on this new theory, we present several theorems which unify more than one hundred known generalizations of the Brouwer theorem related to topological vector spaces or topological fixed point theory. The following are simple descriptions of some of those theorems:

(1) An abstract form of the KKM theorem is equivalent to a Fan-Browder type fixed point theorem and to a maximal element theorem.

(2) The Zima-Hadžić type fixed point theorems on the KKM spaces.

(3) Any compact $\mathfrak{KC}$-map on a Φ-space has a fixed point.

(4) Any closed compact $\mathfrak{B}$-map on an abstract convex space has a fixed point whenever its range is Klee approximable.

(5) Any generalized upper hemicontinuous map from a convex subset to a locally convex t.v.s. having closed convex values has a fixed point whenever it satisfies certain compactness and boundary conditions.

Park 2012 — JNAS 51 [36]

Abstract: In the last two decades, we introduced the admissible multimap class $\mathfrak{A}_c^\kappa$, the better admissible class $\mathfrak{B}$, and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$. In our previous work [30], we reviewed applications of our fixed point theorems for the multimap class of compact compositions of acyclic maps and, in [34], we collected most of fixed point theorems related to the KKM theory due to the author. Moreover, applications of our versions of the Fan-Browder fixed point theorem were introduced in [35]. In the present work, we review applications of our fixed point theorems and our multimap classes, appeared mainly in other authors' works. Most of them are not treated in our previous reviews.

Park 2013 — JNAS 52 [38]

Abstract: In this paper, from a general form of the KKM type theorems or some properties of KKM type maps on abstract convex spaces, we deduce several Fan-Browder type alternatives, coincidence or fixed point theorems, and other results. These theorems unify and generalize various particular results of the same kinds recently due to a number of authors for particular types of abstract convex spaces.

Section 2 is a preliminary on our abstract convex spaces with one of the most general KKM theorems. In Section 3, we obtain a Fan-Browder type alternative or coincidence theorem and also give another versions of the theorem. Section 4 deals with new particular forms of them on G-convex spaces. In Section 5, we investigate the contents appeared in related papers on topological vector spaces (t.v.s.). Section 6 deals with related papers on G-convex spaces. In section 7, particular maximal element theorems and their applications. We show that basic theorems of the papers mentioned in Sections 5-7 are consequences of our new results. Finally, in Section

8, we add some historical remarks and further comments to improve many of the known results and their applications.

Park 2014 — JNAS 53 [41]

Abstract: In our previous review [37], we gave a short history of the KKM theory and reviewed its current study by recalling our previous comments or surveys in a sequence of papers. Moreover, in another review [40], we continued to review some recent works on the theory mainly due to other authors. On this occasion, we gave some corrections on [33]. The present paper is the third part of our review of recent studies on the KKM theory and concerns with comments on recent works of some authors.

3. Recent results related to our works

In this section, we introduce other authors' recent works closely related to our previous works. Actually we list some of our previous works, from which some other authors deduced their applications.

Park: FPTA (ed. Tan) (1992) [17]

The following is originally given as a particular form of Park [17, Theorem 1] or [18, Theorem 5]:

Theorem 3.1. *Let X be a nonempty convex subset of a topological vector space E, $A, B : X \multimap X$ two multimaps, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, $A(x) \subset B(x)$ and $B(x)$ is convex;*
(2) *for each $x \in K$, $A(x) \neq \emptyset$;*
(3) *for each $y \in X$, $A^-(y)$ is open in X;*
(4) *for each finite subset N of X, there exists a compact convex subset L_N of X containing N such that $A(x) \cap L_N \neq \emptyset$ for all $x \in L_N \setminus K$.*

Then B has a fixed point.

Note that the Fan-Browder fixed point theorem follows from Theorem 3.1. In order to prove some existence results for noncompact settings, many authors applied Theorem 3.1. See [38].

