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REMARKS ON FIXED POINT AND GENERALIZED VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In the first half of this paper, we introduce the contents of some of our previous papers on fixed point problems related to the Schauder conjecture. Some of them contain incorrect statements. The second half devotes to improve or correct the results in certain papers of other authors based on one of our incorrect statements related to a fixed point conjecture. These results are mainly concerned with generalized vector equilibrium problems.

1. Introduction

One of the most important unsolved problems in the analytical fixed point theory (other than metric or topological fixed point theories) is the Schauder conjecture raised in 1935. Since the appearance of the incorrect proof of the conjecture by Cauty in 2001, several papers related to the problem have appeared by a number of authors. Actually, the present author also had published on some topics related to the conjecture. Some of such papers contain incorrect statements. Influenced by one of our incorrect statements, some authors have published another papers applying it. Therefore, we should clarify the situations and try to improve or correct such statements of ours or other authors.

In the first half of this paper, we introduce the contents of some of our previous papers related to fixed point problems. Some of them include incorrect statements; see [7-12, 14, 15]. The second half devotes to improve or correct the results in other authors' papers [1, 5, 6] based on one of our incorrect statements related to fixed points. These results are mainly concerned with generalized vector equilibrium problems.

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Key words and phrases: Schauder conjecture, fixed point, generalized equilibrium problem.

2. Our previous papers related to the Schauder conjecture

In this section, we introduce the contents of some of our previous papers related to fixed point problems. Some of them include incorrect statements.

2.1. Park: NACA 2001 [7]

Abstract: We discuss the current state of research related to the Schauder conjecture and other problems in analytical fixed point theory. We introduce then various results and conjectures closely related to fixed points of compact Browder maps. An example is as follows: Let X be a nonempty convex subset of a topological vector space (t.v.s.) and $T : X \multimap X$ a multimap with nonempty convex values $T(x)$ for $x \in X$ and open [resp. closed] fibers $T^{-}(y)$ for $y \in X$. If $T(X)$ is covered by a finite number of fibers of T, then either T has a fixed point or T^{-} has a maximal element. Moreover, we obtain several generalizations of the Browder fixed point theorem.

2.2. Park: RIMS 2004 [8]

Abstract: We discuss the current state of research related to the Schauder conjecture and other problems in analytical fixed point theory. We revise and update the contents of the previous version [7].

From the text: The following famous long-standing conjecture is known to be the compact AR problem:

Conjecture 2. *Every compact convex subset X of a metrizable t.v.s. is an AR.*

This is also resolved affirmatively [10].

Comments: This last statement is incorrect. Actually the proofs in the following two papers [9, 10] are incorrect.

2.3. Park: FPTA 2003 [9]

Abstract: We give affirmative solutions to the compact AR problem and the Banach problem of whether every infinite dimensional compact convex subset X of a metrizable t.v.s. is homeomorphic to the Hilbert cube. Some related results are also discussed.

2.4. Park: NACA 2003 [10]

Abstract: We discuss current state of research related to some well-known problems in infinite dimensional topology—the Schauder conjecture, the compact AR problem, and the Banach problem of whether every infinite dimensional compact convex subset X of a metrizable t.v.s. is homeomorphic to the Hilbert cube. We give affirmative answers to the second and third problems. Moreover, for every such X, we show that a closed multimap $F \in \mathfrak{B}(X, X)$ has a fixed point, where $\mathfrak{B}$ denotes the class of the better admissible multimaps. Our new results are applied to Roberts spaces. Some related problems are also discussed.

2.5. Park: IJM 2004 [11]

Abstract: Based on a fixed point theorem for the multimap class $\mathfrak{B}$, we generalize or correct our previous results on homeomorphically convex sets, on openness of multimaps, and on the Birkhoff-Kellogg type theorems.

2.6. Park: NA-TMA 2005 [12]

Abstract: We survey recent results in analytical fixed point theory. Firstly, we state resolutions of long-standing problems in infinite dimensional topology—the Schauder conjecture, the compact AR problem, and the Banach problem on the Hilbert cube. Secondly, we list some fixed point theorems on Kakutani maps, generalized upper hemicontinuous maps, Fan-Browder maps, approximable maps, acyclic maps, and admissible or better admissible class $\mathfrak{B}$ of maps. Some related results are also given.

