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EXISTENCE THEOREMS FOR GENERALIZED NASH EQUILIBRIUM PROBLEMS

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Abstract In a recent paper [7], Dutang provided a thorough study of theorems guaranteeing existence of generalized Nash equilibria and analyze the assumptions on practical parametric feasible sets. Motivated by [7], firstly in this paper, we obtain a generalization (Theorem 1) of an equilibrium theorem for a competitive economy due to Arrow and Debreu [1]. Secondly, we show that our previous generalization (Theorem 2) [14] of a social equilibrium existence theorem of Debreu [6] unifies the previous results due to von Neumann, Nash, Debreu, Park, and Dutang. Our arguments are based on the celebrated Eilenberg-Montgomery fixed point theorem. Finally, we add some historical remarks and the contents of our previous works related to our main results.

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1. Introduction

In our earlier work [14], we gave an acyclic version of the social equilibrium existence theorem of Debreu [6]. Moreover, our main result there was applied to acyclic versions of a saddle point theorem, a minimax theorem, and the Nash equilibrium theorem.

Recently, the generalized Nash equilibrium, where the feasible sets of the players depend on other players' action, becomes increasingly popular among academics and practitioners. In a recent paper [7], Dutang provided a thorough study of theorems guaranteeing existence of generalized Nash equilibria and analyze the assumptions on practical parametric feasible sets.

In this paper, firstly, we obtain a generalization (Theorem 1) of an equilibrium theorem for a competitive economy due to Arrow and Debreu [1]. Secondly, we show that our

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previous generalization (Theorem 2) [14] of a social equilibrium existence theorem of Debreu [6] unifies the previous results due to von Neumann, Nash, Debreu, Park, and Dutang. Our arguments are based on the celebrated Eilenberg-Montgomery fixed point theorem. At the end of this paper, we add some historical remarks and the contents of our previous works related to our main results in this article.

In Section 2, we recall Berge's theorem, the Eilenberg-Montgomery fixed point theorem, and related matters. Section 3 devotes basic definitions in Dutang [7] which we need later. In Section 4, our main results, Theorems 1 and 2, are given with several consequences of them. Finally, Section 5 deals with some historical remarks and the abstracts of our previous articles related to the present article.

2. Preliminaries

In this section we follow [14].

For topological spaces X and Y, a *multimap* or *correspondence* $F : X \multimap Y$ is a function from X into the power set 2^Y of Y with nonempty values $F(x) \subset Y$ for $x \in X$. A map $F : X \multimap Y$ is said to be *closed* if its graph $\mathrm{Gr}(F) = \{(x, y) \mid x \in X,\ y \in F(x)\}$ is closed in $X \times Y$; *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$ is closed; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $F^-(B)$ is open; and *continuous* if it is u.s.c. and l.s.c. If F is u.s.c. with closed values, then F is closed. The converse is true whenever Y is compact.

The concepts of upper or lower semicontinuity of extended real-valued functions are standard.

The following form of the well-known theorem [3] is adopted in [7]:

Berge's maximum theorem. *Let X, Y be two metric spaces, $f : X \times Y \to \mathbb{R}$ be an objection function and $F : X \multimap Y$ a constraint correspondence. Assume that f is continuous, F is both l.s.c. and u.s.c.; and F is nonempty and compact valued. Then we have*

(i) $\phi : x \longmapsto \max_{y \in F(x)} f(x, y)$ *is a continuous function from X into $\mathbb{R}$.*

(ii) $\Phi : x \longmapsto \mathrm{argmax}_{y \in F(x)} f(x, y)$ *is a u.s.c. map from X into 2^Y and compact valued.*

Note that $\Phi(x)$ is sometimes written $\{y \in F(x) \mid f(x, y) = \phi(x)\}$.

A *polyhedron* is a set in $\mathbb{R}^n$ homeomorphic to a union of a finite number of compact convex sets in $\mathbb{R}^n$. The product of two polyhedra is a polyhedron [6].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery [4] or, more generally, to Begle [5]:

Lemma 1. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map (that is, u.s.c. with acyclic values). Then T has a fixed point $\hat{x} \in Z$; that is, $\hat{x} \in T(\hat{x})$.*

Recall that Lemma 1 reduces to Kakutani's theorem [10] whenever Z is a simplex and T has nonempty convex values.

The following is a collectively fixed point theorem equivalent to Lemma 1.

