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A UNIFICATION OF GENERALIZED FAN-BROWDER TYPE ALTERNATIVES

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Abstract. In our previous works, we obtained some characterizations of (partial) KKM spaces. In the present article, we begin with a modification of a such characterization using a general Fan-Browder type fixed point property and show that this characterization implies an alternative theorem. This theorem unifies and contains a number of historically well-known important fixed point or maximal element theorems. We list some of them chronologically and give simple proofs. Finally, we introduce some recent works related to the generalized Fan-Browder type alternatives.

1. Introduction

A partial KKM space is an abstract convex space satisfying an abstract form of the celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem and a KKM space is the one also satisfying the open-valued version of the form. In our previous works [22, 24, 27], we obtained some characterizations of (partial) KKM spaces and one of them is closely related to the Fan-Browder fixed point theorem [9, 4]. Moreover, in our previous work [23] in 2008, KKM theorems or coincidence theorems on abstract convex spaces were applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtained generalizations or improvements of a number of known equilibria results.

In the present paper, we begin with a modification of the characterizations of (partial) KKM spaces using a general Fan-Browder type fixed point property and show that this characterization unifies and implies an alternative theorem containing a number of historically well-known important fixed point or maximal element theorems. We list some of them chronologically and give simple proofs.

In Section 2, definitions and some basic facts on abstract convex spaces are introduced. Section 3 deals with a characterization of the (partial) KKM spaces with its application to very general forms of a fixed point theorem and an alternative theorem of the Fan-Browder type. Section 4 deals with various existence theorems on fixed points or maximal elements given by Fan, Browder, Borglin and Keiding, Yannelis and Prabhakar, Lassonde, Chang, Horvath, and others for various types of abstract convex spaces. These are all simple consequences of our alternative or fixed point theorem and we give simple proofs of them. Finally, in Section 5, we introduce some related works and state historical comments on them.

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2. Abstract convex spaces

Recall the following in [27] and the references therein.

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Now the following diagram for triples $(E, D; \Gamma)$ is well-known:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space

$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. A generalized Fan-Browder fixed point theorem

In our previous works [22, 24, 27], we obtained some characterizations of (partial) KKM spaces and one of them is closely related to the Fan-Browder fixed point theorem [9, 4]. We need the following for an abstract convex space $(E, D; \Gamma)$:

The Fan-Browder fixed point property. *For any maps $S : E \multimap D$ and $T : E \multimap E$ satisfying*

(1) *$S^-(y)$ is open (resp., closed) for each $y \in D$;*
(2) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*
(3) *$E = S^-(M) = \bigcup_{y\in M} S^-(y)$ for some $M \in \langle D\rangle$,*

T has a point $\bar{x} \in E$ such that $\bar{x} \in T(\bar{x})$.

The (partial) KKM principle is equivalent to the preceding property as follows:

Theorem I. *An abstract convex space $(E, D; \Gamma)$ is a partial KKM space iff the open version of the Fan-Browder fixed point property holds.*

A partial KKM space $(E, D; \Gamma)$ is a KKM space iff the closed version of the Fan-Browder fixed point property holds.

Proof. Necessity. Let $(E, D; \Gamma)$ be a partial KKM space [resp., KKM space] and $S : E \multimap D$, $T : E \multimap E$ be maps satisfying (1)-(3). Suppose that $x \notin T(x)$ for all $x \in E$. Define a closed-valued [resp., open-valued] map $F : D \multimap E$ by $F(y) := E \setminus S^-(y)$ for each $y \in D$. Since $E = S^-(M)$ and

$$\bigcap_{y \in M} F(y) = \bigcap_{y \in M} (E \setminus S^-(y)) = E \setminus \bigcup_{y \in M} S^-(y) = \emptyset,$$

$\{F(y)\}_{y \in D}$ does not have the finite intersection property. Hence F is not a KKM map. Therefore, $\Gamma(N) \not\subset F(N)$ for some $N \in \langle D \rangle$. Let $x_0 \in \Gamma(N) \setminus F(N)$. Then $x_0 \in \mathrm{co}_\Gamma N$ and $x_0 \notin F(N) = \bigcup_{y \in N}(E \setminus S^-(y))$. Hence $x_0 \in S^-(y)$ or $y \in S(x_0)$ for all $y \in N$, that is, $N \subset S(x_0)$. Then

$$x_0 \in \mathrm{co}_\Gamma(N) \subset \mathrm{co}_\Gamma S(x_0) \subset T(x_0).$$

This contradicts the non-existence of fixed points of T.