Comments: Recall that a *Kakutani map* is an upper semicontinuous (u.s.c.) multimap with nonempty closed convex values. The following was claimed in this paper, but withdrawn later. Hence it would be better to call it a conjecture as follows:

Conjecture (Dobrowolski [2]). *Let X be a convex subset of a metrizable t.v.s. Then every compact Kakutani map $T : X \multimap X$ has a fixed point.*

2.7. Park: NACA 2007 [14]

Abstract: We show that some fixed point theorems and related results in our previous works [8-12] need additional requirements for their validities. Some of the new correct results will appear in [13].

From the text: Based on the works of Cauty and Dobrowolski, we claimed or announced in [8-12] to obtain some fixed point theorems on multimaps (maps) and related results. Later, it is known that there are some gaps in the proofs in the works of Cauty and Dobrowolski, and so some of our results might be groundless. Our aim in this paper is to clarify this and to give additional requirements which guarantee the validity of each of such results.

We add some new related results in our forthcoming work [13].

(II) [2] *A compact convex subset of a metrizable t.v.s. has the simplicial approximation property.*

It is also known that there is a gap in the proof of Statement (II).

From a statement of Dobrowolski, we deduced the following in [9]:

(III) *A compact convex subset of a metrizable t.v.s. is admissible.*

From Statement (III), the author [9, 10] deduced the affirmative resolutions to the compact AR problem and the Banach problem as follows:

(IV) *A compact convex subset of a metrizable t.v.s. is an AR.*

(V) *An infinite dimensional compact convex subset of a metrizable t.v.s. is homeomorphic to the Hilbert cube Q.*

Since Statements (III)-(V) are based on Statement (II), their validities are not sufficiently justified. In view of Statements (IV) and (V), we claimed in [9, 10] that every *Roberts spaces* – that is, compact convex sets with no extreme points constructed by Roberts' method of needle point spaces – are AR and homeomorphic to the Hilbert cube Q.

Remarks. 1. Tavernise and Trombetta [18] pointed out that "it is not so clear if the Schauder's problem has been solved in its generality. In particular, the original proof given by Cauty contains some unsolved gaps." And they suggested to see three references and Zentralblatt Math review of [14].

2. Cauty has tried to renew or correct his original proof in several papers. In the "Symposium on Schauder Conjecture and Fixed Point Theorems of Cauty", May 15-17, 2011, Kielce, Poland (organized by A. Idzik), Cauty presented several papers on this matter, but most of participants disagreed with his ways.

2.8. Park: EAJM 2008 [15]

Abstract: We survey recent results and some conjectures in analytical fixed point theory. We list some known fixed point theorems for Kakutani maps, Fan-Browder maps, locally selectionable maps, approximable maps, admissible maps, and the better admissible class $\mathfrak{B}$ of maps. We also give 16 conjectures related to that theory.

3. Generalized vector equilibrium problems

In this section, we improve main results on generalized vector equilibrium problems appeared in three papers of other authors [1, 5, 6]. These results are deduced from the Dobrowolski conjecture stated in our [12]; see Section 2.

Instead of the conjecture, we apply the well-known Himmelberg fixed point theorem as follows:

Theorem (Himmelberg)**.** *Let X be a convex subset of a Hausdorff locally convex t.v.s. Then every compact Kakutani map $T : X \multimap X$ has a fixed point.*

Note that any finite dimensional subspace of a topological vector space is locally convex and Hausdorff.

Recall that the following due to Ky Fan:

The 1984 KKM Theorem (Ky Fan)**.** *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x\in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Here the Hausdorffness is indispensable. For the evolution of this theorem, see [16].

Definition. Let K_0 be a nonempty subset of K. A set-valued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if, $\mathrm{co}A \subset \bigcup_{x\in A}\Gamma(x)$ for every finite subset A of K_0.

The authors of [1, 5, 6] take the following form of the 1984 theorem:

Lemma. *Let K be a nonempty subset of a topological vector space X and $\Gamma : K \to 2^X$ be a KKM map with closed values. Assume that there exist a nonempty compact convex subset $D \subset K$ such that $B = \bigcap_{x\in D}\Gamma(x)$ is compact. Then $\bigcap_{x\in A}\Gamma(x) \neq \emptyset$.*

In [17], we noted that the more general forms of the 1984 theorem or Lemma are equivalent to the one with the simple coercivity condition that the intersection of a finite number of map values is compact.

3.1. Farajzadeh et al.: AAA 2008 [5]

Abstract: We first define upper sign continuity for a set-valued mapping and then we consider two types of generalized vector equilibrium problems in topological vector spaces and provide sufficient conditions under which the solution sets are nonempty and compact. Finally, we give an application of our main results. The paper generalizes and improves results obtained by Fang and Huang [4].

