REMARKS ON THE CONCEPT OF ABSTRACT CONVEX SPACES

Sehie Park

The National Academy of Sciences, Republic of Korea, Seoul 137–044; and
Department of Mathematical Sciences
Seoul National University, Seoul 151–747, KOREA
E-mail : park35@snu.ac.kr, sehiepark@gmail.com

Abstract. Since we introduced the concept of abstract convex spaces in the KKM theory, some readers raised certain questions or comments on them. In the present note, we want to clarify such things on the concept of abstract convex spaces raised by Ben-El-Mechaiekh [*Thoughts on KKM*, Personal Communications, 2013] and Kulpa and Szymmanski [12]. A number of examples and related matters are also added.

1. Introduction

The KKM theory, originally called by the author, is nowadays the study of applications of various equivalent formulations or generalizations of the Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929. In the last two decades, the theory has been extensively studied for generalized convex spaces (simply, G-convex spaces) and abstract convex spaces in the sense of ourselves in a sequence of our papers; for details, see [16-21] and the references therein.

Since the concept of G-convex spaces first appeared in 1993, a number of its modifications or imitations have followed. In order to unify such things, we introduced the so-called ϕ_A-spaces in 2007 [17]. Moreover, in our previous works [16-21], we introduced a new concept of abstract convex spaces and multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ having certain KKM property. These new spaces and multimap classes are known to be adequate to establish the KKM theory; see [22-26]. Especially, in [24], we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. It is noticed there that the class of abstract convex spaces $(E, D; \Gamma)$ satisfying the KKM principle play the major role in the KKM theory. Therefore, it seems to be quite natural to call such spaces the KKM spaces. In our works [24-27], we showed that a large number of well-known

2010 Mathematics Subject Classification: 47H04, 47H10, 47J20, 47N10, 49J53, 52A99, 54C60, 54H25, 58E35, 90C47, 91A13, 91B50.

Key words and phrases: Convex space, H-space, G-convex space, ϕ_A-space, abstract convex space, (partial) KKM space, KKM map.

results in the KKM theory on G-convex spaces also hold on the KKM spaces. Now it is evident that the class of abstract convex spaces contains many subclasses on which it is convenient to establish the KKM theory.

Since we introduced some classes of abstract convex spaces in the KKM theory, some readers raised certain questions or comments on them. In the present note, we want to clarify such things on the concept of abstract convex spaces raised by Ben-El-Mechaiekh [2] and Kulpa and Szymmanski [12]. A number of examples and related matters are also added.

2. Abstract convex spaces

We recall a short history of the abstract convex spaces.

In the KKM theory, motivated by the convex spaces of Lassonde in 1983 and c-spaces or H-spaces of Horvath in 1990-1993, Park and Kim introduced generalized (G-)convex spaces in 1993. Since 1998, we adopted the following definition; see [23]:

Definition. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ is the class of all nonempty subsets of a set D, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

In [20], we clearly stated that, in certain cases, it is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$.

Example. Recall that Horvath introduced a large number of examples of his c-spaces. Major examples of other G-convex spaces than convex spaces or c-spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, topological semilattices with path-connected intervals, hyperconvex metric spaces, Takahashi's convexity in metric spaces, L-spaces due to Ben-El-Mechaiekh et al., and so on. For the literature, see [15,36] and the references therein.

Moreover, a number of authors investigated another abstract convexities particular to G-convex spaces for various purposes. All of those authors considered the case $X = D$, contrary to the classical works of Knaster-Kuratowski-Mazurkiewicz and Fan for the case $X \neq D$; see [23]. This fact should be recognized by all peoples working in generalized abstract convexities.

Since G-convex spaces first appeared in 1993, a large number of modifications or imitations have followed.

Example. Such examples are L-spaces, B'-simplicial convexity, G-H-spaces, pseudo-H-spaces, spaces having property (H), FC-spaces, M-spaces, another L-spaces, simplicial spaces, $P_{1,1}$-spaces, generalized H-spaces, mc-spaces, L*-spaces, minimal G-convex spaces, GFC-spaces, FWC-spaces, and so on. See [15,17,29-32,35] and references therein.