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over the rational field vanish. A u.s.c. map is said to be *acyclic* if it has compact acyclic values. In [17], we obtained the following fixed point theorem:

Theorem 3.2. *Let X be a nonempty convex subset of a locally convex Hausdorff t.v.s. E. Then any compact acyclic map $F : X \multimap X$ has a fixed point.*

This reduces to Himmelberg's theorem when F is convex-valued. We obtained a large number of generalizations of Theorem 3.2; see [29,31,32] and the references therein. Especially, in [29], we reviewed various generalizations of the Himmelberg fixed point theorem within topological vector spaces.

This paper [17] has been cited by many authors; see [30,34,36,38]. One of the typical ones is the following:

Agarwal-Balaj-O'Regan (JOTA 155 (2012) [1]) — *Abstract*: "The purpose of this paper is to present a unified approach to study the existence of solutions for two types of variational relation problems, which encompass several generalized equilibrium problems, variational inequalities and variational inclusions investigated in the recent literature. By using two well-known fixed point theorems, we establish several existence criteria for the solutions of these problems."

Comments to [1]: A particular form of Theorem 3.1 and the Himmelberg fixed point theorem were applied. In fact, two well-known fixed point theorems mentioned by the authors are a consequence of Theorem 3.1 and Theorem 3.2. Moreover, the authors applied the concept of KKM maps with respect to a multimap F due to Park [18].

Park: JKMS 35 (1998) [20]

In this paper [20] the old notion of the admissible multimap class $\mathfrak{A}_c^\kappa$ is recalled as follows. Let X and Y be topological spaces. A polytope is a homeomorphic image of a simplex.

Definition 3.3. An *admissible class* $\mathfrak{A}_c^\kappa(X,Y)$ of maps $T : X \multimap Y$ is the one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K,Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite compositions of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:
(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(3) for each polytope P, each $T \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of compositions are suitably chosen for each $\mathfrak{A}$.

A nonempty subset X of a Hausdorff topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E.

Note that every nonempty convex subset of a locally convex topological vector space is admissible. Other examples of admissible maps can be found in [20].

Definition 3.4. A map $F : X \multimap Y$ for topological spaces X and Y belongs to the better admissible class $\mathfrak{B}(X,Y)$ if and only if for any polytope $P \subset X$ and any continuous function $f : F(P) \to P$ the composition $f \circ F|P : P \multimap P$ has a fixed point.

In [20], the following is obtained:

Theorem 3.5. *Let E be a Hausdorff topological vector space and X an admissible, convex subset of E. Then any closed, compact map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Agarwal-O'Regan-Precup (TMNA 22 (2003) [5]) — *Abstract*: "Some new fixed point theorems for approximable maps are obtained in this paper. Homotopy results, via essential maps, are also presented for approximable maps."

From Text of [5] : Let $\mathcal{A}_0$ and $\mathcal{A}$ denote the class of approachable and approximable maps, resp. From a fixed point result obtained by Ben-El-Mechaiekh and Deguire, the following is deduced in [5]:

Theorem 3.6 ([5])**.** *Let X be a convex subset of a (Hausdorff) complete locally convex topological vector space E and let $F \in \mathcal{A}(X,X)$ be a upper semicontinuous compact map. Then F has a fixed point.*

Remark. It is possible to consider a more general space E in Theorem 3.6 if one works in a larger class of maps (see [20,21]).

Comments on [5]: The following concept due to Park is introduced in this paper:

A nonempty subset X of a Hausdorff topological vector space E is said to be *q-admissible* if any nonempty compact, convex subset K of X is admissible.

This is defined in [20, Corrections]. Its motivation is stated there as follows: "Note that even if X is admissible, we can not say that K is admissible in E. Therefore, we need the concept of q-admissible sets."

Shahzad (TMNA 24 (2004) [44]) — *Abstract*: "The paper presents new approximation and fixed point results for $\mathfrak{A}_c^\kappa$ maps in Hausdorff locally convex spaces."

Comments on [44]: A particular form of Theorem 3.5 for $\mathfrak{A}_c^\kappa$ maps is applied.

