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COMMENTS ON "SOME REMARKS ON PARK'S ABSTRACT CONVEX SPACES"

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Abstract. Recently, Kulpa and Szymanski published an article entitled "Some remarks on Park's abstract convex spaces" [Top. Meth. Nonlinear Anal. **44**(2) (2014) 369–379]. The present short note is to trace out the history of that article and to respond to some remarks given there.

1. Introduction

Since we began to study the KKM theory in 1992, we introduced several types of abstract convex spaces like G-convex spaces, ϕ_A-spaces, (partial) KKM spaces, and of multimap classes like acyclic maps, the admissible $\mathfrak{A}_c^\kappa$-maps, the better admissible $\mathfrak{B}$-maps, the KKM-class $\mathfrak{KC}$, $\mathfrak{KD}$. Each occasion of such appearances, many authors tried to generalize, modify or imitate our new things, some of them are constructive and some deconstructive. In some cases, we reviewed or criticized or commented to related papers in order to defend our stand-point to improve our new-born theory.

Recently, Kulpa and Szymanski [3] published an article entitled "Some remarks on Park's abstract convex spaces". The present short note is to trace out the history of that article and to respond to some remarks given there.

In Section 2, we give only a small portion of basic concepts in our KKM theory on abstract convex spaces as a preliminary. Section 3 is devoted to introduce abstracts of previous papers of Kulpa-Szymanski and ours related to [3]. Finally, in Section 4, we state our responses to some remarks given in [3].

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2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of nonempty finite subsets of a set D.

In order to unify various types of convexity spaces appeared in the KKM theory, we introduced the following in 2006 [4]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Note that we clearly stated the following in 2006 [4]:

"Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [15], where the bibliography lists 283 papers. For any subset $X \subset E$, its $\mathcal{C}$-*convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is $\mathcal{C}$-*convex* if $X = \mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E\rangle \multimap E$ given by $\Gamma_A := \mathrm{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

Notice that our abstract convex space $(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E."

Later, we add to assume E is a topological space in an abstract convex space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

3. Previous articles related to [3]

In this section, we introduce abstracts of previous articles of Kulpa - Szymanski and ours related to [3].

3.1. Kulpa and Szymanski: Set-valued Anal. 2008 [1]

Abstract: The main result of the paper is a series of theorems, called here Infimum Principles. As applications, we derive some well-known results related to fixed point, minimax, and equilibrium theorems including the Nash equilibrium theorem and Gale-Nikaido theorem. Our study is based on and utilizes the techniques of simplicial structures and CO families. This approach enables us to derive not only classical theorems but also stimulates new research.

3.2. Park: CANA 2012 [7]

Abstract: In a recent paper, Kulpa and Szymanski [1] introduced a series of theorems called Infimum Principles in simplicial spaces. As applications, they derive fixed point theorems due to Schauder, Tychonoff, Kakutani, and Fan-Browder; minimax theorems; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Ky Fan minimax inequality. Their study is based on and utilizes the techniques of simplicial structure and the Fan-Browder map. In this paper, we recall that for any abstract convex spaces satisfying abstract KKM principle, we can deduce such classical theorems without using any Infimum Principles. Moreover, we note that the newly defined L^*-spaces in [1] are particular types of abstract convex spaces satisfying the abstract KKM principle, and add some remarks on them.

3.3. Kulpa and Szymanski: Preprint [2]

Abstract: We discuss S. Park's abstract convex spaces and their relevance to convexities and L^*-operators. We construct an example of a space satisfying the partial KKM principle that is not a KKM space, thus solving a problem by Park. We show that if a compact Hausdorff space admits a 2-continuous L^*-operator, then the space must be locally connected continuum and it has the fixed point property provided the covering dimension is 1. We also show that the unit circle admits no 2-continuous L^*-operators.

3.4. Park: RIMS 2014 [8]

Abstract: Since we introduced the concept of abstract convex spaces in the KKM theory, some readers raised certain questions or comments on them. In the present note, we want to clarify such things on the concept of abstract convex spaces raised by Ben-El-Mechaiekh [*Thoughts on KKM*, Personal Communications, 2013] and Kulpa and Szymmanski [2]. A number of examples and related matters are also added.