Sufficiency. Suppose that $(E, D; \Gamma)$ is not a partial KKM space [resp., KKM space]. Then there exists a closed-valued [resp., open-valued] KKM map $F : D \multimap E$ such that $\{F(y)\}_{y \in D}$ does not have the finite intersection property, that is, $\bigcap_{y \in M} F(y) = \emptyset$ for some $M \in \langle D \rangle$. Define $S : E \multimap D$ and $T : E \multimap E$ such that $S^-(y) = E \setminus F(y)$ for $y \in D$ and $T(x) = \mathrm{co}_\Gamma S(x)$ for $x \in E$. Then the requirements (1) and (2) are satisfied. Moreover,

$$E = E \setminus \bigcap_{y \in M} F(y) = \bigcup_{y \in M} (E \setminus F(y)) = \bigcup_{y \in M} S^-(y) = S^-(M)$$

and hence the requirement (3) also holds. Therefore, there exists $x_0 \in E$ such that $x_0 \in \mathrm{co}_\Gamma S(x_0) = T(x_0)$, that is, there exists $A \in \langle S(x_0) \rangle$ such that $x_0 \in \Gamma(A) \subset \mathrm{co}_\Gamma S(x_0)$. Since F is a KKM map, we have $x_0 \in \Gamma(A) \subset F(A)$. Hence $x_0 \in F(y) = E \setminus S^-(y)$ for some $y \in A$, that is, $x_0 \notin S^-(y)$ or $y \notin S(x_0)$. This contradicts $y \in A \subset S(x_0)$. □

Recall that Theorem I is a slightly modified form of one of the characterizations of (partial) KKM spaces given in [22, 24, 27] with a new proof.

From Theorem I, we deduce the following slightly modified form of [23, Theorem 4.3]:

Theorem II. *Let $(E, D; \Gamma)$ be a partial KKM space [resp., a KKM space], and $S : E \multimap D$, $T : E \multimap E$ maps. Suppose that*

(1) *S^- is open-valued [resp., closed-valued];*
(2) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
(3) *there exists a nonempty subset K of E such that $K \subset S^-(N)$ for some $N \in \langle D \rangle$; and*

(4) *either*
   (i) $E \setminus K \subset S^-(M)$ *for some* $M \in \langle D \rangle$*; or*
   (ii) *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and* $L_N \setminus K \subset S^-(M)$ *for some* $M \in \langle D' \rangle$*.*

Then there exists $\bar{x} \in E$ *such that* $\bar{x} \in T(\bar{x})$*.*

Proof. Case (i): Since

$$E = K \cup (E \setminus K) = S^-(N) \cup S^-(M) = S^-(M \cup N),$$

all of the requirements of the Fan-Browder fixed point property are satisfied.

Case (ii): It is well-known that $(L_N, D'; \Gamma|_{\langle D' \rangle})$ is a (partial) KKM space as a subspace of the (partial) KKM space $(E, D; \Gamma)$, respectively. Since

$$L_N \subset K \cup (L_N \setminus K) \subset S^-(N) \cup S^-(M) = S^-(M \cup N),$$

$(L_N, D'; \Gamma|_{\langle D' \rangle})$ satisfies all of the requirements of Theorem I with $(S|_{L_N}, T|_{L_N}, D')$ instead of (S, T, D).

Then, for $(L_N, D'; \Gamma|_{\langle D' \rangle})$, all of the requirements of the Fan-Browder fixed point property holds. Hence the conclusion follows from Theorem I. □

Remark. 1. Condition (i) or (ii) is called a *coercivity or compactness* condition and aims to replace the compact case of E.

2. The closed version of the Fan-Browder fixed point property is not popular. However, the open version is very useful as the subsequent results shows.