These are all unified by the following concept in 2007 [17]:

Definition. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

In order to unify these concepts, we introduced the following in 2006 [16]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Note that we clearly stated the following in 2006 [16]:

"Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [34], where the bibliography lists 283 papers. For any subset $X \subset E$, its $\mathcal{C}$-*convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is $\mathcal{C}$-*convex* if $X{=}\mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E\rangle \multimap E$ given by $\Gamma_A{:=}\mathrm{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

Notice that our abstract convex space $(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E."

Even in 2013, some authors still adopt the above concepts; see [42].

Later, we add to assume E is a topological space in an abstract convex space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp, $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Example. We give examples of KKM spaces:

1. Every ϕ_A-space is a KKM space [30].

2. A connected ordered space $(X, \leq)$ can be made into an abstract convex space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A, \max A] = \{x \in X \mid \min A \leq x \leq \max A\}$ for each $A \in \langle D\rangle$. Further, it is a KKM space; see [19, Theorem 5(i)].

3. The extended long line L^* can be made into a KKM space $(L^* \supset D; \Gamma)$; see [19]. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D\rangle \multimap L^*$ be the one as in Example 2 above.

4. A ϕ_A-space is a KKM space and the converse does not hold; for example, the extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$, but not a ϕ_A-space.

In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^*\rangle$ and $\Delta_1 := [0, 1]$, there does not exist a continuous function $\phi_A : [0, 1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not a ϕ_A-space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned}\text{Simplex} &\Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Convex space} \Longrightarrow \text{H-space}\\ &\Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}\\ &\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}\end{aligned}$$

3. Earlier concepts of abstract convex spaces

In this section, we recall some (not all) earlier concepts of particular spaces of our abstract convex spaces:

(1) In 1971, Kay and Womble [8] defined an abstract convexity on a set X as a family $\mathfrak{C} = \{A_i\}_{i\in I}$, of subsets of X, stable under arbitrary intersections $(\bigcap_{i\in J} \in \mathfrak{C},\ J \subset I)$ and which contains the empty and total set $(\emptyset,\ X \in \mathfrak{C})$.

(2) In 1979, Penot [37] was successful in giving an abstract formulation of Kirk's theorem via the *convexity structures* as follows:
Let M be an abstract set. A family Σ of subsets of M is called a *convexity structure* if
(i) the empty set $\emptyset \in \Sigma$;
(ii) $M \in \Sigma$;
(iii) Σ is closed under arbitrary intersection.
The convex subsets of M are the elements of Σ.

(3) In 1984, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* due to Sortan [38] was appeared; see Section 2.

(4) In 1987, Bielawski [4] adopted the convexity as in (2).

(5) In 1988, Keimel and Wieczorek [9] worked in an abstract setting in which a convexity $\mathcal{K}$ is just any family of closed sets stable under arbitrary intersections; its members may be interpreted sets which are "closed and convex".

(6) In 1989, Krynski [10] noted that the concept of the above convexity was known in other fields of mathematics under various names, e.g., a "Moore family" (G. Birkhoff), or a "closure system" (P. Cohn); also cf. a "cyrtology" (S. Dolecki and G. Greco).

(7) In 1992, Wieczorek [41] adopted the following: A *convexity* on a topological space X is a family $\mathcal{K}$ of closed subsets of X which contains X as an element and which is closed under arbitrary intersections. Elements of $\mathcal{K}$ are called *closed convex* sets (there might be subsets of X not in $\mathcal{K}$ also interpreted as convex sets).

(8) In 1998, Ben-El-Mechaiekh et al. [3] adopted a convexity structure as above.

(9) In 2007, Amini et al. [1] adopted the same as above. Here the authors noted that this kind of convexity was widely studied and cited works by Ben-El-Mechaiekh et al. [3], Kay and Womble [8], J. V. Linares, and M. L. J. Van De Vel [39].

(10) According to Horvath [7], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as

singletons and which is closed under arbitrary intersections and updirected unions.

(11) In 2013, S. Xiang, S. Xia and J. Chen [42] adopted the concept of abstract convexity spaces as in [9,38].

4. Making new spaces from old

(1) In 2000 [15], we gave a new class of G-convex spaces as follows:

Proposition 4.1. *Any continuous images of G-convex spaces are G-convex spaces.*

This answers to a question raised by George Yuan at the NACA '98, Niigata, Japan, whether there are non-trivial examples of G-convex spaces which are not c-spaces. This is because ω-connectedness is not a continuous invariant.