Agarwal-O'Regan-Taoudi (Asian-Europ. J. Math. 4 (2011) [7]) — *Abstract*: "The authors present new fixed point theorems for $\mathfrak{A}_c^\kappa$-admissible maps acting on locally convex t.v.s. They considered multimaps need not be compact, and merely assume that multimaps are weakly compact and map weakly compact sets into relatively compact sets. Their fixed point results are obtained under Schauder, Leray-Schauder and Furi-Pera type conditions."

From Text of [7]: The following result was proved in Park [20].

Theorem 3.7 ([20])**.** *Let E be a Hausdorff topological vector space and X an admissible, convex subset of E. Then any compact map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

This is the basis of [7].

O'Regan and Shahzad (Adv. Fixed Point Theory 2 (2012) [16]) — *Abstract*: "A new Krasnoselskii fixed point result is presented for weakly sequentially upper semicontinuous maps. The proof is immediate from results of O'Regan. The authors also extend the results for a general class of maps, namely the $\mathfrak{B}^\kappa$ maps of Park."

From Text of [16] : We will establish a result for a very general class of maps, namely the $\mathfrak{B}^\kappa$ maps of Park (1998).

In 1998, Park proved Theorem 3.5.

Examples of $\mathfrak{B}^\kappa$ maps can be found in Park [20].

Rutsky (arXiv:1409.3871v2 (2014) [43]) — *From Text of* [43]: 3. Preparations. We begin by stating a very general fixed point theorem from [20] used in the main result. . . . admissible (in the sense of Klee) . . . In other words, X is admissible if any compact set $K \subset X$ can be continuously and uniformly approximated by a family of finite-dimensional sets of X. . . . better admissible class . . . . Observe that admissibility refers in this notion to the existence of fixed points in a restricted sense. The class $\mathfrak{B}(X,Y)$ encompasses a large number of particular classes of maps that are known to have fixed points; in the present work we will only use the fact that this class contains finite compositions of acyclic maps. The corresponding fixed point theorem was established *in previous literature*, and it is possible to use it directly with minor adaptations; the powerful result [20], however, allows us to keep the necessary topological explanations to a minimum.

And Rutsky stated Theorem 3.5 ([20, Corollary 1.1]).

Park: JKMS 37 (2000) [22]

Zhang-Liu-Cheng (NA 71 (2009) [47]) — *Abstract*: "In this paper, the notions of generalized loose (weak) saddle points for set-valued mappings are introduced. By applying the fixed point theorem and the scalarization method, some new existence theorems for generalized loose (weak) saddle points are established in locally G-convex space. As applications, some vector minimax theorems are given also."

From Text of [47] : After adopting the concept of a generalized convex space or a G-convex space $(X, D; \Gamma)$ from [21], [46] states as follows:

A G-convex space $(X, D; \Gamma)$ is called a locally G-convex space (see [22]) if X is a separated uniform space with the basis β for symmetric entourages, D is a dense subset of X and for each $V \in \beta$ and each $x \in X$, the set $V[x] = \{y \in X : [x, y] \in V\}$ is Γ-convex.

The following result ([46, Lemma 2.2]) is a special case of Theorem 5.2 of Park [22].

Lemma 3.8. *Let $(X; \Gamma)$ be a locally G-convex space and $T : X \to 2^X$ a compact u.s.c. set-valued mapping with closed Γ-convex values. Then T has a fixed point $x_0 \in X$, that is $x_0 \in T(x_0)$.*

This is basic in [46].

Park: IJMMS 24 (2000) [23]

Agarwal and O'Regan (TMNA 21 (2003) [2]) — *From Text*: Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in PK(Z, W)$ if W is convex, and there exists a map $S : Z \to W$ with

$$Z = \bigcup \{\mathrm{Int} S^-(w) : w \in W\}, \quad \mathrm{co}(S(x)) \subset F(x) \text{ for } x \in Z,$$

and $S(x) \neq \emptyset$ for each $x \in Z$, here $S^-(w) = \{z : w \in S(z)\}$.

We recall the following selection theorem (see [23]).

