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RECOLLECTING JOINT WORKS WITH B. E. RHOADES

Sehie Park

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In this paper, we recall briefly the contents of all of the joint papers of S. Park and B. E. Rhoades in the period 1979-1993 for the readers' convenience. Each title of Sections 1-11 represents the main topic of the corresponding joint work.

Introduction

Since the Banach contraction principle appeared in 1922, several hundreds of generalizations or imitations of the contractive condition have followed. The first attempt to compare or unify or classify such various definitions of contractive mappings was given by Billy E. Rhoades in 1977 [1]. Motivated by this work, the present author also tried to continue to classify such definitions in 1980 [2] with some new fixed point theorems.

Before that time the present author recognized that certain collaboration with Billy should be necessary. In the period 1979-1993, Billy and the present author published 11 joint works, and the aims of most of them were to get generalizations or unifications of many of known fixed point theorems on metric spaces.

According to Billy's recollection: "Our joint collaboration began in 1979, when Sehie came to Bloomington during part of his sabbatical leave. That collaboration resulted in papers [3], [5], [6],

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and [7]. During his next visit to Indiana in 1986, we prepared papers [12], [13], and [14]. Papers [8], [11], and [15] were written by correspondence.

During the time of our collaboration Sehie was also doing joint work with others, as well as overseeing the work of his own doctoral students. Our collaboration has ceased only because our research interests have moved in different directions."(*Mathematical Works of Sehie Park* in a publication of commemorating the retirement of S. Park in 2001.)

In this paper we recall briefly the contents of all of the joint efforts of Park and Rhoades for the readers' convenience. Each title of the following sections represents the main topic of the corresponding paper. These papers appeared in journals of Hungary, Japan, Korea, Poland, India, Romania, and England in the period 1979-1993, and most of them are hard to locate, nowadays. Therefore, the present paper should be helpful to certain interested readers.

Here the present author would like to express that he has very pleasant memories from working together, and he admires Billy's continuing to work very hard.

1. General Fixed Point Theorems - Acta Sci. Math. (1980) [3]

In this paper we establish several fixed point theorems involving hypotheses weak enough to include a number of known theorems as special cases.

Let f be a selfmap of a topological space X. The orbit $O(x)$ of $x \in X$ under f is defined by $O(x) = \{x, f(x), f^2(x), \dots\}$. A function $G : X \longrightarrow [0, \infty)$ is called f-orbitally lower semicontinuous at a point $p \in X$ if, for every $x_0 \in X$, $x_{n_k} \longrightarrow p$ implies $G(p) \leq \liminf_k G(x_{n_k})$, where $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ is defined by $x_{n+1} = f(x_n)$, i.e. $\{x_n\}_{n=1}^{\infty} = O(x_0)$.

Theorem 1. *Let d be a nonnegative real valued function defined on $X \times X$ such that $d(x, y) = d(y, x)$ and $d(x, y) = 0$ iff $x = y$. If there exists a point $u \in X$ such that $\lim_n d(f^{n+1}(u), f^n(u)) = 0$, and if $\{f^n(u)\}$ has a convergent subsequence with limit $p \in X$, then p is a fixed point of f iff $G(x) = d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

Theorem 2. *Let f be a selfmap of a metric space (X, d) satisfying*:

(i) $\delta(O(x)) < \infty$ *for each $x \in X$, where δ denotes the diameter.*

(ii) *There exists a $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(iii) *There exists a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\varphi(t) < t$ for each $t > 0$ and the inequality*

$$d(f(x), f^2(x)) \leq \varphi(\delta(O(x) \cup O(f(y)))) \quad for\ each\ x, y \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

These results extend works of Pal-Maiti, Park, Hegedüs and Daneš. A 2-metric space version of Theorem 2 is added.

2. Theorems of Hegedüs and Kasahara - Math. Sem. Notes (1981) [5]

Let ω denote the set of nonnegative integers, $\mathbb{R}_+$ the set of nonnegative real numbers, f and g selfmaps of a metric space (X, d).

