EVOLUTION OF THE FAN-BROWDER TYPE ALTERNATIVES

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Abstract. Corresponding to each stage of development of the KKM theory, the Fan-Browder fixed point theorem on Fan-Browder type multimaps has been generalized to hundreds of different forms or reformulated to the maximal element theorem with numerous generalizations. Recall that the theorem can be stated as an alternative form; that is, its conclusion is "the Fan-Browder map has either a fixed point or a maximal element. Our aim in this paper is to trace the evolution of the Fan-Browder type alternatives from the origin to the most recent generalization of them.

1. Introduction

The KKM theory is originated from the Knaster-Kuratowski-Mazurkiewicz (KKM for short) theorem of 1929 [17]. Since then, it has been found a large number of results which are equivalent to the KKM theorem; see [24,31]. Typical examples of the most remarkable and useful equivalent formulations are Ky Fan's KKM lemma of 1961 [6] and his geometric property of 1961 and 1969 [6,9]. This property was applied by Fan to many problems in nonlinear analysis.

In 1968 Browder [4] independently obtained a fixed point theorem in the convenient form of Fan's geometric lemma. He applied his theorem to a systematic treatment of various results in the KKM theory. This is also applied by Borglin and Keiding [3] and Yannelis and Frabhakar [41] to the existence of maximal elements in mathematical economics. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to Park [24,38] and the references therein.

At the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84 [6-12]. A number of intersection theorems and their applications to various equilibrium problems followed. Then, the KKM theory was extended to convex spaces by Lassonde in 1983 [19], and to c-spaces (or H-spaces) by Horvath in 1983-93 [13-16] and others. Since 1993 the theory has been extended to generalized convex (G-convex) spaces; see [23,39,40]. Since 2006 the main theme of the theory has became abstract convex spaces in the sense of Park; see [25,26]. The basic theorems in the theory have many applications to various equilibrium problems in nonlinear analysis and other fields.

Parallel to such development of the KKM theory, the Fan-Browder fixed point theorem on Fan-Browder type multimaps has been generalized to hundreds of different forms or reformulated to the maximal element theorem with numerous generalizations. Therefore the theorem can be stated alternative form; that is, its conclusion is "either (i) the Fan-Browder map has a fixed point or (ii) a maximal

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2010 *Mathematics Subject Classification.* Primary 47H04, 47H10; Secondary 46A16, 46A55, 49J27, 49J35, 52A07, 54C60, 54H25, 55M20, 91B50.

Key words and phrases. KKM theorem, Fan's geometric property, Fan-Browder fixed point theorem, maximal element, convex space, H-space, G-convex space, ϕ_A-space, (partial) KKM space, abstract convex space.

element. Our aim in this paper is to trace the evolution of the Fan-Browder type alternatives from the origin to the most recent generalization of them.

This paper is organized as follows. In Section 2, we introduce Fan's 1961 geometric property, Browder's 1968 theorem, and the related works of Borglin and Keiding [3] and Yannelis and Frabhakar [41] on maximal elements. Section 3 deals with the evolution of Fan-Browder type alternatives. Actually, we give its generalizations to convex spaces of Lassonde, convex spaces with the coercivity due to S. Y. Chang, H-spaces originated by Horvath, and G-convex spaces and ϕ_A-spaces of Park. In Section 4, we introduce recently obtained basic concepts and results of the KKM theory such as the abstract convex spaces, KKM spaces, and generalized KKM theorems. In Section 5, we show that one of our general alternative subsumes previously obtained inequalities on various types of KKM spaces in Section 3. Finally, we add some related historical remarks in Section 6.

2. The origin of the Fan-Browder type alternatives

2.1. **Fan's geometric property.** A milestone on the history of the KKM theory was erected by Ky Fan [6]. He extended the KKM theorem to infinite dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma 2.1 (Fan [6]). *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem.

Fan also obtained the following geometric or section property of convex sets, which is equivalent to the preceding Lemma.

Lemma 2.2 (Fan [6]). *Let X be a compact convex set in a Hausdorff topological vector space. Let A be a closed subset of $X \times X$ with the following properties:*

(i) *$(x, x) \in A$ for every $x \in X$.*
(ii) *For any fixed $y \in X$, the set $\{x \in X : (x, y) \notin A\}$ is convex (or empty).*

Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

Fan applied this Lemma to give a simple proof [61] of the Tychonoff theorem and to prove two results generalizing the Pontrjagin-Iohvidov-Kreĭn theorem on existence of invariant subspaces of certain linear operators. Also, Fan [7] applied his KKMF lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff theorem.