Theorem 3.9. *If Z is paracompact, W is convex, and $F \in PK(Z, W)$. Then there exists a continuous (single valued) function $f : Z \to W$ with $f(x) \in F(x)$ for each $x \in Z$. Moreover, if Z is compact, then $f(Z) \subset \mathrm{co}(A)$ for some finite subset A of W.*

Park: Math. Comput. Modelling 34 (2001) [24]

Farajzadeh (JMAA 322 (2006) [11]) — *Abstract*: "In this paper we, using a particular technique, consider the symmetric vector quasi-equilibrium problems in the Hausdorff topological vector space. As applications of our existence theorem, a coincidence point theorem and the existence of vector optimization problem for a pair of vector-valued mappings are obtained. Moreover, we answer an open question raised by Fu [12]."

Comments: This paper is entirely based on the main result of [24].

Agarwal-O'Regan-Park 2002 — JKMS 39 [5]

Agarwal and O'Regan (TMNA 21 (2003) [2]) — *Abstract*: "This paper presents a continuation theory for $\mathfrak{A}_c^\kappa$ maps. The analysis is elementary and relies on properties of retractions and fixed point theory for selfmaps. Also we present a separate theory for a certain subclass of $\mathfrak{A}_c^\kappa$ maps, namely the PK maps."

From Text of [2]: For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K) = \mathrm{Cov}_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in \mathrm{Cov}(Y)$, F and G are said to be α-close, if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. By a space we mean a Hausdorff topological space. In what follows Q denotes a class of topological spaces.

A space Y is an *extension space* for Q (written $Y \in \mathrm{ES}(Q)$) if for any pair (X, K) in Q with $K \subset X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

A space Y is an *approximate extension space* for Q (and we write $Y \in \mathrm{AES}(Q)$) if for any $\alpha \in \mathrm{Cov}(Y)$, any pair (X, K) in Q with $K \subset X$ closed, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|K$ is α-close to f_0.

Definition 3.10. Let V be a subset of a Hausdorff topological space E. Then we say V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in \mathrm{Cov}_V(K)$, there exists a continuous function (called the *Schauder projection*) $\pi_\alpha : K \to V$ such that

(a) π_α and $i : K \to V$ are α-close,

(b) $\pi_\alpha(K)$ is contained in a subset $C \subset V$ with $C \in$ AES(compact). If $V \in$ AES(compact) then V is trivially Schauder admissible.

If V is an open convex subset of a Hausdorff locally convex topological space E, then it is well known that V is Schauder admissible.

The following fixed point result was established in [5].

Theorem 3.11. *Let V be a Schauder admissible subset of a Hausdorff topological space E and $F \in \mathfrak{A}_c^\kappa(V, V)$ a compact map. Then F has a fixed point.*

Agarwal and O'Regan (AAA 2003 [3]) — *Abstract*: "We establish Birkhoff-Kellogg type theorems on invariant directions for a general class of maps. Our results, in particular, apply to Kakutani, acyclic, O'Neill, approximable, admissible, and $\mathfrak{A}_c^\kappa$ maps."

Comments: The preceding material in [2] from [5] also appear in [3].

Agarwal-O'Regan (Comm. Math. XLIV(1) (2004) [4])—*Abstract*: "New fixed point theory is presented for compact $\mathfrak{A}_c^\kappa(X, X)$ maps where X is an admissible subset of a Hausdorff topological vector spaces."

Comments on [4]: The aim of this paper is to generalize results of [20, 5] and others. The authors defined extension spaces (ES), ES admissible subsets, Borsuk ES admissible subsets, Klee approximable extension spaces (KAES), Borsuk KAES admissible spaces, q-Borsuk KAES admissible subsets, etc. They show that any compact $\mathfrak{A}_c^\kappa(X, X)$ map on these spaces has a fixed point. Finally, they present a continuation theorem for particular types of admissible spaces considered in previous works of Park.