A point $x_0 \in X$ is called *regular for* f *and* g, or, simply, *regular*, if a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfying $fx_{n+1} = gx_n$ always exists.

Let $O(x_0) = \{fx_n \mid n \in \omega\}$, and let $\delta[O(x_0)]$ denote the diameter of $O(x_0)$.

THEOREM 1. *Let f and g be commuting selfmaps of a metric space (X, d), $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, ϕ nondecreasing, continuous from the right, and satisfying $\phi(t) < t$ for each $t > 0$. Suppose that there exists a regular point $x_0 \in X$ such that $\{fx_n\}_{n=1}^{\infty}$ has a cluster point a_0, which is regular. If*

(1) $d(gx, gy) \leq \phi(\delta[O(x) \cup O(y)])$

for each $x, y \in \{x_n\}_{n=1}^{\infty} \cup \{a_n\} \cup \{fa_0\}$, where $\{a_n\}$ is defined by $fa_{n+1} = ga_n$ for $n \in \omega$ and, if f is continuous at a_0, then fa_0 is a common fixed point for f and g and $\{fx_n\}_{n=1}^{\infty}$ converges to a_0.

If (1) is satisfied for all regular points $x, y \in X$, then fa_0 is the unique common fixed point of f and g.

Theorem 1 extends works of Kasahara, Hegedüs, and Park.

THEOREM 2. *Let f and g be commuting selfmaps of a complete metric space (X, d), f continuous, and $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, ϕ nondecreasing, continuous from the right, and satisfying $\phi(t) < t$ for each $t > 0$. If every point of X is a regular point and (1) is satisfied for each $x, y \in X$, then f and g have a unique common fixed point and $\{fgx_n\}$ converges to the fixed point for each $x \in X$.*

Theorem 2 extends works of Ciric, Kasahara and Hegedüs.

3. Meir-Keeler Type Contractions - Math. Jap. (1981) [6]

Two fixed point theorems are established. The first result considers pairs of selfmaps f, g satisfying a contractive condition which, for $f = g$, properly generalizes the Meir-Keeler type. The second result involves an analogous contractive definition for a pair of commuting maps, and extends a result of Park-Bae.

For any set X, $x_0 \in X$, f, g self-maps of X, define $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1})$, $n = 0, 1, 2, \dots$. Then $\{x_n\}_{n=1}^{\infty}$ is called the (f, g)-*orbit* of x_0.

Theorem 2. *Let X be an (f, g)-orbitally complete metric space, f, g orbitally continuous selfmaps of X satisfying: Given an $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\varepsilon \le \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2} < \varepsilon + \delta$$

$$\Longrightarrow d(fx,gy) < \varepsilon.$$

Then for each $x_0 \in X$, either (1) f or g has a fixed point in the (f, g)-orbit $\{x_n\}_{n=1}^{\infty}$ of x_0, or (2) f and g have a common unique fixed point p and $\lim x_n = p$.

Let f be a continuous selfmap of X, $C_f = \{g : X \longrightarrow X \mid fg = gf \text{ and } gX \subset fX\}$. For $x_0 \in X$, the sequence $\{fx_n\}_{n=1}^{\infty}$ is called the f-iteration of x_0 under g, and is defined by $fx_n = gx_{n-1}$, $n = 0, 1, 2, \dots$, with the understanding that, if $fx_n = fx_{n+1}$ for some n, then $fx_{n+j} = fx_n$ for each $j \ge 0$.

Theorem 4. *Let f be a continuous selfmap of a complete metric space (X, d), $g \in C_f$ and continuous and satisfying the following condition: Given an $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\varepsilon \le \max\{d(fx,fy), d(fx,gx), d(fy,gy), \frac{d(fx,gy)+d(fy,gx)}{2} < \varepsilon + \delta$$

$$\Longrightarrow d(gx,gy) < \varepsilon.$$

Then f and g have a unique common fixed point p in X, and, for any $x_0 \in X$, any f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = p$.