Lemma 2. *Let $\{X_i\}_{i \in I}$ be a finite family of acyclic polyhedra, and $T_i : X \multimap X_i$ an acyclic map for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

Proof. Note that X itself is an acyclic polyhedron. Define $T : X \multimap X$ by $T(x) = \Pi_{i\in I}T_i(x)$ for each $x \in X$. Then T is a acyclic map. In fact, each T_i is u.s.c. for each $i \in I$ and hence T is also u.s.c.; see Ky Fan [7, Lemma 3]. Note that each $T(x)$ is acyclic. Therefore, by Lemma 1, T has a fixed point $\hat{x} \in X$; that is, $\hat{x} \in T(\hat{x})$ and hence $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$. This completes our proof. ∎

3. Generalized Nash equilibrium

In this section we follow [7].

Let $I = \{1, 2, \ldots, n\}$ be a finite set of players, $\{X_i\}_{i\in I}$ be a family of strategy sets $X_i \subset \mathbb{R}^{n_i}$ of the player $i \in I$. We denote the payoff function by $\theta_i : X \to \mathbb{R}$ (to be maximized), where

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n.$$

A game is thus described by $(I, X_i, \theta_i(.))$.

Player i's (pure) strategy is denoted by $x_i \in X_i$ while $x_{-i} \in X_{-i}$ denotes the other players' action. A game is thus described by $(I, X_i, \theta_i(.))$.

Definition. A Nash equilibrium is a strategy point $x^* \in X$ such that no player has an incentive to deviate, i.e. for all $i \in I$,

$$\forall x_i \in X_i, \quad \theta_i(x_i, x^*_{-i}) \leq \theta_i(x^*_i, x^*_{-i}).$$

Since the introduction of games $(I, X_i, \theta_i(.))$, many extensions have been proposed in the literature; see [7]. In the following, we consider the extension dealing with games where each player has a range of actions which depends on the actions of other players. This new extension leads to the so called generalized Nash equilibrium.

Let $C_i : X_{-i} \multimap X_i$ be the constraint correspondence of Player i, i.e. a multimap sending a point in X_{-i} to a subset of X_i. Thus, $C_i(x_{-i})$ defines the ith player action space given other players' action x_{-i}. Typically, the constraint correspondence C_i is defined by a parametrized action space as $C_i(x_{-i}) = \{x_i \in X_i \mid g_i(x_i, x_{-i}) \geq 0\}$, where $g_i : X \to \mathbb{R}^{m_i}$ is a constraint function. When g_i does not depend on x_{-i}, we get back to standard game. A generalized game is described by $(I, X_i, C_i(.), \theta_i(.))$ and is also called an abstract economy in reference to Debreu's economic work [1, 6].

Definition. The generalized Nash equilibrium for a generalized game (I, X_i, C_i, θ_i) is defined as a point x^* solving for all $i \in I$,

$$x^*_i \in \text{argmax}_{x_i\in C_i(x^*_{-i})}\theta_i(x_i, x^*_{-i}).$$

4. Main Results

The following is our main result:

Theorem 1. *Let n players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, we have*
(i) *X_i is an acyclic polyhedron,*
(ii) *C_i is continuous on X_{-i},*
(iii) *$\forall x_{-i} \in X_{-i}$, $C_i(x_{-i})$ is nonempty, closed, convex,*
(iv) *θ_i is continuous on the graph $\text{Gr}(C_i)$,*
(v) *$\forall x \in X$, $\text{argmax}_{x_i\in C_i(x_{-i})}\theta_i(x_i, x_{-i})$ is acyclic.*

Then there exists a generalized Nash equilibrium.

Proof. Since θ_i is continuous, C_i is both l.s.c. and u.s.c.; and C_i is nonempty and compact valued, the maximum theorem implies that the best response correspondence defined as

$$P_i : x_{-i} \longmapsto \text{argmax}_{x_i \in C_i(x_{-i})} \theta_i(x_i, x_{-i})$$

is u.s.c. and compact valued. Further, by (v), P_i is acyclic valued. Now, consider the Cartesian product of $P_i(x_{-i})$ to define $\Phi : X \multimap X$ as

$$x \longmapsto P_1(x_{-1}) \times \cdots \times P_n(x_{-n})$$

where X is an acyclic polyhedron. This multiplayer best response is acyclic and compact valued. In our finite dimensional setting and with a finite Cartesian product, the upper semicontinuity of each component P_i implies the upper semicontinuity of Φ, see Prop 3.6 of [8]. Finally, Lemma 2 gives the existence result. ∎

The following is [7, Theorem 3.1]:

Corollary 1. *Let n players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, we have*
(i) *X_i is nonempty, convex and compact subset of a Euclidean space,*
(ii) *C_i is continuous on X_{-i},*
(iii) *$\forall x_{-i} \in X_{-i}$, $C_i(x_{-i})$ is nonempty, closed, convex,*
(iv) *θ_i is continuous on the graph $\text{Gr}(C_i)$,*
(v) *$\forall x \in X$, $x_i \longmapsto \theta_i(x_i, x_{-i})$ is quasiconcave on $C_i(x_{-i})$, i.e., $\forall r \in \mathbb{R}$, $U_{\theta_i}(r) = \{x_i \in C_i(x_{-i}) \mid \theta_i(x_i, x_{-i}) \geq r\}$ is convex.*
Then there exists a generalized Nash equilibrium.

Proof. In view of Theorem 1, as θ_i is quasiconcave, it suffices to show that P_i is convex valued. Let $z_i, y_i \in P_i(x_{-i})$. By definition of maximal points, $\forall x_i \in C_i(x_{-i})$, we have $\theta_i(y_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$ and $\theta_i(z_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$. Let $r \in]0,1[$. By the quasiconcaveness assumption, we get $\theta_i(ry_i + (1-r)z_i, x_{-i}) \geq \min(\theta_i(y_i, x_{-i}), \theta_i(z_i, x_{-i})) \geq \theta_i(x_i, x_{-i})$. Hence, $ry_i + (1-r)z_i \in P_i(x_{-i})$, i.e., $P_i(x_{-i})$ is a convex set. Furthermore, P_i is also nonempty valued since $C_i(x_{-i})$ is nonempty. ∎

According to Dutang [7], Corollary 1 was established by [1] in the context of abstract economy, so a simplified version by [9] is reported above. Note that [2] propose a different version, called the Arrow-Debreu-Nash theorem, where objective functions are player concave rather than player quasiconcave.

From Lemma 2, we have the following version [14, Theorem 2] of the social equilibrium existence theorem of Debreu [6]. For the completeness we give a proof as in [14].

Theorem 2. *Let $\{X_i\}_{i \in I}$ be a family of acyclic polyhedra, $A_i : X_{-i} \multimap X_i$ closed maps, and $f_i, g_i : \text{Gr}(A_i) \to \overline{\mathbb{R}}$ u.s.c. functions for each $i \in I$ such that*
(1) *$g_i(x) \leq f_i(x)$ for all $x \in \text{Gr}(A_i)$;*
(2) *$\varphi_i(x_{-i}) = \max_{y \in A_i(x_{-i})} g_i(x_{-i}, y)$ is a l.s.c. function of $x_{-i} \in X_{-i}$; and*
(3) *for each $i \in I$ and $x_{-i} \in X_{-i}$, the set*

$$M(x_{-i}) = \{x_i \in A_i(x_{-i}) \mid f_i(x_{-i}, x_i) \geq \varphi_i(x_{-i})\}$$

is acyclic.

Then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$ *for all* $i \in I$*; that is,*

$$\hat{a}_i \in A_i(\hat{a}_{-i}) \quad \text{and} \quad f_i(\hat{a}) = \max_{a_i \in A(\hat{a}_{-i})} g_i(\hat{a}_{-i}, a_i) \quad \text{for all} \quad i \in I.$$

Proof. For each $i \in I$, define a map $T_i : X \multimap X_i$ by

$$T_i(x) = \{y \in A_i(x_{-i}) \mid f_i(x_{-i}, y) \geq \varphi_i(x_{-i})\}$$

for $x \in X$. Then $T_i(x) \neq \emptyset$ by (1) since $A_i(x_{-i})$ is compact and $g_i(x_{-i}, \cdot)$ is u.s.c. on $A_i(x_{-i})$. We show that $\mathrm{Gr}(T_i)$ is closed in $X \times X_i$. In fact, let $(x^\alpha, y^\alpha) \in \mathrm{Gr}(T_i)$ and $(x^\alpha, y^\alpha) \to (x, y)$.