Similarly, we have the following:

Proposition 4.2. *Any continuous image of a ϕ_A-space is a ϕ_A-space.*

Proof. Let $(X, D; \phi_A)$ be a ϕ_A-space; that is, for each $A \in \langle D \rangle$, there exists a continuous function $\phi_A : \Delta_n \to X$ with $|A| = n+1$. Let Y be a topological space with a continuous surjection $f : X \to Y$. Let $\psi_A := f \circ \phi_A : \Delta_n \to Y$ for each $A \in \langle D \rangle$. Then

$$(Y, D; \{\psi_A\}_{A \in \langle D \rangle}) \text{ or simply } (Y, D; \psi_A)$$

is a ϕ_A-space. □

(2) We introduce another way of making a new abstract convex space from old as in [34]:

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$ a map. Let $\Lambda_A := F(\Gamma_A)$ for each $A \in \langle D \rangle$. Then $(Z, D; \Lambda)$ is called the *abstract convex space induced by F.*

Let $Y \subset Z$ and $D' \subset D$ such that $\Lambda_B \subset Y$ for each $B \in \langle D' \rangle$. Then Y is called a Λ-*convex subset* of Z relative to D', and $(Y, D'; \Lambda')$ a *subspace* of $(Z, D; \Lambda)$ whenever $\Lambda' = \Lambda|_{\langle D' \rangle}$.

An abstract convex space without any nontrivial KKM map seems to be useless in the KKM theory; see [33].

Proposition 4.4. [34] *A KKM map $G : D \multimap Z$ on an abstract convex space $(E, D; \Gamma)$ with respect to $F : D \multimap Z$ is simply a KKM map on the corresponding abstract convex space $(Z, D; \Lambda)$ induced by F.*

Proposition 4.5. [34] *For an abstract convex space $(E, D; \Gamma)$, the corresponding abstract convex space $(Z, D; \Lambda)$ induced by $F : D \multimap Z$ is a partial KKM space if and only if $F \in \mathfrak{KC}(E, Z)$.*

The abstract convex space $(Z, D; \Lambda)$ induced by $F : D \multimap Z$ is a KKM space if and only if $F \in \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z)$.

Any cartesian product of abstract convex spaces can be made into an abstract convex spaces:

Proposition 4.6. [25] *Let* $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ *be a family of abstract convex spaces. Let* $X := \prod_{i\in I} X_i$ *be equipped with the product topology, and let* $D := \prod_{i\in I} D_i$*. For each* $i \in I$*, let* $\pi_i : D \to D_i$ *be the projection. For each* $A \in \langle D\rangle$*, define* $\Gamma_A := \prod_{i\in I} \Gamma_i(\pi_i(A))$*. Then,* $(X, D; \Gamma)$ *is an abstract convex space.*

5. Some questions raised by Ben-El-Mechaiekh

The following comments are given by Ben-EL-Mechaiekh [2] and some of them are also frequently asked questions by other readers. Here the present author gives short responses to them.

Comment 1: On abstract convex spaces I do not understand how a multimap $\Gamma : \langle D\rangle \multimap E$ with non-empty values is enough to define an abstract convex structure. This cannot be and is not sufficient. I believe that, to justify the terminology "convexity" the maps must directly (and not through subsequent properties) define a convexity structure. To me, it makes more sense (even if this appears "less general" than what have been written elsewhere) to say:

Definition 1. A *convexity structure* on a given non-empty set E is a collection $\mathcal{C}_E$ of subsets of E closed for arbitrary intersections and containing $\emptyset$ and E.

Definition 2. Let $E,\ D$ be non-empty sets and $\Gamma : \langle D\rangle \multimap E$ a multimap with non-empty values defined on $\langle D\rangle$, the collection of all non-empty finite subsets of D:

(i) For any given $D' \subset D$, let the Γ-*convex envelope* of D' in E be denoted and defined by

$$\mathrm{co}_\Gamma(D') := \bigcup\{\Gamma(A) \mid A \in \langle D'\rangle\} \subset E.$$

(ii) Call a subset X of E a Γ-convex subset of E relative to some $D' \subset D$ if for any $A \in \langle D'\rangle$, we have $\Gamma(A) \subset X$, that is, $\mathrm{co}_\Gamma(D') \subset X$.