From Theorem II, we have the following alternative:

Theorem III. *Let* $(E, D; \Gamma)$ *be a partial KKM space, and* $S : E \multimap D$*,* $T : E \multimap E$ *maps. Suppose that*

(1) S^- *is open-valued;*
(2) *for each* $x \in E$*,* $\mathrm{co}_\Gamma S(x) \subset T(x)$*;*
(3) *there exists a compact subset* K *of* E*; and*
(4) *either*
   (i) $E \setminus K \subset S^-(M)$ *for some* $M \in \langle D \rangle$*; or*
   (ii) *for each* $N \in \langle D \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *containing* N *such that*

$$L_N \setminus K \subset \bigcup_{y \in D'} S^-(y).$$

Then either (a) *there exists* x_0 *in* E *such that* $S(x_0) = \emptyset$*; or* (b) *there exists* x_1 *in* X *such that* $x_1 \in T(x_1)$*.*

Proof. Suppose that the conclusion (a) does not hold, that is, $S(x) \neq \emptyset$ for all $x \in E$. Then $y \in S(x)$ or $x \in S^-(y)$ for some $y \in D$. Then $E = S^-(D)$. Since K is compact and each $S^-(y)$ is open and nonempty, $K \subset S^-(N)$ for some $N \in \langle D \rangle$.

Case (i): Since $E \setminus K \subset S^-(M)$ for some $M \in \langle D \rangle$, the conclusion (b) follows from Theorem II(i).

Case (ii): Since there exists a compact subset L_N of E which is Γ-convex relative to some $D' \subset D$ containing N such that $L_N \setminus K \subset S^-(D')$. Therefore

$$L_N = K \cup (L_N \setminus K) \subset S^-(N) \cup S^-(D').$$

Since $K \subset S^-(N)$ and L_N is compact, $L_N \setminus K \subset S^-(M)$ for some $M \in \langle D' \rangle$. Now, by Theorem II(ii), the conclusion (b) follows. □

Remark. 1. Note that (a) tells that S has a *maximal element* and (b) tells that T has a *fixed point*.

2. Theorem III improves [23, Theorem 4.4].

4. Particular forms

In this section, we list some (not all) historically well-known particular forms of Theorem II or III for various types of partial KKM spaces and some comments on their applications.

4.1. **Fan's geometric property 1961** [9]**.** A milestone on the history of the KKM theory was erected by Ky Fan [9]. He extended the KKM theorem to infinite dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space. Fan also obtained the following geometric or section property of convex sets, which is equivalent to his KKM lemma.

Theorem 4.1 (Fan [9])**.** *] Let X be a compact convex set in a Hausdorff topological vector space. Let A be a closed subset of $X \times X$ with the following properties:*

(i) $(x, x) \in A$ *for every* $x \in X$.

(ii) *For any fixed* $y \in X$*, the set* $\{x \in X : (x, y) \notin A\}$ *is convex (or empty).*

Then there exists a point $y_0 \in X$ *such that* $X \times \{y_0\} \subset A$.

We give the following proof:

Proof (Theorem II(i) $\Longrightarrow$ Theorem 4.1). Let $E = D = K = X$ in Theorem II(i) and let $S = T : X \to X$ be defined by $S(x) = \{y \in X : (x, y) \notin A\}$ for $x \in X$. Then $S(x)$ is convex (or empty). Moreover, $S^-(y) = \{x \in X : (x, y) \notin A\}$ is open for $y \in X$.

Suppose that $S(x) \neq \emptyset$ for all $x \in X$ contrary to the conclusion. Then there exists $y \in X$ such that $y \in S(x)$ and hence $X \subset S^-(X)$. Since X is compact and each $S^-(y)$ is open, there exists $N \in \langle X \rangle$ such that $X \subset S^-(N)$.

Therefore, by Theorem II(i) with $E = K$, there exists an $\overline{x} \in X$ such that $\overline{x} \in S(\overline{x})$, that is, $(\overline{x}, \overline{x}) \notin A$. This contradicts condition (i) of Theorem 4.1. □

Fan applied this result to give a simple proof [9] of the Tychonoff theorem and to prove two results generalizing the Pontrjagin-Iohvidov-Kreĭn theorem on existence of invariant subspaces of certain linear operators. Also, Fan [10] applied his KKMF lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff theorem. Moreover, "a theorem concerning sets with convex sections" was applied to prove various results in Fan [11]. Note that Hausdorffness is redundant in Theorem 4.1.