Then

$$f_i(x_{-i}, y) \geq \overline{\lim_\alpha} f_i(x^\alpha_{-i}, y^\alpha) \geq \overline{\lim_\alpha} \varphi_i(x^\alpha_{-i}) \geq \underline{\lim_\alpha} \varphi_i(x^\alpha_{-i}) \geq \varphi_i(x_{-i})$$

and, since $\mathrm{Gr}(A_i)$ is closed in $X_{-i} \times X_i$, $y^\alpha \in A_i(x^\alpha_{-i})$ implies $y \in A_i(x_{-i})$. Hence $(x, y) \in \mathrm{Gr}(T_i)$. Moreover, each $T_i(x) = M(x_{-i})$ is acyclic by (3). Now we apply Lemma 2. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for all $i \in I$; that is, $\hat{x}_i \in A_i(\hat{x}_{-i})$ and $f_i(\hat{x}_{-i}, \hat{x}_i) \geq \varphi_i(\hat{x}_{-i})$. ∎

Remarks 1. If X_i and $M(x_{-i})$ in (3) are contractible, if $f_i = g_i$ is continuous, and if φ_i is continuous for each $i \in I$, then Theorem 3.2 reduces to Debreu [6, Theorem]. Note that our proof is much simpler than his.

2. Since A_i and g_i are u.s.c., by Berge's theorem, φ_i are automatically u.s.c. Hence, condition (2) implies continuity of φ_i.

3. If A_i and g_i are continuous, condition (2) holds immediately by Berge's theorem, and hence each φ_i is continuous. This fact is noted by Debreu [6, Remark].

4. As was also noted by Debreu, instead of acyclic polyhedra, one might take for example absolute retracts or others.

The Debreu theorem [6] based on contractible sets is now given. Originally, the uppersemicontinuity is replaced by the closedness of the graph $\mathrm{Gr}(C_i)$, but these are equivalent since contractible sets are closed and compact sets.

The following is [7, Theorem 3.2]:

Corollary 2. *Let* N *agents be characterized by an action space* X_i *and* $X = X_1 \times \cdots \times X_N$*. Let a payoff function* $\theta_i : X \to \mathbb{R}$ *and a restricted action space* $C_i(x_{-i})$ *given other player actions* x_{-i}*. Each agent i maximizes its payoff on* $C_i(x_{-i})$*. If for all agents, we have*

(i) X_i *is a contractible polyhedron,*

(ii) $C_i : X_{-i} \multimap X_i$ *is u.s.c.,*

(iii) $\theta_i : \mathrm{Gr}(C_i) \to \mathbb{R}$ *is continuous,*

(iv) $\phi_i : x_{-i} \rightarrowtail \max_{x_i \in C_i(x_{-i})} \theta_i(x_i, x_{-i})$ *is continuous,*

(v) *for all* $x_i \in X_i$*, the best response set*

$$M_{x_{-i}} = \{x_{-i} \in X_i(x_{-i}) \mid \theta_i(x_i, x_{-i}) \geq \phi_i(x_{-i})\}$$

is contractible.

Then there exists a generalized Nash equilibrium.

The following is our previous generalization [14, Corollary 3] of the Nash equilibrium theorem.

Corollary 3. *Let n agents be characterized by an action space X_i and an objective function θ_i. If $\forall i \in \{1, \ldots, n\}$, X_i is an acyclic polyhedron; $\theta_i : X \to \mathbb{R}$ is continuous with $X = X_1 \times \cdots \times X_n$ and*

(0) *for each $x_{-i} \in X_{-i}$ and each $\alpha \in \overline{\mathbb{R}}$, the set*

$$\{x_i \in X_i \mid f_i(x_{-i}, x_i) \geq \alpha\}$$

is empty or acyclic.

Then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i \in X_i} f_i(\hat{a}_{-i}, y_i) \quad \textit{for all} \quad i \in I.$$

Proof. We apply Theorem 2 with $f_i = g_i$ and $A_i : X_{-i} \multimap X_i$ defined by $A_i(x_{-i}) = X_i$ for $x_{-i} \in X_{-i}$. Then condition (2) of Theorem 2 follows from Berge's theorem, and the set in condition (3) is nonempty and acyclic by (0). Therefore, we have the conclusion. ∎

Originally, Nash introduced the equilibrium concept to finite games in [11, 12], i.e. X_i is a finite set. Therefore, he used the mixed strategy concept (i.e. a probability distribution over the pure strategies) and proved the existence of such an equilibrium in that context. Dutang [7] reported the existence theorem of [13] for infinite games as follows:

Corollary 4. (Nash) *Let n agents be characterized by an action space X_i and an objective function θ_i. If $\forall i \in \{1, \ldots, n\}$, X_i is nonempty, convex and compact; $\theta_i : X \to \mathbb{R}$ is continuous with $X = X_1 \times \cdots \times X_n$ and $\forall x_{-i} \in X_{-i}$, $x_i \mapsto \theta_i(x_i, x_{-i})$ is concave on X_i, then there exists a Nash equilibrium.*

5. Historical Remarks

The first half of this section is from [19], see the references therein.