Kim-Kim-Park 2002 — MCM 35 [14]

Agarwal-O'Regan-Precup (TMNA 22 (2003) [6]) — *From Text*: Recently in [14] another type of fixed point result for approximable maps defined on Hausdorff topological vector spaces was presented. One of the main reasons for examining this approximable type map was that the proof of the fixed point result is elementary and just relies on Brouwer's fixed point theorem. For completeness the authors present the proof of the following:

Theorem 3.12 ([14])**.** *Let X be an admissible convex set in a Hausdorff topological vector space E and suppose $F : X \to 2^X$ is a closed compact map with $F \in \mathcal{A}(X, F(X))$. Then F has a fixed point.*

Comment: Theorem 3.12 extends Theorem 3.5.

Park 2002 — NA 51 [25]

Barroso-Kalenda-Reboucas (JMAA 401 (2013) [8]) — *Abstract*: "Let C be a convex subset of a locally convex space. We provide optimal approximate fixed point results for sequentially continuous maps $f : C \to C$. First, we prove that, if $f(C)$ is totally bounded, then it has an approximate fixed point net. Next, it is shown that, if C is bounded but not totally bounded, then there is a uniformly continuous map $f : C \to C$ without approximate fixed point nets. We also exhibit an example of a sequentially continuous map defined on a compact convex set with no approximate fixed point sequence. In contrast, it is observed that every affine (not-necessarily continuous) self-mapping of a bounded convex subset of a topological vector space has an approximate fixed point sequence. Moreover, we construct an affine sequentially continuous map from a compact convex set into itself without fixed points."

From text of [8]:

Theorem 3.13 ([8]). *Let C be a convex subset of a locally convex space X, and let $f : C \to C$ be a sequentially continuous map such that $f(C)$ is totally bounded. Then f has an approximate fixed point net.*

The same result, under the stronger assumption that f is continuous, was proved by Park in [25, Corollary 2.1].

Corollary 3.14 ([8]). *Let C be a totally bounded convex subset of a normed space X. Then every continuous map $f : C \to C$ admits an approximate fixed point sequence.*

This is already a consequence of [25, Corollary 2.1]. However, we find it interesting to formulate it explicitly, as this was not done in [25].

We will use a slight generalization of a result of Fan's KKM Lemma or [42, Lemma]).

Lemma 3.15. *Let C be a subset of a topological vector space $(X, \mathcal{T})$, D a nonempty finite subset of C such that $\operatorname{co} D \subset C$, and $F : D \to 2^C$ a multivalued map with the following two properties.*

(a) *$F(z)$ is sequentially closed in C for all $z \in D$.*

(b) *$\operatorname{co} N \subset \bigcup_{z \in N} F(z)$ for all $N \subset D$.*

Then $\bigcap_{z \in D} F(z) \neq \emptyset$.

Park 2004 — FPTA [26]

Fujimoto (Metroeconomica 64 (2013) [13]) — *Abstract*: "This paper introduces economists to some fixed point theorems for discontinuous mappings with non-convex images on a non-convex domain. These theorems have recently been developed based on a new approach by mathematical economists and mathematicians. The new method of proof is first transformed into a sort of metatheorem, which is then used to obtain a set of necessary and sufficient conditions for a map to have a fixed point. Some fixed point theorems for discontinuous maps are then explained in more concrete cases. The formulations are intended for easier applications towards economic models involving discontinuity as well as non-convexity."

From Text of [13]: The most important element in the method of Urai [44, Corollary 1.1] is the use of an auxiliary map, which is linked to a given map only through the subset of the domain which consists of non-fixed points. An improvement on some of Urai's results has been made by Park [26], using a generalization of the KKM (Knaster-Kuratowski-Mazurkiewicz) theorem. . . . Thus we also present further extensions of some of Urai-Park theorems, allowing for non-convexity of a domain. . . . Now we prove our first theorem, which is a slight generalization of Theorem 1 (the case (K*)) of Urai [45, p.13] and Corollary 3.6 in Park [26, p.153]. . . . We do not have to give up, in the face of these difficulties, the applications of fixed point theorems, at least in proving the existence of a solution, thanks to the recent developments made notably by Urai and Park.