Moreover, "a theorem concerning sets with convex sections" was applied to prove the following results in Fan [8]:

> An intersection theorem (which generalizes the von Neumann lemma (1937)).
> An analytic formulation (which generalizes the equilibrium theorem of Nash [20,21](1951) and the minimax theorem of Sion (1958)).
> A theorem on systems of convex inequalities of Fan (1957).
> Extremum problems for matrices.
> A theorem of Hardy-Littlewood-Pólya concerning doubly stochastic matrices.

A fixed point theorem generalizing Tychonoff (1935) and Iohvidov (1964).
Extensions of monotone sets.
Invariant vector subspaces.
An analogue of Helly's intersection theorem for convex sets.

Motivated by Browder's work [4] on fixed point theorems, Fan [9] deduced the following from his geometric lemma:

Theorem 2.3 (Fan [9]). *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$||y_0 - f(y_0)|| = \underset{x \in X}{\to} \mathrm{Min}||x - f(y_0)||.$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.)

Fan also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. Those are known as best approximation theorems and applied to obtain generalizations of the Brouwer theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps in Fan [9].

2.2. Origin of fixed point and maximal element theorems

In 1968 Browder [4] independently obtained Fan's geometric lemma [6] in the convenient form of a fixed point theorem. Since then the following is known as the Fan-Browder fixed point theorem:

Theorem 2.4 (Browder [4]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^-(y) = \{x \in K \mid y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later this is also known to be equivalent to the Brouwer fixed point theorem. Browder [4] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding [3] and Yannelis and Frabhakar [41], to the existence of maximal elements in mathematical economics. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to [24,38] and the references therein.

Browder proved his theorem by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem.

Later the Hausdorffness in Fan's KKM lemma, geometric property, and Browder's theorem was known to be redundant by Lassonde in 1983. Moreover, the Fan property and Browder's theorem are known to be equivalent to the KKM theorem. Consequently Browder's theorem can be obtained by a simple KKM method.

In 1976 Borglin and Keiding [3] restated Fan's geometric property [6,8] as follows:

Let X be a non-empty, convex, compact subset of some (Hausdorff) topological vector space V. A correspondence $\varphi : X \multimap X$ is KF if φ is convex-valued (possibly empty-valued), graph of φ is open in $X \times X$ and $x \notin \varphi(x)$ for $x \in X$. One has the following:

Theorem 2.5 (Ky Fan). *Let $\varphi : X \multimap X$ be KF then there is $\overline{x} \in X$ such that $\varphi(\overline{x}) = \emptyset$.*

In [3] a corollary to this theorem is obtained which on the one hand allows a slight generalization of the results of Gale and Mas-Colell (1975) on a competitive equilibrium and Shafer and Sonnenschein (1975) in the same direction and their (1974) 'the Arrow-Debreu lemma for abstract economies' on the other hand makes it possible to prove them in a rather different way. The general idea is to reduce the search for an equilibrium to the search for an equilibrium action for a suitably chosen fictitious agent.

In 1983 Yannelis and Prabhakar [41] presented some mathematical theorems including Browder's theorem which are used to generalize previous results on the existence of maximal elements and of equilibrium. Their main theorem in [41] is a new existence proof for an equilibrium in an abstract economy, which is closely related to a previous result of Borglin-Keiding, and Shafer-Sonneschein, but allows for an infinite number of commodities and a countably infinite number of agents.

Any binary relation R in a set $X \times Y$ can be regarded as a multimap $T : X \multimap Y$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map T if $T(x_0) = \emptyset$.

3. Evolution of the Fan-Browder type alternatives

3.1. **Fan-Browder type.** Two theorems in Subsection 2.2 can be modified to the following *Fan-Browder alternative*:

Theorem 3.1. *Let K be a nonempty compact convex subset of a (Hausdorff) topological vector space. Let $T : K \multimap K$ be a map, where for each $x \in K, T(x)$ is a convex subset of K. Suppose further that for each y in K, $T^-(y)$ is open in K.*

Then either (i) *there exists x_0 in K such that $x_0 \in T(x_0)$; or* (ii) *there exists y_0 in K such that $T(y_0) = \emptyset$.*

Remark 3.2. Note that (i) tells that T has a *fixed point* and (ii) tells that T has a *maximal element.*

In 1979, Fan obtained a theorem [11, Theorem 10] which is equivalent to the following:

Theorem 3.3. *Let X be a nonempty convex set in a Hausdorff topological vector space. Let $T : X \multimap X$ be a map such that*