4.2. **Browder 1968** [4]**.** In 1968 Browder [4] independently obtained Fan's geometric lemma [9] in the convenient form of a fixed point theorem. Since then the following is known as the Fan-Browder fixed point theorem:

Theorem 4.2 (Browder [4])**.** *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^{-}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later this is also known to be equivalent to the Brouwer fixed point theorem. Browder [4] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding [3] and Yannelis and Prabhakar [38], to the existence of maximal elements in mathematical economics. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to [20, 30] and the references therein.

Browder proved his theorem by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem.

Later the Hausdorffness in Fan's KKM lemma, geometric property, and Browder's theorem was known to be redundant by Lassonde in 1983 [17]. Moreover, the Fan property and Browder's theorem are known to be equivalent to the KKM theorem. Consequently Browder's theorem can be obtained by a simple KKM method.

4.3. **Borgin and Keiding 1976** [3]**.** In 1976 Borgin and Keiding [3] restated Fan's geometric property [9, 11] as follows:

Let X be a non-empty, convex, compact subset of some (Hausdorff) topological vector space V. A correspondence $\varphi : X \multimap X$ is *KF* if φ is convex-valued (possibly empty-valued), graph of φ is open in $X \times X$ and $x \notin \varphi(x)$ for $x \in X$. One has the following:

Theorem 4.3 (Ky Fan)**.** *Let $\varphi : X \multimap X$ be KF then there is $\overline{x} \in X$ such that $\varphi(\overline{x}) = \emptyset$.*

In [3] a corollary to this theorem is obtained which on the one hand allows a slight generalization of the results of Gale and Mas-Cole11 (1975) on a competitive equilibrium and Shafer and Sonnenschein (1975) in the same direction and their (1974) 'the Arrow-Debreu lemma for abstract economies' on the other hand makes it possible to prove them in a rather different way. The general idea is to reduce the search for an equilibrium to the search for an equilibrium action for a suitably chosen fictitious agent.

4.4. **Yannelis and Prabhakar 1983** [38]**.** In 1983 Yannelis and Prabhakar [38] presented some mathematical theorems including Browder's theorem which are used to generalize previous results on the existence of maximal elements and of equilibrium. Their main theorem in [38] is a new existence proof for an equilibrium in an abstract economy, which is closely related to a previous result of Borglin-Keiding and Shafer-Sonneschein, but allows for an infinite number of commodities and a countably infinite number of agents.

Theorems 4.2, 4.3 and the main result of [38] can be modified to the following *Fan-Browder alternative*:

Theorem 4.4. *Let K be a nonempty compact convex subset of a topological vector space. Let $T : K \multimap K$ be a map, where for each $x \in K$, $T(x)$ is a convex subset of K. Suppose further that for each y in K, $T^-(y)$ is open in K.*

Then either (a) *there exists x_0 in K such that $x_0 \in T(x_0)$; or* (b) *there exists y_0 in K such that $T(y_0) = \emptyset$.*

Proof (Theorem III(i) $\Longrightarrow$ Theorem 4.4). Let $E = D = K$ and consider $S = T$ in Theorem III(i). Suppose that the conclusion (b) does not hold, that is, $T(x) \neq \emptyset$ for all $x \in K$. Then $y \in T(x)$ or $x \in T^-(y)$ for some $y \in K$, and hence $K = T^-(K)$. Since K is compact and each $T^-(y)$ is open and nonempty, $K = T^-(N)$ for some $N \in \langle K \rangle$. Moreover each $T(x)$ is convex. Therefore, by Theorem III(i), (a) T^- has a fixed point. □

4.5. **Fan 1979** [12]**.** In 1979, Fan obtained a theorem [12, Theorem 10] which is equivalent to the following:

Theorem 4.5. *Let X be a nonempty convex set in a Hausdorff topological vector space. Let $T : X \multimap X$ be a map such that*

(1) *for each $x \in X$, $T(x)$ is convex,*
(2) *for each $y \in X$, $T^-(y)$ is open, and*
(3) *there is a nonempty compact convex set $L \subset X$ such that $L \cap T(x) \neq \emptyset$ for every $x \in X \setminus L$.*

Then either (a) *there exists x_0 in X such that $T(x_0) = \emptyset$; or* (b) *there exists x_1 in X such that $x_1 \in T(x_1)$.*

Remark. Note that (a) implies T^- is not surjective.