(iii) The triple $(E, D; \Gamma)$ is said to be an *abstract convex space* if the collection of all Γ-convex subsets of E relative to some $D' \subset D$ generates a convexity structure $\mathcal{C}_E$ on E.

We would then proceed, as you did, to outline only the most important convex spaces and relate them to one another: hyperconvex metric spaces, convex spaces of Lassonde, *c*-spaces of Horvath, spaces with simplicial convexities of Bielawski, your G-convex spaces, L-spaces of Ben-El-Mechaiekh et al., ϕ_A-spaces, etc.

I claim then many of these are EQUIVALENT. If they are not, we would provide counterexamples (preferably stemming from "concrete situations" brought up by applications, rather than artificial constructions). On another matter: I cannot see the importance of the auxiliary set D. Perhaps I am not aware of concrete situation where the existence of a set $D \neq E$

on which Γ acts is absolutely needed. In such case, kindly send me references. Otherwise, there is the danger (which has already happened) to see the beauty and the simplicity of the KKM principle being overwhelmed by theorems that are impossible to read as they impose unjustified detours through auxiliary sets, spaces, and additional mappings.

Author's Response: 1. The convexity structure and my abstract convex space are equivalent in a sense. My definition (where E is a topological space) is enough to establish many statements in the KKM theory just following Section 2.

2. According to Hichem's suggestion, we can modify our definition as follows:

Definition. Let E be a topological space, D a nonempty set, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

3. Importance of the set D —— The set D was originally appeared in the KKM theorem, and Fan-KKM lemma. KKM maps of the form $F : D \multimap E$ on abstract convex spaces $(E, D; \Gamma)$ had appeared in earlier works of Dugundji-Granas and Granas-Lassonde; for example, see [5,6,13] and many others. See also my paper [23] where examples of $D \neq E$ were given. In fact, well-known theorems due to Sperner, Alexandroff-Pasynkoff, and Shapley adopt non-trivial D, and we can make another examples by applying various types of the Stone-Weierstrass approximation theorem.

Comment 2: On KKM maps I do not understand the concept of a map G being *KKM with respect to F*. Indeed, in the diagram:

$$\begin{array}{ccc} \langle D\rangle & \overset{\Gamma}{\rightrightarrows} & E \\ & & \downdownarrows \; F \\ D & \underset{G}{\rightrightarrows} & Z \end{array}$$

why do we need to factorize through E, when the space $(Z, D; \Psi = F \circ \Gamma)$ is also an abstract convex space in the sense of your definition. Aren't we making things more complicated? Is there a compelling reason, beyond generality?

The same remark holds true for the situation $D \overset{G}{\rightrightarrows} E \overset{s}{\longrightarrow} Z$ if we let $\tilde{G} = s \circ G$ and $\Psi = s \circ \Gamma : \langle D\rangle \overset{\Gamma}{\rightrightarrows} E \overset{s}{\longrightarrow} Z$, then:

- the triple $(Z, D; \Psi)$ is also an abstract convex space;
- the Ψ-convex hull of $D' \subset D = s(\Gamma$-convex hull of $D' \subset D)$.

In summary: do the introduction of these concepts involving auxiliary mappings really needed? In the absence of real motivation beyond generality, I claim that the answer is No! And that in fact, all goes back to the simplest case.

Author's Response: 1. The concept of KKM maps G w.r.t. F —– Your $(Z, D; F \circ \Gamma)$ was already noted in one of our recent papers [34]. See also Section 3. However, this concept is necessary to define the multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$. Note that there appeared some reasonable amount of works on such classes. See [31] and references therein.

2. On the same remark w.r.t. s —– Such s is motivated from the well-known 1983 paper of Lassonde [13], where many useful related results can be seen.

6. Remarks on abstract convex spaces by Kulpa and Szymansky

The following is recently given by Kulpa and Szymansky [12].

Abstract: We discuss S. Park's abstract convex spaces and their relevance to convexities and L*-operators. We construct an example of a space satisfying the partial KKM principle that is not a KKM space, thus solving a problem by Park. We show that if a compact Hausdorff space admits a 2-continuous L*-operator, then the space must be locally connected continuum and it has the fixed point property provided the covering dimension is 1. We also show that the unit circle admits no 2-continuous L*-operators.

In the following, we give some responses to comments raised by Kulpa and Szymanski [12].