4. Some Fixed Point Theorems - Yonsei Univ. 1981 [7]

This paper is a survey of some of the contractive definitions that have appeared in the literature in the past twenty years (1961-80). Emphasis should be placed on the word *some*, since we do not aim for complete coverage. We have attempted to point out some of the interesting and useful new idears which have appeared on the scene, and to state some of the more general fixed point theorems.

Unless otherwise stated, (X, d) will denote a complete metric space. For each of the contractive definitions discussed herein, the selfmap f has the property that the unique fixed point p can be found by repeated iteration of f, applied to some initial choice x_0.

In 1977, Rhoades [1] compared and classified a large number of contractive definitions, and in 1980, Park [2] considered additional fourteen definitions, all of which contain the quasicontraction of Ciric as a special case.

In [7], the following in [2] are introduced:

THEOREM 1. (Aδ). *Let f be a continuous compact selfmap of a metric space X satisfying*

(Aδ) $d(fx, fy) < \delta(x, y) = \text{diam}[O(x) \cup O(y)]$.

Then f has a unique fixed point, and furthermore, for any $\alpha \in (0, 1)$ there exists a metric ρ on X relative to which f satisfies $\rho(fx, fy) \leq \alpha\rho(x, y)$ for all $x, y \in X$.

A point $x \in X$ is said to be regular if $O(x)$ has a finite diameter.

Theorem 2. (Cδ). *Let f be a selfmap of a metric space x. Suppose that there exists a regular point $u \in X$ such that*

(1) *$O(u)$ has a regular cluster point $p \in X$, and*

(2) *for each $\varepsilon > 0$ there exists a positive number $\varepsilon_0 < \varepsilon$ and a $\delta_0 > 0$ such that, for each $x, y \in O(u) \cup O(p)$,*

(Cδ) $\varepsilon \leq \delta(x,y) < \varepsilon + \delta_0$ *implies* $d(fx, fy) \leq \varepsilon_0$

holds.

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n(u) \longrightarrow p$.

Moreover, Theorem 1 of [5] and Theorem 2 of [6] are introduced as another key results.

This paper was presented by B. E. Rhoades at the 775$^{\text{th}}$ meeting of the American Mathematical Society, Special Session on Fixed Point Theory, in Bloomington, IN, on April 12-13, 1980, and by S. Park at the Conference on Mathematics, Yonsei University, Seoul, on November 27-28, 1981.

5. Theorems of Fisher and Janos - Bull. Polon. (1982) [8]

Two fixed point theorems, which are substantial generalizations of a similar result of Fisher and Janos, are established.

Theorem 1. *Let f be a continuous selfmap of a metric space x. Suppose there exists a regular point $u \in X$ such that*

(1) *$O(u)$ has a regular cluster point $a \in X$, and*

(2) *for fixed integers p and q, given $\varepsilon > 0$ there exists a positive number $\varepsilon_0 < \varepsilon$ and a $\delta_0 > 0$ such that, for each $x, y \in O(u)$ $\cup O(p)$,*

$$\varepsilon \leq \delta(x,y) < \varepsilon + \delta_0 \textit{ implies } d(f^p x, f^q y) \leq \varepsilon_0$$

holds.

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n(u) \longrightarrow p$.

An example showing that condition (2) is indeed a proper extension of Fisher's extension of Ciric's quasicontraction.

Moreover, the following extension of Theorem 1 (Aδ) of [2] is given:

THEOREM 2. *Let f be a continuous compact selfmap of a metric space X satisfying*

$$d(f^p x, f^q y) < \delta(x,y) = \mathrm{diam}[O(x) \cup O(y)]$$

for some fixed integers p and q. Then f has a unique fixed point, and furthermore, for any $\alpha \in (0,1)$ there exists a metric ρ on X relative to which f satisfies $\rho(fx, fy) \leq \alpha\rho(x,y)$ for all $x, y \in X$.

Theorem 2 contains some results of Janos and Rhoades.

6. Characterizations for Metric Completeness - Math. Jap. (1986) [11]

Several authors have characterized completeness of a metric space by using a fixed point theorem. The two theorems of this paper encompas some previous results as well as future theorems of this type.