The following is due to Ben-El-Mechaiekh et al. [2, Theorem 5]:

Theorem 4.6. *In Theorem 4.5, replace (3) by the following:*

(3)′ *there exist a compact subset K of X and a compact convex subset L of X such that $L \cap T(x) \neq \emptyset$ for every $x \in X \setminus K$.*

Then the same conclusion holds.

Note that for $K = L$, Theorem 4.6 reduces to Theorem 4.5.

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.6). Let $E = D = X$ and consider $S = T$ in Theorem III(ii). Then each $T^-(y)$ is open and each $T(x)$ is convex. Suppose that the conclusion (a) does not hold, that is, T^- is surjective and hence $K \subset T^-(X)$. Since K is compact and each $T^-(y)$ is open, $K \subset T^-(N)$ for some $N \in \langle X \rangle$.

Since L is a compact convex subset of a Hausdorff t.v.s., there is a compact convex subset L_N of X containing L and N (see [17]). Moreover, condition (3)′ of Theorem 4.6 implies the existence of a $y \in L \cap T(x) \subset L_N \cap T(x)$ for each $x \in L_N \setminus K \subset X \setminus K$. Hence

$$x \in T^-(y) \;\Rightarrow\; L_N \setminus K \subset T^-(L) \;\Rightarrow\; L_N = K \cup (L_N \setminus K) \subset T^-(N) \cup T^-(L).$$

Since L_N is compact, $L_N \setminus K \subset T^-(M)$ for some $M \in \langle L \rangle \subset \langle L_N \rangle$ with $D' = L_N$. Now, by Theorem III(ii), the conclusion (b) follows. □

The "coercivity" or "compactness" condition (3) is first appeared in [12]. Note that the Hausdorffness in Theorems 4.5 and 4.6 is essential.

4.6. **Lassonde 1983** [17]**.** The concept of convex sets in a topological vector space is extended to convex spaces by Lassonde in 1983 [17], and further to c-spaces by Horvath in 1983-93 [13, 14, 15, 16]. A number of other authors also extended the concept of convexity for various purposes.

Definition 4.7. Let X be a subset of a vector space and D be a nonempty subset of X. We call (X, D) a *convex space* if $\operatorname{co} D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$ (see Park [19]). For a convex space (X, D), a subset C of X is said to be D*-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\operatorname{co} A \subset C$.

A nonempty subset L of a convex space X is called a *c-compact* set [17] if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [17].

Lassonde [17] presented a simple and unified treatment of a large variety of minimax and fixed point problems. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem.

The following is a variant of [17, Theorem I.1.1]:

Theorem 4.8. *Let (X, D) be a convex space and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*
(2) *$T(x)$ is convex for each $x \in X$,*
(3) *$S^-(y)$ is open for each $y \in D$, and*
(4) *for some c-compact set $L \subset X$, $X \setminus S^-(L \cap D)$ is compact.*

Then either (a) *there exists x_0 in X such that $S(x_0) = \emptyset$; or* (b) *there exists x_1 in X such that $x_1 \in T(x_1)$.*

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.8). Let $E = X$ in Theorem III(ii). Then S^- is open-valued and T is convex-valued. Note that condition (1) of Theorem 4.8 simply tells that $S(x) \subset T(x)$ for all $x \in X$, and implies condition (2) of Theorem III. Suppose that the conclusion (a) does not hold, that is, S^- is surjective and hence $X = S^-(D)$. Let $K = X \setminus S^-(L \cap D) \subset S^-(D)$. Since K is compact and each $S^-(y)$ is open, $K \subset S^-(N)$ for some $N \in \langle D \rangle$. Note that $X \setminus K \subset S^-(L \cap D)$ by (4).