From the text: As an application of Theorem 2.11 [5], we derive the existence result for a solution of the following problem which consists of finding a $u \in K$ such that

$$\langle A(u,u), v - g(u)\rangle \nsubseteq -C(u) \setminus \{0\}, \forall v \in K,$$

where $A : K \times K \to 2^{L(X,Y)}$ and $g : K \to K$.

This problem was considered by Fang and Huang [4] in reflexive Banach spaces setting for a set-valued mapping which is demi-C-pseudomonotone.

Theorem 2.12. *Let X be a* (metriziable) *topological vector space, K nonempty convex subset of X, $A : K \times K \to 2^{L(X,Y)}$, and $g : K \to K$ be two mappings. Assume that*

(i) *for each fixed $w \in K$, the mapping $(u, v) \to \langle A(w,u), v - g(u)\rangle$ is C-pseudomonotone and C-upper sign continuous;*

(ii) $\langle A(w,u), u - g(u)\rangle \cap C(u) \neq \emptyset$ *for each (w,u);*

(iii) *for each fixed $v \in K$, the mapping $(w,u) \to \langle A(w,u), u - g(v)\rangle$ is lower semicontinuous;*

(iv) *for each finite dimensional subspace M of X with $K_M = K \cap M \neq \emptyset$, there exist compact subset B_M and compact convex subset D_M of K_M such that $\forall (w,z) \in K_M \times (K_M \setminus B_M)$, $\exists u \in D_M$ such that $\langle A(w,u), z - g(u)\rangle \nsubseteq -C(u)$.*

Then there exists $u \in K$ such that

$$\langle A(u,u), v - g(u)\rangle \nsubseteq -C(u) \setminus \{0\}, \forall v \in K.$$

Now in this paper, we can give a proof by modifying the original one for Theorem 2.12 in [5] without assuming the metrizability:

Modified Proof. Let $M \subseteq X$ be a finite dimensional subspace with $K_M = K \cap M \neq \emptyset$. For each fixed $w \in K$, consider the problem of finding a $u \in K_M$ such that

$$\langle A(w,u), v = g(u)\rangle \nsubseteq -C(u) \setminus 0, \forall v \in K_M.$$

By Theorem 2.11, the problem has a nonempty compact solution set in K. For $w \in K_M$, we define a set-valued mapping $T : K_M \to 2^{K_M}$ by

$$T(w) = \{u \in K_M : \langle A(w,u), v - g(u)\rangle \nsubseteq -C(u) \setminus 0, \forall v \in K_M.$$

Then T is a compact Kakutani map on a convex subset K_M of a finite dimensional subspace as in the original proof. Hence by the Himmelberg theorem, it has a fixed point $w_0 \in K_M$. For the remainder of the proof, just follow that of [5].

Remark. This theorem also based on Ky Fan's 1984 KKM theorem; especially, see the coercivity condition (iv). However, there appeared more general coercivity conditions in the literature; see our recent work [17].

3.2. Ansari et al.: JGO 2009 [1]

Abstract: In this paper, we consider vector variational inequality and vector F-complementarity problems in the setting of topological vector spaces. We extend the concept of upper sign continuity for vector-valued functions and provide some existence results for solutions of vector variational inequalities and vector F-complementarity problems. Moreover, the nonemptyness and compactness of solution sets of these problems are investigated under suitable assumptions. We use a version of Fan-KKM theorem and Dobrowolski's fixed point theorem to establish our results.

From the text: In Section 4: Existence of solutions of GSVVI begins with the following:

Now we establish the following existence result for a solution of GSVVI under C_x-pseudomonotonicity and C_x-upper sign continuity but without demipseudomonotonicity assumption. This theorem generalizes and improves Theorem 3.1 in [3].

Theorem 4. *Let K be a nonempty closed convex subset of a* (metrizable) *topological vector space X and $C : K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a proper solid convex cone. Let $F : K \to Y$ be a P-convex and continuous mapping and $W : K \to 2^Y$ be a closed set-valued mapping defined as $W(x) = Y \setminus (-intC(x))$ for all $x \in K$ such that $tW(x) + (1-t)W(y) \subseteq W(tx + (1-t)y)$ for all $x, y \in K$ and $t \in [0,1]$. Let $A : K \times K \to L(X,Y)$ be a mapping. Assume that the following conditions hold:*

(i) *For all* $z \in K$, *the mapping* $A(\cdot, z) : K \to L(X, Y)$ *is finite-dimensional continuous, that is, for any finite dimensional subspace* $M \subseteq X$, $A(\cdot, z) : K \cap M \to L(X, Y)$ *is continuous;*

(ii) A *is* C_x*-pseudomonotone and* C_x*-upper sign continuous in the second argument;*

(iii) *For each finite dimensional subspace* M *of* X *with* $K_M = K \cap M \neq \emptyset$, *there exist compact subset* $B_M \subseteq K_M$ *and compact convex subset* $D_M \subseteq K_M$ *such that* $\forall (x, z) \in K_M \times (K_M \setminus B_M)$, $\exists y \in D_M$ *such that* $\langle A(x, z), y - z\rangle + F(y) - F(z) \in -intC(z)$.