Let B be a class of selfmaps of closed subsets of a metric space X such that if, any $g \in B$ has a fixed point, then X is complete. Examples of B are the classes of Banach contraction (Hu) and the Kannan type contractions (Reich, Subrahmanyam).

Let A be a class of selfmaps of closed subsets of X containing B such that the completeness of X implies the existence of a fixed point for any map in A. Examples of A containing the preceding examples of B are classes of maps satisfying the conditions of Meir-Keeler, Hegedüs-Szilágyi, Caristi, Tasković, and Hikida.

The following is the main result:

Theorem 1. *X is complete if and only if any map in A has a fixed point.*

Let B' be a class of selfmaps defined on X such that if every map in B' satisfies a certain condition Q, then X is complete. An example of B' is the maps satisfying an equivalent formulation of Caristi's theorem as given by Weston.

Let A' be a class of selfmaps of X containing B' such that completeness of X implies that every map in A' satisfies a condition P, where P implies Q. An example of A' is the map satisfying the Ekeland variational principle as given by Sullivan.

Theorem 2. *X is complete if and only if any map in A' satisfies the condition P.*

Some examples of Theorem 2 are added.

7. Istratescu's Convex Contractions - Jñānābha (1987) [12]

In two papers in 1982, V. I. Istratescu considered several classes of maps related to contraction maps by introducing convexity conditions with respect to the iterates of maps, and obtained a number of fixed point theorems for such maps. However, we notice that some of his results are closely related to our previous works. In fact, in [12], we show that some of Istratescu's key results are obvious consequences of our previous results in [4] and [8]. Further, we obtain generalizations of some of his results as follows:

THEOREM 3. *Let f be a continuous selfmap of a metric space X and $F : X \longrightarrow \mathbb{R}$ a continuous function satisfying*:

(i) *$F(fx) \leq F(x)$ for all $x \in X$,*

(ii) *if $x \neq fx$, then $F(fx) < F(x)$, and*

(iii) *for some $x \in X$, the sequence $\{F^n x\}_{n=1}^{\infty}$ has a cluster point $z \in X$.*

Then z is a fixed point of f.

THEOREM 5. *Let f be a selfmap of a topological space X and $F : X \longrightarrow \mathbb{R}$ a l.s.c. function satisfying*:

(i) *if $x \neq fx$, $x \in X$, then $F(fx) < F(x)$, and*

(ii) *the range of f is relatively countably compact.*

Then z is a fixed point of f.

Moreover, since continuous densifying selfmaps of a complete metric space have relatively compact orbits whenever the orbits are bounded, Theorem 5 can also be applied to such maps.

8. For Expansion Mappings - Math. Jap. (1988) [13]

We show that the first four theorems in the paper by Wang, Li, Gao, and Iseki [Math. Japon. 29 (1984), 631–636] are all consequences of the following equivalent formulation of the Caristi-Kirk theorem due to the first author [9]:

PROPOSITION 7. *Let X be a set, M a complete metric space, and $f, g : X \longrightarrow M$ maps such that*

(1) *f is surjective, and*

(2) *there exists a lower semicontinuous function* $\phi : M \longrightarrow \mathbb{R}^+$ *satisfying*

$$d(fx, gx) \leq \phi(fx) - \phi(gx)$$

for each $x \in X$.

Then f *and* g *have a coincidence point.*

9. On the Caristi-Kirk-Browder Theorem - Bull. Roumanie (1988) [14]

In 1985, J. M. Rao [Bull. Math. de la Soc. Math. Roumanie 29 (1985), 79-80] gave a Caristi-type fixed point theorem for multifunctions, ... In [14], we show that Rao's theorem follows from an equivalent formulation of Caristi's theorem due to Maschler and Peleg [SIAM J. Control 14 (1976), 985–995].