3.5. Kulpa and Szymanski: TMNA 2014 [3]

Abstract: We discuss S. Park's abstract convex spaces and their relevance to classical convexities and L^*-operators. We construct an example of a space satisfying the partial KKM principle that is not a KKM space. The existence of such a space solves a problem by Park.

4. Comments on some remarks in [3]

After our preprint of [9] was sent to Kulpa and Szymanski, their preprint [2] was revised to [3]. In this new version, some remarks against our abstract convex spaces appear. We give comments on such remarks as follows.

Remark 2.6(a) ([3]). Park's general notion of *abstract convex space* $(E, D; \Gamma)$ seem to be a bit superfluous. For if $|D| \leq |E|$, then we may rename the elements of D by elements of a subset of E. If so, then the original abstract convex space $(E, D; \Gamma)$ can be regarded as given in the form $(E \supset D; \Gamma)$. No example of abstract convex space $(E, D; \Gamma)$ with $|D| > |E|$ has ever been considered.

Comments by the present author. This remark is not adequate as the following example in [6, Example 3(3)] shows:

Exmaple. Let $\mathcal{C} := \mathcal{C}[0,1]$ be the class of all real continuous functions on $[0,1]$ and $\mathcal{P} := \mathcal{P}[0,1]$ the subclass of all polynomials $p(x)$ on $x \in [0,1]$ with real coefficients. Let $\varepsilon > 0$. For each $f \in \mathcal{C}$, choose a fixed $p_f \in \mathcal{P}$ which is ε-near to f, that is, $\max_{x\in[0,1]} |f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle \mathcal{C} \rangle \multimap \mathcal{P}$ be defined by $\Gamma_A := \text{co}\,\{p_{f_i}\}_{i=0}^n \subset \mathcal{P}$ for each $A = \{f_i\}_{i=0}^n \in \langle \mathcal{C} \rangle$. Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear map such that $e_i \mapsto p_{f_i}$. Then $(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$ is a G-convex space satisfying $X \subsetneq D$.

Note that the renaming is another superfluous job.

Here we give an example of abstract convex space $(E, D; \Gamma)$ with $|D| > |E|$:

Exmaple. Let $E = \{0, 1\}$ with the discrete topology, $D = \mathbb{R}$ the set of real numbers, and $\Gamma : \langle D \rangle \multimap E$ is defined by $\Gamma(A) = \{0\}$ for $A \subset \mathbb{Q}$ the set of rational numbers, $\Gamma(A) = \{1\}$ for $A \subset \mathbb{R} \setminus \mathbb{Q}$, and $\Gamma(A) = E$ for other cases.

Let $G : D \multimap E$ be defined by $G(x) = \{0\}$ if $x \in \mathbb{Q}$ and $G(x) = \{1\}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then G is a KKM map, but $(E, D; \Gamma)$ is not a (partial) KKM space.

Remark 2.6(b) ([3]). By Lemma 2.2 and Theorem 2.5 (of [3]), any convexity space can be considered as an abstract convex space and vice versa. Consequently, classifying (known and previously considered) convexity spaces as abstract convex spaces (cf. [9] through [23] in References of [3]) is to some extend obsolete, unless one wants to distinguish a special multifunction Γ.

Comments by the present author. This remark is also obsolete. We already noted in 2006 [4] as follows:

The convexity space $(E, \mathcal{C})$ in the classical sense of Sortan [15] "becomes our abstract convex space $(E; \Gamma)$. Notice that our abstract convex space

$(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E."

However, the authors of [3] gave too long arguments to show this simple fact for only convexity space of van de Vel in 1993. Recall that there are many similar convexities (see [9]), but all of them could not reach to meet the KKM theory.

Note also that, recently we introduced a large number of applications of our abstract convex spaces to various fields in mathematics; see [10]-[14] and the references therein.

Remark 2.10 ([3])**.** Our Example 2.8 [3] along with Proposition 2.7 [3] shows that, e.g.. Corollary 3.4 from [7] or Theorem 4.2 from [5] are false as stated and need to be corrected.

Comments by the present author. For Theorem 4.2 in [5], we already noted its incorrectness and apologized to all the readers on this matter [9]. This also works for Corollary 3.4 in [5].

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