Since L is a c-compact subset of X, there is a compact convex subset L_N of X containing L and N. Let $D' = L_N \cap D$. Since

$$L_N = K \cup (L_N \setminus K) \subset K \cup (X \setminus K) \subset S^-(N) \cup S^-(L_N \cap D).$$

Since $K \subset S^-(N)$ and L_N is compact, $L_N \setminus K \subset S^-(M)$ for some $M \in \langle L_N \cap D \rangle = \langle D' \rangle$. Now, by Theorem III(ii), the conclusion (b) follows. □

4.7. **Chang type 1989** [5]**.** In 1989, S.Y. Chang [5] obtained a KKM theorem with a coercivity condition which eliminated the concept of c-compact sets. From a Fan-Browder type fixed point theorem equivalent to her KKM theorem, we notice that the following modification of [18, Theorem 5] holds:

Theorem 4.9. *Let (X, D) be a convex space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*
(2) *$T(x)$ is convex for each $x \in X$,*
(3) *$S^-(y)$ is open for each $y \in D$, and*
(4) *for each finite subset N of D, there exists a compact convex subset L_N of X containing N such that*

$$L_N \setminus K \subset S^-(L_N \cap D).$$

Then either (a) *S has a maximal element, or* (b) *T has a fixed point.*

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.9). Let $E = X$ in Theorem III(ii). Then S^- is open-valued and T is convex-valued. Suppose that the conclusion (a) does not hold, that is, S^- is surjective and hence $X = S^-(D)$. Since K is compact and each $S^-(y)$ is open, $K \subset S^-(N)$ for some $N \in \langle D \rangle$. Then by (4), there exists a compact convex subset L_N of X containing N such that $L_N \setminus K \subset S^-(L_N \cap D)$. Let $D' = L_N \cap D$. Note that

$$L_N = K \cup (L_N \setminus K) \subset S^-(N) \cup S^-(L_N \cap D).$$

Since $K \subset S^-(N)$ and L_N is compact, $L_N \setminus K \subset S^-(M)$ for some $M \in \langle L_N \cap D \rangle = \langle D' \rangle$. Now, by Theorem III(ii), the conclusion (b) follows. □

4.8. **Horvath type 1987-1993** [13, 14, 15, 16]**.** In this subsection, we follow [18].

Definition 4.10. A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

For an H-space $(X, D; \Gamma)$, a subset C of X is said to be H-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

A multimap $F : D \multimap X$ is said to be H-KKM if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$. A subset L of X is called an H-*subspace* of $(X, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D \rangle$, $\Gamma_A \cap L$ is contractible.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [13, 14, 15, 16] or an H-space by Bardaro and Ceppitelli [1]. Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces, and are examples of compact c-spaces without having the fixed point property.

In the frame of H-spaces, there have been appeared several Fan-Browder type fixed point theorems. The following is motivated from [18].

Theorem 4.11. *Let $(X, D; \Gamma)$ be an H-space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*

(2) *$T(x)$ is H-convex for each $x \in X$,*
(3) *$S^-(y)$ is open for each $y \in D$, and*
(4) *for each finite subset N of D, there exists a compact H-convex subset L_N of X containing N such that*

$$L_N \setminus K \subset S^-(L_N \cap D).$$

Then either (i) *S has a maximal element, or* (ii) *T has a fixed point.*

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.11). Recall that H-spaces are partial KKM spaces. Let $E = X$ in Theorem III(ii). Note that S^- is open-valued and T is convex-valued. Suppose that the conclusion (a) does not hold, that is, S^- is surjective and $X = S^-(D)$. Since K is compact and each $S^-(y)$ is open, $K \subset S^-(N)$ for some $N \in \langle D \rangle$. Then there exists a compact H-convex subset L_N of X containing N such that $L_N \setminus K \subset S^-(L_N \cap D)$. Let $D' = L_N \cap D$. Since

$$L_N \subset K \cup (L_N \setminus K) \subset S^-(N) \cup S^-(L_N \cap D).$$

Since $K \subset S^-(N)$ and L_N is compact, $L_N \setminus K \subset S^-(M)$ for some $M \in \langle L_N \cap D \rangle = \langle D' \rangle$. Now, by Theorem III(ii), the conclusion (b) follows. □

4.9. **For G-convex spaces 1993-2006** [34, 35, 36]**.** From 1993, the generalized convex spaces became one of the main themes of the KKM theory [34, 35, 36]. Later we adopted the following definition:

Definition 4.12. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \mathrm{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$.