Then GSVVIP has a solution.

Now in this paper, we can prove Theorem 4 without assuming the metrizability. The following is a simple modification of original one in [1]:

Modified Proof. Let $M \subseteq X$ be a finite dimensional subspace with $K_M = K \cap M \neq \emptyset$. For each fixed $w \in K$, consider the problem of finding a $\bar{u} \in K_M$ such that

$$\langle A(w, \bar{u}), v - \bar{u}\rangle + F(v) - F(\bar{u}) \notin -intC(\bar{u}), \forall v \in K_M.$$

By Theorem 2, the problem has a nonempty compact solution set. For $w \in K_M$, we define a set-valued mapping $T : K_M \to 2^{K_M}$ by

$$T(w) = \{u \in K_M : \langle A(w, u), v - u\rangle + F(v) - F(u) \notin -intC(u), \forall v \in K_M.$$

Then T is a compact Kakutani map on a convex subset K_M of a finite dimensional subspace as in the original proof. Hence by the Himmelberg theorem, it has a fixed point $w_0 \in K_M$. For the remainder of the proof, just follow that of [1].

As an application of Theorem 4, we derive the existence result for a solution of GVCP as in [1].

Theorem 5. *Let* C, W, F *and* A *be the same as in Theorem 4 and let* K *be a nonempty closed convex cone in a* (metrizable) *topological vector space. Assume that all the conditions of Theorem 4 hold such that* $F(\frac{1}{2}x) = \frac{1}{2}F(x)$ *for all* $x \in K$. *Then GVCP has a solution.*

Remark. This theorem also based on Ky Fan's 1984 KKM theorem; especially, see the coercivity condition (iv). However, there appeared more general coercivity conditions in the literature; see our recent work [17].

3.3. Khan: CAMWA 59 (2010) [6]

ABSTRACT: In this paper, we introduced the generalized vector variational inequality-type problem and the generalized vector complementarity-type problem in the setting of topological vector space. By utilizing a modified version of the Fan-KKM theorem, we investigated the nonemptyness and compactness of solution sets of these problems without the demipseudomonotonicity assumption. Further, we prove that solution sets of both the problems are equivalent to each other under some suitable conditions.

From the text: Now we establish the following existence result for a solution of (GVVITP) under pseudomonotonicity and P_x-upper sign continuity with respect to f without the demipseudomonotonicity assumption. This theorem generalizes and improves Theorem 3.1 in [3].

Theorem 3.2. *Let K be a nonempty, closed, convex subset of a* (metrizable) *topological vector space X and $P : K \to 2^Y$ be a set-valued mapping be such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $intP(x) \neq \emptyset$. A set-valued mapping $W : K \to 2^Y$ defined as $W(x) = Y \setminus \{-intP(x)\}$, $\forall x \in K$, is closed and concave. Suppose following conditions hold:*

(i) *$f : K \times K \to Y$ is P_--convex and upper semicontinuous in first and second argument, respectively;*

(ii) *$A : K \times K \to L(X, Y)$ is pseudomonotone and P_x-upper sign continuous in the second argument;*

(iii) *For each fixed $v \in K$, $A(\dot{;}v) : K \cap M \to L(X, Y)$ is continuous on each finite dimensional subspace of X, that is, for any finite dimensional subspace $M \subseteq X$, $A(\dot{;}v) : K \cap M \to L(X, Y)$ is continuous;*

(iv) *For each finite dimensional subspace M of X with $K_M = K \cap M \neq \emptyset$, there exist compact subset $B_M \subseteq K_M$ and compact convex subset $D_M \subseteq K_M$ such that $\forall(x, z) \in K_M \times (K_M \setminus B_M)$, there exists $y \in D_M$ such that $\langle A(x, v), y - v\rangle + f(y, v) - f(v, v) \in -intP(x)$.*

Then (GVVITP) has a solution.