(3.3.1) *for each $x \in X, T(x)$ is convex,*

(3.3.2) *for each $y \in X$, $T^-(y)$ is open, and*

(3.3.3) *there is a nonempty compact convex set $L \subset X$ such that $L \cap T(x) \neq \emptyset$ for every $x \in X \setminus L$.*

Then either (i) *there exists x_0 in X such that $x_0 \in T(x_0)$; or* (ii) *there exists y_0 in X such that $T(y_0) = \emptyset$.*

The following is due to Ben-El-Mechaiekh et al. [2, Theorem 5]:

Theorem 3.3′. *In Theorem 3.3, replace (3.3.3) by the following:*

(3.3.3)′ *there exist a compact subset K of X and a compact convex subset L of X such that $L \cap T(x) \neq \emptyset$ for every $x \in X \setminus K$.*

Then the same conclusion holds.

Note that for $K = L$, Theorem 3.3′ reduces to Theorem 3.3.

Theorem 3.3 ⇒ Theorem 3.1. Put $X = L = K$ in Theorem 3.3. Then (3.3.3) holds trivially. □

The "coercivity" or "compactness" condition (2.3) is first appeared in [11].

4. Lassonde type

The concept of convex sets in a topological vector space is extended to convex spaces by Lassonde in 1983 [19], and further to c-spaces by Horvath in 1983-93 [13-16]. A number of other authors also extended the concept of convexity for various purposes.

Let $\langle D \rangle$ denote the set of nonempty finite subsets of a set D.

Definition 4.1. Let X be a subset of a vector space and D be a nonempty subset of X. We call (X, D) a *convex space* if $\mathrm{co}\, D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$ (see Park [23]). For a convex space (X, D), a subset C of X is said to be *D-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\mathrm{co}\, A \subset C$.

If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [19].

A nonempty subset L of a convex space X is called a *c-compact* set [19] if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

Lassonde [19] presented a simple and unified treatment of a large variety of minimax and fixed point problems. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem; the principal topics treated by him may be listed as follows:

Fixed point theory for multimaps;
Minimax equalities;
Extensions of monotone sets;
Variational inequalities;
Special best approximation problems.

The following variant of [19, Theorem I.1.1] is motivated from our earlier works:

Theorem 4.2. *Let (X, D) be a convex space and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(4.2.1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*
(4.2.2) *$T(x)$ is convex for each $x \in X$,*
(4.2.3) *$S^-(y)$ is open for each $y \in D$, and*
(4.2.4) *for some c-compact set $L \subset X$, $X \setminus S^-(L \cap D)$ is compact.*

Then either (i) *there exists x_0 in X such that $x_0 \in T(x_0)$; or* (ii) *there exists x_1 in X such that $S(x_1) = \emptyset$.*

Remark 4.3. Note that (ii) implies S^- is not surjective.

Theorem 4.2 $\Rightarrow$ Theorem 4.2'. We use Theorem 4.2 with $X = D$ and $S = T$.

Under the assumptions of Theorem 3.3', (4.2.1)-(4.2.3) are satisfied. Since L is a compact convex subset of a Hausdorff t.v.s., it is a c-compact set (see [19]). Moreover, (3.3.3)' of Theorem 3.3' implies the existence of a $y \in L \cap T(x)$ for each $x \in X \setminus K$. Hence

$$x \in T^-(y) \;\Rightarrow\; X \setminus K \subset T^-(L) \;\Rightarrow\; X \setminus T^-(L) \subset K.$$

Since $X \setminus T^-(L)$ is a closed subset of the compact set K, it is compact. Therefore (4.2.4) is satisfied. Now, by Theorem 4.2, the conclusion follows. □

Remark 4.4. As shown in the above proof, the Hausdorffness in Theorems 3.3 and 4.2.3' is essential. However, we show that the Hausdorffness in the Fan-Browder fixed point theorem and Theorem 3.1 is redundant by showing the following:

Theorem 4.2 $\Rightarrow$ Theorem 3.1. It suffices to show that if $X = K = L = D$ is compact, then (4.2.4) holds trivially. In fact, for the c-compact subset $L := X$, $X \setminus S^-(L \cap D) = X \setminus S^-(X)$ is compact since $S^-(X)$ is open. □

4.1. **Chang type.** In 1989, S. Y. Chang [5] obtained a KKM theorem with a coercivity condition eliminating the concept of c-compact sets. From a Fan-Browder type fixed point theorem equivalent to her KKM theorem, we notice that the following modification of [22, Theorem 5] holds:

Theorem 4.5. *Let (X, D) be a convex space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(4.5.1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*

(4.5.2) *$T(x)$ is convex for each $x \in X$,*

(4.5.3) *$S^-(y)$ is open for each $y \in D$, and*

(4.5.4) *for each finite subset N of D, there exists a compact convex subset L_N of X containing N such that*

$$L_N \setminus K \subset S^-(L_N \cap D).$$

Then either (i) *S has a maximal element, or* (ii) *T has a fixed point.*

Theorem 4.5 $\Rightarrow$ Theorem 4.2. It suffices to show that (4.2.4) implies (4.5.4) with $K := X \setminus S^-(L \cap D)$. Since L is c-compact, for each finite $N \subset D$, there exists a compact convex subset $L_N \subset X$ containing $L \cup N$ satisfying (4.2.4). Then

$$L_N \setminus K \subset X \setminus K \subset S^-(L \cap D) \subset S^-(L_N \cap D).$$

Hence (4.5.4) holds. □

4.2. **Horvath type.** In this subsection, we follow [22].

Definition 4.6. A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

For an H-space $(X, D; \Gamma)$, a subset C of X is said to be H-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

A multimap $F : D \multimap X$ is said to be H-KKM if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$. A subset L of X is called an H-*subspace* of $(X, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D \rangle$, $\Gamma_A \cap L$ is contractible.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [13-16] or an H-space by Bardaro and Ceppitelli [1]. Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces, and are examples of compact c-spaces without having the fixed point property.

In the frame of H-spaces, there have been appeared several Fan-Browder fixed point theorems. The following is motivated from [22].

Theorem 4.7. *Let $(X, D; \Gamma)$ be an H-space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(4.7.1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*

(4.7.2) *$T(x)$ is H-convex for each $x \in X$,*

(4.7.3) *$S^-(y)$ is open for each $y \in D$, and*

(4.7.4) *for each finite subset N of D, there exists a compact H-convex subset L_N of X containing N such that*

$$L_N \setminus K \subset S^-(L_N \cap D).$$

Then either (i) *S has a maximal element, or* (ii) *T has a fixed point.*

Theorem 4.7 ⇒ Theorem 4.5. Any convex space is an H-space. □

4.3. **For G-convex spaces.** From 1996, the following became one of the main themes of the KKM theory [39,40].

Definition 4.8. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \mathrm{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}$ $\subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$.

For a G-convex space $(X, D; \Gamma)$, a subset C of X is said to be G-*convex* w.r.t. some $D' \subset D$ if for each $A \in \langle D' \rangle$, we have $\Gamma_A \subset C$.

A map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

There are lots of examples of G-convex spaces; see [31] and the references therein. So, the KKM theory was extended to the study of KKM maps on G-convex spaces.

The following is for G-convex spaces motivated by several works on G-convex spaces [39,40].

Theorem 4.9. *Let $(X, D; \Gamma)$ be a G-convex space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(4.9.1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*

(4.9.2) *$T(x)$ is G-convex w.r.t. $S(x)$ for each $x \in X$,*

(4.9.3) *$S^-(y)$ is open for each $y \in D$, and*

(4.9.4) *for each finite subset N of D, there exists a compact G-convex subset L_N of X w.r.t. some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \setminus K \subset S^-(D').$$

Then either (i) *S has a maximal element, or* (ii) *T has a fixed point.*

Theorem 4.9 ⇒ Theorem 4.7. Any H-space is a G-convex space. Let $D' := L_N \cap D$. Then (4.7.2) and (4.7.4) imply (4.9.2) and (4.9.4), resp. □

4.4. **For ϕ_A-spaces.** Since 2007 the following became one of the main themes of the KKM theory [25,28,29].

Definition 4.10. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle}) \text{ or simply } (X, D; \phi_A)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

For a ϕ_A-space $(X, D; \phi_A)$, a subset C of X is said to be ϕ_A-*convex* with respect to some $D' \subset D$ if for each $A \in \langle D' \rangle$, we have $\phi_A(\Delta_{|A|-1}) \subset C$.

We define a KKM map $G : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ if, for each $N \in \langle D \rangle$ and $J \subset N$, we have

$$\phi_N(\Delta_{|J|-1}) \subset G(J)$$

where $\Delta_{|J|-1}$ is the face of $\Delta_{|N|-1}$ corresponding to J.

There are lots of examples of ϕ_A-spaces; see [25,28,29,31] and the references therein. So, the KKM theory was extended to the study of KKM maps on ϕ_A-spaces.