For a G-convex space $(X, D; \Gamma)$, a subset C of X is said to be G-*convex* w.r.t. some $D' \subset D$ if for each $A \in \langle D' \rangle$, we have $\Gamma_A \subset C$.

A map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

There are lots of examples of G-convex spaces; see [34, 35] and the references therein. So, the KKM theory was extended to the study of KKM maps on G-convex spaces.

The following is motivated by several works on G-convex spaces [35, 36].

Theorem 4.13. *Let $(X, D; \Gamma)$ be a G-convex space, $D \subset X$, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*
(2) *$T(x)$ is G-convex w.r.t. $S(x)$ for each $x \in X$,*
(3) *$S^-(y)$ is open for each $y \in D$, and*
(4) *for each finite subset N of D, there exists a compact G-convex subset L_N of X w.r.t. some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \setminus K \subset S^-(D').$$

Then either (i) S *has a maximal element, or* (ii) T *has a fixed point.*

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.13). A similar proof to that of Theorem 4.11 works. □

4.10. **For ϕ_A-spaces 2007** [21, 23, 25, 26]**.** Since 2007 the following became one of the main themes of the KKM theory [21, 23, 25, 26].

Definition 4.14. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

For a ϕ_A-space $(X, D; \phi_A)$, a subset C of X is said to be ϕ_A-*convex* with respect to some $D' \subset D$ if for each $A \in \langle D'\rangle$, we have $\phi_A(\Delta_{|A|-1}) \subset C$.

We define a KKM map $G : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ if, for each $N \in \langle D\rangle$ and $J \subset N$, we have

$$\phi_N(\Delta_{|J|-1}) \subset G(J)$$

where $\Delta_{|J|-1}$ is the face of $\Delta_{|N|-1}$ corresponding to J.

There are lots of examples of ϕ_A-spaces; see [21, 23, 25, 26] and the references therein. So, the KKM theory was extended to the study of KKM maps on ϕ_A-spaces.

The following Fan-Browder type alternative theorem for ϕ_A-spaces is new:

Theorem 4.15. *Let* $(X, D; \phi_A)$ *be a* ϕ_A*-space,* $D \subset X$*,* K *a compact subset of* X*, and* $S : X \multimap D$*,* $T : X \multimap X$ *maps satisfying the conditions*

(1) $S^-(y) \subset T^-(y)$ *for each* $y \in D$*,*
(2) $T(x)$ *is* ϕ_A*-convex w.r.t.* $S(x)$ *for each* $x \in X$*,*
(3) $S^-(y)$ *is open for each* $y \in D$*, and*
(4) *for each finite subset* N *of* D*, there exists a compact* ϕ_A*-convex subset* L_N *of* X *w.r.t. some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \setminus K \subset S^-(D').$$

Then either (*a*) S *has a maximal element or* (*b*) T *has a fixed point.*

Proof (Theorem III(ii) $\Longrightarrow$ Theorem 4.15). Recall that, by putting $\Gamma_A = \phi_A(\Delta_{|A|-1})$ for each $A \in \langle D\rangle$, any ϕ_A-space becomes a KKM space. Therefore Theorem III(ii) works for any ϕ_A-space. □

In our previous work [29] in 2012, we showed that hyperbolic spaces are ϕ_A-spaces and hence, most of main theorems in the KKM theory on abstract convex spaces can be applied to hyperbolic spaces (and their particular form of Hadamard manifolds).

In the same year, Yang and Pu [37] obtained a very particular form of Theorem 4.15 for particular types of Hadamard manifolds and applied it to obtain a maximal element theorem, a section theorem, a Ky Fan type minimax inequality, and a Nash type equilibrium theorem. All of these results were already noted to hold on arbitrary partial KKM spaces in 2010 [27].