Comment 1: Convexities and Abstract Convex Spaces — Following van de Vel's monograph, a convexity on a set X is a collection $\mathcal{G}$ of subsets of X satisfying certain conditions.

For any set X, let $\langle X \rangle$ and $\exp(X)$ denote, respectively, the set of all finite non-empty subsets of X, and the set of all non-empty subsets of X. Following Park, (see, e.g., [28] for the concept itself as well as for references to other related works), an abstract convex space $(E, D; \Gamma)$ consists of non-empty sets E, D, and a multimap $\Gamma : \langle D \rangle \to \exp(E)$. Even though the definition of abstract convex spaces does not call for any particular connection between the set D and the underlying space E, all the examples of abstract convex spaces considered in the literature (cf. [24] or [27]) have D to be a subset of E. In this setting, Park's approach stays within the framework of the classical one represented by van de Vel.

To wit, let us notice that if $(E, D; \Gamma)$ is an abstract convex space and $D \subset E$, then, without loss of generality, one can extend Γ onto $\langle E \rangle$ by setting $\Gamma(A) = E$ for each $A \in \langle E \rangle \setminus \langle D \rangle$. The abstract convex spaces where

$D = E$ will be denoted by $(E;\Gamma)$. A subset X of E is called a Γ-convex subset of $(E;\Gamma)$ if for any $N \in \langle X\rangle$, $\Gamma(N) \subset X$.

Proposition 1. (a) *Let $(E;\Gamma)$ be an abstract convex space. Then the family $\mathcal{G}_\Gamma = \{X \subset E : X$ is a Γ-convex subset of $(E;\Gamma)\}$ is a convexity on E.*

(b) *Let $(E;\mathcal{G})$ be a convexity space. If $\Gamma : \langle E\rangle \multimap E$ is given by $\Gamma(A) =$ conv A, then the abstract convex space $(E;\Gamma)$ satisfies $\mathcal{G}_\Gamma = \mathcal{G}$.*

Thus any convexity space can be considered as an abstract convex space and vice versa. Consequently, classifying (known and previously considered) convexity spaces as abstract convex spaces (cf. [14,19-21,24,27-29,36]) renders it obsolete, unless one wants to distinguish a special multifunction Γ. The conclusion of Proposition 1, part (b), was mentioned by Park (to the best of our knowledge only in [20], Example 1).

Author's Response: Note that van de Vel's convexity in 1993 is more restrictive than those of Sortan in 1984 [37]. We already showed that there are plenty of examples satisfying $D \not\subset E$. Moreover we showed Proposition 1 early in 2006 [16]. On the final part of Comment 1 of [12], van de Vel's convexity $(X,\mathcal{G})$ can be an abstract convex space in the later sense when X has a topology. Note that we can not construct any KKM theory on van de Vel's convexity, but our $(E,D;\Gamma)$ with a topology on E has so many new results.

Comment 2: L*-operators in 2008 (cf. [11]) — An L*-operator on X is any map $\Lambda : \langle X\rangle \multimap X$ that satisfies the following condition:

(*) If $A \in \langle X\rangle$ and $\{U_x : x \in A\}$ is a cover of X by non-empty open sets, then there exists $B \subset A$ such that $\Lambda(B) \cap \bigcap\{U_x : x \in B\} \neq \emptyset$.

A topological space X together with an L*-operator Λ is referred to as an L*-space and it is denoted by $(X;\Lambda)$. Thus . . . the convex hull operator on a linear topological space is an L*-operator on that space. Examples of L*-operators, and thus of L*-spaces, abound. In fact, one can define an L*-operator on arbitrary topological space X. Simply set $\Lambda(A)$ to be an any dense subset of X.

Let $(E,D;\Gamma)$ be an abstract convex space. Following Park, . . . the contrapositive version (with slight modifications) of the statement asserting that $(E,D;\Gamma)$ satisfies the partial KKM principle, where the closed sets $G(x)$ have been replaced by their complements $S(x)$, has the following form:

(**) If $S : D \multimap E$ is an open-valued multimap and $E = \bigcup_{x\in A} S(x)$ for some $A \in \langle D\rangle$, then there exists a $B \in \langle A\rangle$ such that $\Gamma(B) \cap \bigcap\{S(x) : x \in B\} \neq \emptyset$.

In Park's terminology, an abstract convex space $(E,D;\Gamma)$ satisfying (**) is referred to as possessing *the Fan type matching property* (see [21]).