Park 2007 — JMAA 329 [27]

Chen (NA 71 (2009) [10]) — *Abstract*: "In this paper, we first discuss the properties of the almost convex subsets of a Hausdorff topological vector space. Then, in the setting of the almost convex sets, we establish some fixed point theorems for the two KKM* and BD type set-valued mappings with either Ψ-set contraction or 1-set contraction. Next, we also establish new cycle point theorems for three set-valued mappings. Our results generalize the results of Chang et al. [9]] and Park [27]."

Comments: C.-M. Chen [10] incorrectly stated that "in 1996, many authors had introduced the 'better' admissible class $\mathfrak{B}(X, Y)$." In that year Park gave talks in Korea, Poland, Greece, and Taiwan on that class of maps. This class was first published as a generalization of the admissible class $\mathfrak{A}_c^\kappa(X, Y)$ in Park [19]. Since then a number of authors imitated and modified the class. Moreover, Chen stated that his results generalize the results of Park [27] without giving any evidence.

Moreover, in [10], let X, Y and Z be three nonempty sets, and let $T : X \to 2^Y$, $G : Y \to 2^Z$ and $F : Z \to 2^X$ be three set-valued mappings. Its author call (T, G, F) has a cycle point in (X, Y, Z) if there exists $x_0 \in X, y_0 \in Y$ and $z_0 \in Z$ such that $y_0 \in T(x_0)$, $z_0 \in G(y_0)$ and $x_0 \in F(z_0)$. This concept is simply that of coincidence point, for example, of T and FG. Results on such cycle points are given in Section 3 in [10].

Park 2009 — NA 71 [30]

Yang-Xu-Huang (FPTA 2011 [46]) — *Abstract*: "We first prove that the product of a family of $L\Gamma$-spaces is also an $L\Gamma$-space. Then, by using a Himmelberg type fixed point theorem in $L\Gamma$-spaces, we establish existence theorems of solutions for systems of generalized quasivariational inclusion problems, systems of variational equations, and systems of generalized quasiequilibrium problems in $L\Gamma$-spaces. Applications of the existence theorem of solutions for systems of generalized quasiequilibrium problems to optimization problems are given in $L\Gamma$-spaces."

From Text of [46] :The following are basic:

Definition 3.16 ([31])**.** An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{B})$ is an abstract convex space with a basis $\mathcal{B}$ of a uniformity of E.

Definition 3.17 ([31])**.** An abstract convex uniform space $(E \supset D; \Gamma; \mathcal{B})$ is called an $L\Gamma$-space if

(i) D is dense in E, and

(ii) for each $U \in \mathcal{B}$ and each Γ-convex subset $A \subset E$, the set $\{x \in E : A \cap U[x] \neq \emptyset\}$ is Γ-convex.

Lemma 3.18 ([31])**.** *Let $(E \supset D; \Gamma; \mathcal{B})$ be a Hausdorff KKM $L\Gamma$-space and $T : E \multimap E$ a compact u.s.c. map with nonempty closed Γ-convex values. Then, T has a fixed point.*

Lemma 3.19 ([28])**.** *Let $\{(E_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $E := \prod_{i \in I} E_i$ and $D := \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then, $(E, D; \Gamma)$ is an abstract convex space.*

Moreover, nine papers of Park on the KKM theory of abstract convex spaces are quoted in this paper.

Park 2013 — J. Operators [38]

Lu and Hu (JFSA 2013 [15]) — *Abstract*: "The main purpose of this paper is to establish a new collectively fixed point theorem in noncompact abstract convex spaces. As applications of this theorem, we obtain some new existence theorems of equilibria for generalized abstract economies in noncompact abstract convex spaces."

Comments: In [15], the authors quoted 13 papers of Park on fixed point theory and the KKM theory of abstract convex spaces from 1992 [17] to 2013 [39]. They concentrate mainly basic facts on abstract convex spaces due to Park and the multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$. In a sense, this paper can be regarded as an introduction to the works of Park.

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