The following Fan-Browder fixed point or maximal element theorem for ϕ_A-spaces is new:

Theorem 4.11. *Let $(X, D; \phi_A)$ be a ϕ_A-space, K a compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ maps satisfying the conditions*

(4.11.1) *$S^-(y) \subset T^-(y)$ for each $y \in D$,*

(4.11.2) *$T(x)$ is ϕ_A-convex w.r.t. $S(x)$ for each $x \in X$,*

(4.11.3) *$S^-(y)$ is open for each $y \in D$, and*

(4.11.4) *for each finite subset N of D, there exists a compact ϕ_A-convex subset L_N of X w.r.t. some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \setminus K \subset S^-(D').$$

Then either (i) S has a maximal element or (ii) T has a fixed point.

Theorem 4.11 $\Rightarrow$ Theorem 4.9. Any G-convex space is a ϕ_A-space such that $\phi_A(\Delta_{|A|-1}) \subset \Gamma_A$ for each $A \in \langle D \rangle$. Therefore, by putting $\phi_A(\Delta_{|A|-1}) := \Gamma_A$, Theorem 4.11 reduces to Theorem 4.9. □

5. Abstract convex spaces and general KKM theorems

Recall the following [26,27,31]:

Definition 5.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 5.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 5.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent works [26,27,31], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed that many important results therein are related to the partial KKM principle.

Example. We give known examples of partial KKM spaces; see [31] and the references therein:

(1) Every ϕ_A-space is a KKM space. More precisely, for a ϕ_A-space $(X, D; \phi_A)$, the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle$, $|A| = n + 1$, is a KKM space. This KKM space may not G-convex; see [34].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.

(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.

(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X \rangle$, Kirk and Panyanak showed that the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle.

(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.

(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Recently, Kulpa and Szymanski [18] found some partial KKM spaces which are not KKM spaces.

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems [30,32,33]:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ be a multimap satisfying*
(1) G has closed [resp., open] values; and
(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).
Then $\{G(z)\}_{z \in D}$ has the finite intersection property.
Further, if
(3) $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,
then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Since all of the spaces in Section 4 are ϕ_A-spaces and hence KKM spaces, Theorem A can be applied to them.

Consider the following related four conditions for a map $G : D \multimap E$:

(a) $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z \in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [30,32,33]:

Theorem B. *Let $(E, D; \Gamma)$ be a partial KKM space [that is, $1_E \in \mathfrak{KC}(E, D, E)$] and $G : D \multimap E$ a map such that*
(1) $\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$]; and
(2) there exists a nonempty compact subset K of E such that either
(a) $\bigcap_{z \in M} \overline{G(z)} \subset K$ for some $M \in \langle D \rangle$; or
(b) for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Remark 5.4. 1. Every ϕ_A-space is a (partial) KKM space. Hence, Theorem B works for ϕ_A-spaces.

2. We may assume K is closed. Then the closure notations in each coercivity condition can be eliminated.

There are several generalizations of the Fan-Browder fixed point theorem in abstract convex spaces as follows:

Theorem C (Park [27,31]) *An abstract convex space* $(X, D; \Gamma)$ *is a KKM space iff for any maps* $S : X \multimap D$, $T : X \multimap X$ *satisfying*

(1) $S^-(z)$ *is open [resp., closed] for each* $z \in D$, *and*

(2) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$,

either (i) T *has a fixed point* $x_0 \in X$*; that is* $x_0 \in T(x_0)$, *or* (ii) E *can not be covered by a finite number of* $S^-(z)$*'s for* $z \in D$.

6. The Fan-Browder alternatives in abstract convex spaces

From Theorem B, we have the following general alternative:

Theorem 6.1. *Let* $(E, D; \Gamma)$ *be a partial KKM space, and* $S : E \multimap D$, $T : E \multimap E$ *maps. Suppose that*

(6.1.1) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*;*

(6.1.2) *there exists a nonempty compact subset* K *of* E *such that either*

(a) $\bigcap_{z \in M} \overline{E \setminus S^-(z)} \subset K$ *for some* $M \in \langle D \rangle$*; or*

(b) *for each* $N \in \langle D \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{E \setminus S^-(z)} \subset K.$$

(α) *If* S^- *is transfer open-valued, then either* (i) T *has a fixed point, or* (ii) S *has a maximal element in* K.

(β) *If* S^- *is unionly open-valued, then either* (i) T *has a fixed point, or* (ii) S *has a maximal element in* E.