Now in this paper, we can give a new proof of Theorem 3.2 without assuming the metrizability:

Modified Proof. Let M be a finite dimensional subspace of X and $K_M = K \cap M \neq \emptyset$. For each fixed $v \in K$, we consider the following generalized vector variational inequality-type problem: Find $u_0 \in K_M$ such that

$$\langle A(v, u_0), u - u_0\rangle + f(u, u_0)_f(u_0, u_0) \notin -intP(u_0),\ \forall u \in K_M.$$

By Theorem 2.1 [6], above problem has compact solution set. Define a set-valued mapping $T : K_M \to 2^{K_M}$ as follows:

$$T(v) = \{w \in K_M : \langle A(v, w), u - w\rangle + f(u, w) \notin -intP(w),\ \forall u \in K_M\}.$$

Then T is a compact Kakutani map on a convex subset K_M of a finite dimensional subspace as in the original proof. Hence by the Himmelberg theorem, it has a fixed point $w_0 \in K_M$. For the remainder of the proof, just follow that of [6].

Remark. This theorem also based on Ky Fan's 1984 KKM theorem; especially, see the coercivity condition (iv). However, there appeared more general coercivity conditions equivalent to a simple one in the literature; see our recent work [17].

References

[1] Q. H. Ansari, A. P. Farajzadeh and S. Schaible, *Existence of solutions of vector variational inequalities and vector complementarity problems*, J. Glob. Optim. **45** (2009), 297–307. DOI 10.1007/s10898-008-9375-x.

[2] T. Dobrowolski, *Revisiting Cautys proof of the Schauder conjecture*, Abstr. Appl. Anal. **7** (2003), 407–433.

[3] Y. P. Fang and N. J. Huang, *The vector F-complementarity problems with demipseudomonotone mappings in Banach spaces*, Appl. Math. Lett. **16** (2003), 1019–1024.

[4] ———, *Existence results for generalized implicit vector variational inequalities with multivalued mappings*, Indian J. Pure . Appl. Math. **36**(11) (2005), 629–640.

[5] A. P. Farajzadeh, A. Amini-Harandi and D. ORegan, *Existence results for generalized vector equilibrium problems with multivalued mappings via KKM Theory*, Abst. Appl. Anal. vol. 2008, Article ID 968478, 8 pages. doi:10.1155/2008/968478.

[6] S. A. Khan, *Generalized vector complementarity-type problems in topological vector spaces*, Comput. Math. Appl. **59** (2010), 3595–3602.

[7] S. Park, *On some conjectures and problems in analytical fixed point theory*, Nonlinear Analysis and Convex Analysis (Hirosaki, 2001), 383–392, Yokohama Publ., 2003.

[8] ———, *On some conjectures and problems in analytical fixed point theory, Revisited*, RIMS Kôkyûroku, Kyoto Univ. **1365** (2004), 166–175.

[9] ———, *The compact AR problem and related topics*, Fixed Point Theory and Its Applications (Valencia, Spain, 2003), 199–206, Yokohama Publ. 2004.

[10] ———, *Fixed points, Roberts spaces, and compact AR problem*, Nonlinear Analysis and Convex Analysis (NACA 2003, Tokyo), 461–468, Yokohama Publ. 2004.

[11] ———, *Multimaps having openness and the Birkhoff-Kellogg theorems*, Indian J. Math. **46** (2004), 223–232.

[12] ———, *Recent results in analytical fixed point theory*, Nonlinear Anal. TMA **63** (2005), 977–986.

[13] ———, *Fixed points of approximable or Kakutani maps in generalized convex spaces*, J. Nonlinear Convex Anal. **7** (2006), 1–17.

[14] ———, *Remarks on recent results in analytical fixed point theory*, Nonlinear Analysis and Convex Analysis (NACA 2005, Okinawa), 517–525, Yokohama Publ., Yokohama, 2007.

[15] ———, *Recent results and conjectures in analytical fixed point theory*, East Asian J. Math. **24**(1) (2008), 11–20.

[16] ———, *Evolution of the 1984 KKM theorem of Ky Fan*, Fixed Point Theory Appl. vol. 2012, 2012:146. DOI:10.1186/1687-1812-2012-146.

[17] ———, *A genesis of general KKM theorems for abstract convex spaces: Revisited*, J. Nonlinear Anal. Optim. **4**(1) (2013), 127–132.

[18] M. Tavernise and A. Trombetta, *On convex total bounded sets in the space of measurable functions*, J. Function Sp. Appl. vol. 2012, Article ID 174856, 9 pages. doi:10.1155/2012/174856.