John von Neumann's 1928 minimax theorem [26] and 1937 intersection lemma [27] have numerous generalizations and applications. Kakutani's 1941 fixed point theorem [10] was to give simple proofs of the above-mentioned results. John Nash [12] obtained his 1951 equilibrium theorem based on the Brouwer or Kakutani fixed point theorem. In 1952, Fan and Glicksberg extended the Kakutani theorem to locally convex Hausdorff topological vector spaces. This result was applied by its authors to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma and others.

An upper semicontinuous (u.s.c.) multimaps with nonempty compact convex values is called a *Kakutani map*. The Fan-Glicksberg theorem was extended by Himmelberg in 1972 for compact Kakutani maps instead of assuming compactness of domains. In 1990, Lassonde extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in Hausdorff topological vector spaces. Moreover, Lassonde applied his theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem comparable to Debreu's social equilibrium existence theorem [6].

On the other hand, in 1946, the Kakutani fixed point theorem was extended for acyclic maps by Eilenberg and Montgomery [4]. This result was applied by Park [14] to give acyclic versions of the social equilibrium existence theorem due to Debreu [6], saddle point theorems, minimax theorems, and the Nash equilibrium theorem. Moreover, Park [15,16] obtained a fixed point theorem for compact compositions of acyclic maps defined on admissible (in the sense of Klee) convex subsets of topological vector spaces. This new theorem was applied in [16] to deduce acyclic versions of the von Neumann intersection lemma, the minimax theorem, the Nash equilibrium theorem, and others. Further, in [18], Park obtained a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorems.

In the following second half of this section, we introduce the abstracts of our previous articles related to the present paper:

1. AML 11 (1998) [14] — An acyclic version of the social equilibrium existence theorem of Debreu [6] is obtained. This is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the Nash equilibrium.

2. JCAM 113 (2000) [16] — Applying a fixed point theorem for compact compositions of acyclic maps, we obtain acyclic versions of the von Neumann intersection theorem, the minimax theorem, the Nash equilibrium theorem, and others.

3. JKMS 38 (2001) [17] — Generalized forms of the von Neumann–Sion type minimax theorem, the Fan–Ma intersection theorem, the Fan–Ma type analytic alternative, and the Nash–Ma equilibrium theorem hold for generalized convex spaces without having any linear structure.

4. AML 15 (2002) [18] — A fixed-point theorem on compact compositions of acyclic maps on admissible (in the sense of Klee) convex subset of a t.v.s. is applied to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem. Our new results generalize earlier works of Lassonde, Simons, and Park.

5. IJMS 6 (2010) [19] — The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we clearly show that a sequence of statements from the partial KKM principle to the Nash equilibria can be obtained for any KKM spaces. This unifies previously known several proper examples of such sequences for particular types of KKM spaces.

6. RIMS Kôkyûroku 1685 (2010) [20] — In 1966, Ky Fan first applied the KKM theorem to the Nash equilibrium theorem. Since then there have appeared several generalizations of the Nash theorem on various types of abstract convex spaces satisfying abstract forms of the KKM theorem. In this review, we introduce the most general results with examples appeared in each of several stages of such developments.

7. FPTA (2010) [21] — In this paper, we derive generalized forms of the Ky Fan minimax inequality, the von Neumann–Sion minimax theorem, the von Neumann–Fan intersection theorem, the Fan type analytic alternative, and the Nash equilibrium theorem for abstract convex spaces satisfying the partial KKM principle. These results are compared with previously known cases for G-convex spaces. Consequently, our results unify and generalize most of previously known particular cases of the same nature. Finally, we add some detailed historical remarks on related topics.

8. NA 73 (2010) [22] — The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In this paper, we

clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

9. JNAO 1 (2010) [23] — The existence theorem of pure-strategy Nash equilibrium due to H. Lu [Economics Letters 94 (2007) 459–462] is extended to generalized convex spaces. Consequently, our version can be applied to a broad class of abstract strategy spaces.

10. RIMS Kôkyûroku 1755 (2011) [25] — In 1950, John Nash [11,12] established his celebrated equilibrium theorem by applying the Brouwer or the Kakutani fixed point theorem. Since then there have appeared several fixed point theorems from which generalizations of the Nash theorem, the Debreu theorem, and many related results can be derived. In this paper, we introduce several stages of such developments.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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