Proof. Suppose that $x \notin T(x)$ for all $x \in E$. Let $G(z) := E \setminus S^-(z)$ for $z \in D$. Then $G : D \multimap E$ is a KKM map.

In fact, suppose on the contrary that there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset G(N)$; that is, there exists an $x \in \Gamma_N$ such that $x \notin G(z)$ for all $z \in N$. In other words, $N \in \langle D \setminus G^-(x) \rangle$ and

$$z \notin G^-(x) \Leftrightarrow x \notin G(z) = E \setminus S^-(z) \Leftrightarrow x \in S^-(z) \Leftrightarrow z \in S(x)$$

for all $z \in N$. Hence $N \subset S(x)$ and, by (6.1.1), we have $x \in \Gamma_N \subset T(x)$. This is a contradiction.

Note that (6.1.2) implies (2) of Theorem B. Therefore, by Theorem B, we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

(α) If S^- is transfer open-valued, then G is transfer closed-valued, then

$$K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$$

by Theorem B. Then we have a $y_0 \in K$ and $y_0 \in G(z) = E \setminus S^-(z)$ or $z \notin S(y_0)$ for all $z \in D$. Hence S has a maximal element in K.

(β) Similarly, S has a maximal element in E. □

Theorem 6.1 $\Rightarrow$ Theorem 4.11. Note that every ϕ_A-space is a KKM space and all requirements of Theorem 6.1(α) is satisfied by Theorem 4.11. □

Remark 6.2. 1. The above proof shows that Theorem 6.1 subsumes all of Theorems 3.1-4.11 and numerous variants or particular forms of them appeared in previous works.

2. In Section 5 of [35], we give some known Fan-Browder-type fixed point theorems and maximal element theorems mainly due to the present author and indicate their known applications by other authors.

From Theorem 6.1, we have the following Fan-Browder type fixed fixed point theorem:

Corollary 6.3. *Let $(X, D; \Gamma)$ be a partial KKM space, X compact, and $S : D \to 2^X$, $T : X \to 2^X$ maps. Suppose that*
(1) *S is unionly open-valued (that is, S^c is intersectionally closed-valued);*
(2) *for each $x \in X$, $M \in \langle S^-(x)\rangle$ implies $\Gamma_M \subset T^-(x)$; and*
(3) *$X = S(D)$.*
Then T has a fixed point.

The following is one of our versions of Fan's geometric lemma:

Corollary 6.4 ([34])**.** *Let $(X, D; \Gamma)$ be a compact partial KKM space and $A \subset X \times X$, $C \subset D \times X$. Suppose that*
(1) *$(x, x) \in A$ for every $x \in X$;*
(2) *for each $z \in D$, $\{y \in X \mid (z, y) \in C\}$ is intersectionally closed;*
(3) *for any fixed $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid (z, y) \notin C\} \subset \{x \in X \mid (x, y) \notin A\}$.*
Then there exists a point $y_0 \in X$ such that $D \times \{y_0\} \subset C$.

7. Applications and remarks

Recall that the main applications of recent generalized KKM theorems are as follows; see [36,37]:

Vector variational-type inequalities;
Various quasi-equilibrium problems;
Eigenvector problems;
Set-valued minimax inequality;
Fixed point theorems;
Generalizations of Nash equilibrium theorem;
Variational inclusion problem;
Simultaneous nonlinear inequalities problem;
Differential inclusion problem;
(Vector mixed) quasi-variational inequality;
(Vector mixed) quasi-complementarity problem;
Traffic network problem;
Quasi-monotone vector equilibrium problem;
Generalized vector equilibrium problem;
Generalized (implicit) vector variational-like inequality;
Set equilibrium problem;
Set-valued mixed (quasi-)variational inequalities;
Variational-hemivariational inequalities.

In 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [26,27,31] and the references therein. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. Now the KKM theory becomes the study of spaces satisfying the partial KKM principle.

In our work [31], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. We add more than a dozen statements as their applications, including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [31] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces. Note that [31] contains incorrect statements such as (V), (VI), Theorem 4.5, (XVI), (XVII). These can be easily corrected.

Finally, in a recent work [38], from a general form of the KKM type theorems or some properties of KKM type maps on abstract convex spaces, we deduced several Fan-Browder type alternatives, coincidence or fixed point theorems, and other results. These theorems unify and generalize various particular results of the same kinds recently due to a number of authors for particular types of abstract convex spaces. We added there some historical remarks and further comments to improve many of those results and their applications.

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