The following is a simplified form of the main result of Park [10]:

Theorem 1. *Let* M *be a complete metric space,* $F : M \longrightarrow \mathbb{R}^+$ *a lower semicontinuous function. Then the following statements are equivalent*:

(i) *There exists a point* $v \in M$ *such that* $d(v, w) > F(v) - F(w)$ *for each* $w \in M, w \neq v$.

(ii) *If* f *is a selfmap of* M *such that* $d(x, fx) \leq F(x) - F(fx)$ *for each* $x \in M$*, then* f *has a fixed point* $v \in M$.

(iii) *If* $T : M \longrightarrow 2^M$ *is a multimap such that, for all* $x \in M$*,* $Tx \neq \emptyset$ *and, for every* $y \in T(x)$*,* $d(x, y) \leq F(x) - F(y)$*, then* T *has a stationary point* $v \in M$*, that is,* $T(v) = \{v\}$.

Statement (i) is due to Ekeland, (ii) to Caristi, and (iii) to Maschler and Peleg. We showed that Rao's theorem is a consequence of Theorem 1 (iii).

10. On Generalizations of the Meir-Keeler Type Contractions - JMAA (1990) [15]

In 1969, Meir and Keeler obtained a remarkable generalization of the Banach contraction principle. Since then, there have appeared a number of generalizations of their result. In 1981, Park and Bae extended the Meir-Keeler theorem to two commuting maps by adopting Jungck's method. This influenced many authors, and, consequently, a number of new results in this line followed. Recent works of Sessa and others contain common fixed point theorems of four maps satisfying certain contractive type conditions. In [15], we give a new result which encompasses most of such generalizations of the Meir-Keeler theorem. Further our result also includes many other generalizations of the Banach contraction principle.

Our main result is the following corrected form.

THEOREM 1. *Let* (X, d) *be a complete metric space, and* S, T *selfmaps of* X *with* S *or* T *continuous. Suppose there exists a sequence* $\{A_i\}_{n=1}^{\infty}$ *of selfmaps of* X *satisfying*

(i) *either* $A_i : X \longrightarrow SX \cap TX$ *for each* i*, or*

(i′) $S, T : X \longrightarrow \bigcap_i A_i X$*,*

(ii) *each* A_i *is compatible with* S *and* T*, and each*

(iii) A_i *weakly commutes with* S *at each point* ξ *for which* $A_i\xi = S\xi$ *and each* A_i *weakly commutes with* T *at each point* η *for which* $A_i\eta = T\eta$*, and*

(iv) *for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that, for each* $x, y \in X$,

$$\varepsilon \leq M_{ij}(x,y) < \varepsilon + \delta \textit{ implies } d(A_i x, A_j y) < \varepsilon,$$

where

$$M_{ij}(x,y) = \max\left\{ d(Sx,Ty), d(Sx,A_i x), d(Ty,A_j y), \frac{d(Sx,A_j y) + d(Ty,A_i x)}{2} \right\}.$$

Then A_i, S, *and* T *have a unique common fixed point.*

11. On Generalizations of the Meir-Keeler Type Contractions - JMAA (1993) [16]

The original paper by three of the authors [15] contains two mistakes. It is the purpose of this note to provide corrections for these errors. The reader should consult [15] for any special definitions or terminology not defined in this paper.

In this paper, it is shown that one way to correct [15] is to modify the contractive definition (iv) as follows:

(iv) there exists a lower semicontinuous function $\delta : (0,\infty) \longrightarrow (0,\infty)$ such that, for any $\varepsilon > 0$, $\delta(\varepsilon) > \varepsilon$, and for $x, y \in X$,

$$\varepsilon \leq M_{ij}(x,y) < \delta(\varepsilon) \textit{ implies } d(A_i x, A_j y) < \varepsilon.$$

Remark. The following references are only related to Park and Rhoades and given in chronological order. For other authors' works, see the references in the ones listed here.

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THE NATIONAL ACADEMY OF SCIENCES
SEOUL 137-044
REPUBLIC OF KOREA

AND

Department of Mathematical Sciences
Seoul National University
Seoul 151-747
Republic of Korea.

E-mail: shpark@snu.ac.kr; sehiepark@gmail.com