Theorem 4. *Let $(E;\Gamma)$ be an abstract convex space, where E is a topological space. $(E;\Gamma)$ satisfies the partial KKM principle if and only if Γ is an L^*-operator on E.*

Author's Response: This part of [12] is already clarified in [29] as follows:

For an abstract convex space $(E, D; \Gamma)$, let us consider the following statements:

(A) The KKM principle. *For any closed-valued* [*resp., open-valued*] *KKM map* $G : D \to 2^E$*, the family* $\{G(z)\}_{z \in D}$ *has the finite intersection property.*

(B) The Fan matching property. *Let* $S : D \to 2^E$ *be a map satisfying*
(B.1) $S(z)$ *is open* [*resp., closed*] *for each* $z \in D$*; and*
(B.2) $E = \bigcup_{z \in M} S(z)$ *for some* $M \in \langle D \rangle$.
Then there exists an $N \in \langle M \rangle$ *such that*

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

(C) The Fan-Browder fixed point property. *Let* $S : E \to 2^D$, $T : E \to 2^E$ *be maps satisfying*
(C.1) $S^-(z) := \{x \in E \mid z \in S(x)\}$ *is open* [*resp., closed*] *for each* $z \in D$*;*
(C.2) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*
(C.3) $E = \bigcup_{z \in M} S^-(z)$ *for some* $M \in \langle D \rangle$.
Then T *has a fixed point* $x_0 \in X$*; that is,* $x_0 \in T(x_0)$.

Theorem 1. (Characterizations of the KKM spaces) *For an abstract convex space* $(E, D; \Gamma)$*, the statements* (A), (B), *and* (C) *are equivalent.*

Our Theorem 1 is more general than [11, Theorem 4]. For more details, see [21], where some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

Comment 3: The KKM principle is the statement that the property (**) also holds for any open-valued KKM map. An abstract convex space is called a KKM space if it satisfies the KKM principle. It's been an open problem whether there is a space satisfying the partial KKM principle that is not a KKM space (see, e.g., [24,26]). Example 1 in [12] provides an affirmative answer to this problem.

Author's Response: At first seen [12], the present author thought Example 1 in [12] is incorrect because of the following:

Theorem 4.2. [18] *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space, and* $F : E \multimap Z$*. Suppose that for any* $A \in \langle D \rangle$ *with* $|A| = n + 1$*, the set* $F(\Gamma_A)$ *in its induced topology is a normal space. If* $F \in \mathfrak{KC}(E, Z)$*, then* $F \in \mathfrak{KO}(E, Z)$*. The converse also holds.*

However, Szymanski's examples in [12] shows that the above statement is wrong. Moreover, he also found that its proof is incorrect. The present author appreciates his efforts for this and apologizes to all the readers on this matter.

Acknowledgements. The author would like to express his gratitude to Professors Ben-El-Mechaiekh and Szymansky for their kind comments and subsequent communications with the author.

References

[1] A. Amini, M. Fakhar, J. Zafarani, *Fixed point theorems for the class S-KKM mappings in abstract convex spaces*, Nonlinear Anal. **66** (2007), 14–21.
[2] H. Ben-El-Mechaiekh, *Thoughts on KKM*, Personal Communications, 2013.
[3] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, J.-V. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–150.
[4] R. Bielawski, *Simplicial convexity and its applications*, J. Math. Anal. Appl. **127** (1987), 155–171.
[5] A. Granas, *Quelques méthodes topologique èn analyse convexe*, Méthdes topologiques en analyse convexe, Sém. Math. Supér. 110, 11–77, Press. Univ. Montréal., 1990.
[6] A. Granas, M. Lassonde, *Sur un principe géométrique en analyse convexe*, (French. English summary) [*On a geometric principle in convex analysis*], Studia Math. **101** (1991), 1–18.
[7] C. D. Horvath, *Topological convexities, selections and fixed points*, Topology Appl. **155** (2008), 830–850.
[8] D. C. Kay and E. W. Womble, *Axiomatic convexity theory and relationships between the Caratheodory, Helly and Radon numbers*, Pacific J. Math. **38**(2) (1971), 471–485.
[9] K. Keimel and A. Wieczorek, *Kakutani property of the polytopes implies Kakutani property of the whole space*, J. Math. Anal. Appl. **130** (1988), 97–109.
[10] S. Krynski, *A characterization of discrete topological spaces as convexity spaces*, Demonstratio Math. **22**(2) (1989), 475–478.
[11] W. Kulpa, A. Szymanski, *Applications of general infimum principles to fixed-point theory and game theory*, Set-valued Anal. **16** (2008), 375–398.
[12] W. Kulpa, A. Szymanski, *Some remarks on Park's abstract convex spaces*, Preprint.
[13] M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.
[14] S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. Appl. Math. **7** (2000), 1–28.
[15] S. Park, *New subclasses of generalized convex spaces*, Fixed Point Theory and Applications (Y. J. Cho, ed.), 91–98, Nova Sci. Publ., New York, 2000.
[16] S. Park, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum **11** (2006), 67–77.
[17] S. Park, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. National Inst. Math. Sci. **2**(2) (2007), 35–47.
[18] S. Park, *Remarks on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps in abstract convex spaces*, Nonlinear Anal. Forum **12**(1) (2007), 29–40.
[19] S. Park, *Examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps on abstract convex spaces*, Soochow J. Math. **33** (2007), 477–486.
[20] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008), 1–27.
[21] S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex Anal. **9** (2008), 1–20.
[22] S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. **69** (2008), 4352–4364.