4.11. **Du 2012** [8]**.** Recently Y.-M. Du [8] obtained generalizations of results in [6, 7] to abstract convex spaces in the sense of Park. Du obtained the following particular form of our Theorem III as [8, Theorem 3.1]:

Theorem 4.16. *Let $(E, D; \Gamma)$ be a partial KKM space, and K be a nonempty compact subset of E. Suppose that $F : E \to 2^D,\ G : E \to 2^E$ be mappings such that*
(i) *$F(x) \subset G(x)$ for each $x \in E$;*
(ii) *for each $y \in D,\ F^-(y)$ is compactly open in E and for each $x \in K,\ F(x) \neq \emptyset$;*
(iii) *for each $N \in \langle D\rangle$, there exists a compact abstract convex subset L_N such that*

$$L_N \setminus K \subset \bigcup\{\operatorname{cint} G^-(y) \mid y \in L_N\}.$$

Then, there exists a point $\hat{x} \in E$, such that $\hat{x} \in \operatorname{co} G(\hat{x})$.

Remark. 1. Note that it should be assumed $E \supset D$ here. Moreover obsolete terminology like compactly open sets and cint should be destroyed.

2. [8, Theorem 3.1] generalizes Theorem 3.1 of Ding and Feng [6] and Theorem 3.1 of Ding and Wang [7] from FC-spaces to abstract convex spaces, and the coercivity condition (iii) is weaker than the condition (3) in Theorem 3.1 of Ding and Wang [7].

5. Other related works

In the present paper, we listed only twelve results that can be deduced from Theorem I. However there are many general forms of the Fan-Browder type theorems and their applications. In this section, we introduce several recent works of ours related to the Fan-Browder type theorems.

(1) In our previous work [23], KKM theorems or coincidence theorems on abstract convex spaces were applied to the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtained generalizations or improvements of a number of known equilibria results, especially, in a work of Ding and Wang [7] on the so-called FC-spaces. Since then Ding and Feng [6] obtained similar results to [7].

(2) In [28], we stated: Since we introduced the KKM theory in 1992, there have appeared more than twelve hundred publications related to the theory. Many of them are concerned with KKM type theorems on particular spaces, their equivalent formulations, and their applications to various problems. Recently the KKM theory tends to the study on abstract convex spaces properly including such particular spaces. In [28], we gave a short history of the theory and review the current study through our previous comments or surveys.

(3) In [30], from a general form of the KKM type theorems or some properties of KKM type maps on abstract convex spaces, we deduce several Fan-Browder type alternatives, coincidence or fixed point theorems, and other results. These theorems unify and generalize various particular results of the same kinds recently due to a number of authors for particular types of abstract convex spaces.

In [30], Section 2 is a preliminary on our abstract convex spaces with one of the most general KKM theorems. In Section 3, we obtain a Fan-Browder type alternative or coincidence theorem and also give another versions of the theorem. Section 4 deals with new particular forms of them on G-convex spaces. In Section 5, we investigate the contents appeared in related papers on topological vector spaces (t.v.s.). Section 6 deals with related papers on G-convex spaces. In section 7, particular maximal element theorems and their applications. We show that basic theorems of the papers mentioned in Sections 5-7 are consequences of our new results. Finally, in Section 8, we add some historical remarks and further comments to improve many of the known results and their applications.

(4) Corresponding to each stage of development of the KKM theory, the Fan-Browder fixed point theorem on Fan-Browder type multimaps has been generalized to hundreds of different forms or reformulated to the maximal element theorem with numerous generalizations. Therefore the theorem can be stated as an alternative form; that is, its conclusion is "the Fan-Browder map has either a fixed point or a maximal element. Our aim in [31] is to trace the evolution of the Fan-Browder type alternatives from the origin to the most recent generalization of them.

(5) In our earlier foundational works on the KKM theory, we were based on several KKM type theorems or the Fan-Browder type coincidence theorems. Recently, we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In [31], we obtain a new coincidence theorem (Theorem D) and recollect that several particular forms of Theorems A-D were applied to establish our earlier foundational works for each of convex spaces, H-spaces, G-convex spaces, and abstract convex spaces.

(6) In our previous review [28], we gave a short history of the KKM theory and reviewed its current study by recalling our previous comments or surveys in a sequence of papers. Its continuation [33] is to review some recent works on the theory mainly due to other authors. On this occasion, we gave some corrections on [27].

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