[23] S. Park, *Generalized convex spaces, L-spaces, and FC-spaces*, J. Glob. Optim. **45**(2) (2009), 203–210.

[24] S. Park, *The KKM principle in abstract convex spaces: equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.

[25] S. Park, *The Fan minimax inequality implies the Nash equilibrium theorem*, Appl. Math. Lett. **24**(12) (2011), 2206–2210.

[26] S. Park, *Generalizations of the Nash equilibrium theorem in the KKM theory*, Fixed Point Theory Appl. 2010, Article ID 234706.

[27] S. Park, *New generalizations of basic theorems in the KKM theory*, Nonlinear Anal. **74** (2011), 3000–3010.

[28] S. Park, *A genesis of general KKM theorems for abstract convex spaces*, J. Nonlinear Anal. Optim. **2**(1) (2011), 121–132 .

[29] S. Park, *Remarks on simplicial spaces and L^*-spaces of Kulpa and Szymanski*, Comm. Appl. Nonlinear Anal. **19**(1) (2012), 59–69.

[30] S. Park, *Abstract convex spaces, KKM spaces and ϕ_A-spaces*, Nonlinear Anal. Forum **17** (2012) 1–10.

[31] S. Park, *Applications of multimap classes in abstract convex spaces*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 51(2) (2012), 1–27. http://www.nas.go.kr/data/thesis /list.jsp?

[32] S. Park, *Comments on the FWC-spaces of H. Lu and J. Zhang*, Nonlinear Anal. Forum **18** (2013), 33–38.

[33] S. Park, *Remarks on an example of the GFC-spaces of Khanh, Long and Quan*, Nonlinear Anal. Forum **18** (2013), 39–44.

[34] S. Park, *A genesis of general KKM theorems for abstract convex spaces: Revisited*, J. Nonlinear Anal. Optim. **4**(1) (2013), 127–132.

[35] S. Park, *A review of the KKM theory on ϕ_A-spaces or GFC-spaces*, Advances in Fixed Point Theory **3**(2) (2013), 355–382.

[36] S. Park, H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU **18** (1993), 1–21.

[37] J. P. Penot, *Fixed point theorems without convexity*, Bull. Soc. Math. France Mémoire **60** (1979), 129–152.

[38] B. P. Sortan, *Introduction to Axiomatic Theory of Convexity*, Kishyneff, 1984. [Russian with English summary].

[39] M. van de Vel, *A selection theorem for topological convex structures*, Trans. Amer. Math. Soc. **336** (1993), 463-496.

[40] M.L.J. van de Vel, *Theory of convex structures*, North-Holland Mathematical Library, 50, 1993.

[41] A. Wieczorek, *The Kakutani property and the fixed point property of topological spaces with abstract convexity*, J. Math. Anal. Appl. **168** (1992), 483–499.

[42] S. Xiang, S. Xia, J. Chen, *KKM lemmas and minimax inequality theorems in abstract convexity spaces*, Fixed Point Theory and Applications 2013, 2013:209 doi:10.1186/1687-1812-2013-209.