RECENT APPLICATIONS OF THE GENERALIZED KKM THEOREMS

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Abstract. In this review, firstly, we recall our versions of general KKM type theorems for abstract convex spaces. Secondly, we introduce relatively recent applications of various generalized KKM type theorems due to other authors in the 21st century. Finally, some general comments to improve such applications are added.

1. Introduction

One of the earliest equivalent formulations of the Brouwer fixed point theorem (1912) is a theorem of Knaster, Kuratowski, and Mazurkiewicz (simply, the KKM theorem) in 1929 [27], which was concerned with a particular type of multimaps called KKM maps later.

The KKM theory, first called by the author in 1992 [32,33], is the study of applications of various equivalent formulations of the KKM theorem and their generalizations. At the beginning, the basic theorems in the theory and their applications were established for convex subsets of topological vector spaces mainly by Fan (1961-84). A number of intersection theorems and their applications to equilibrium problems followed. Then, the KKM theory has been extended to convex spaces by Lassonde (1983), and to c-spaces (or H-spaces) by Horvath (1984-93) and others. Since 1993 the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. Those basic theorems have many applications to various equilibrium problems in nonlinear analysis and other fields.

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In 2006-09, we proposed new concepts of abstract convex spaces and the partial KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [41] and the references therein. Now the KKM theory becomes the study of abstract convex spaces satisfying the partial KKM principle, that is, an abstract form of the KKM theorem.

In our previous survey [44], firstly, we recalled Ky Fan's contribution to the KKM theory based on his celebrated 1961 KKM lemma (or the KKMF theorem). Secondly, we introduced applications of the Fan lemma due to other authors in the 21st century. Finally, some historical remarks were added.

This review is a continuation of [44]. Since our G-convex space theory appeared, many authors tried to obtain various KKM type theorems and their applications in that frame. Our aim in this review is to introduce such applications in the 21st century and to indicate that such applications should be included in the recent study of abstract convex spaces initiated by ourselves.

Section 2 of this paper deals with the basic notions on abstract convex spaces. In Section 3, we introduce recently obtained general KKM type theorems for abstract convex spaces. Recall that in 1990's, we established the foundations of the KKM theory on generalized convex spaces (G-convex spaces); see [34,40,41] and references therein. Motivated by our G-convex space theory or others, a large number of authors obtained applications of KKM type theorems other than Fan's. Section 4 is devoted to such applications. Finally, in Section 5, some general comments to improve such applications are added.

All references given by the form (year) can be found in [34] or references of [35-44].

2. Abstract convex spaces and partial KKM spaces

Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [41]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our recent work [35-37], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example. We give known examples of partial KKM spaces; see [41] and the references therein:

(1) The original KKM theorem [27] is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

(2) A triple $(X, D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [14] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

(3) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde (1984) for $X = D$. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.

(4) A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected)

subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X;\Gamma)$ is called a *c*-space by Horvath (1984-83).

(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi are particular ones of *c*-spaces.

(6) Hyperbolic spaces due to Reich and Shafrir [45] are particular *c*-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.

(7) A *generalized convex space* or a G-*convex space* $(X, D;\Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

For $X = D$, G-convex spaces reduce to L-spaces due to Ben-El-Mechaiekh et al. Recall that all examples (1)-(7) are all G-convex spaces.

(8) A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n+1$. Every G-convex space is a ϕ_A-space and every ϕ_A-space can be made into G-convex spaces in several ways. Later ϕ_A-spaces are called GFC-spaces by Khanh et al. [25].

Every ϕ_A-space is a KKM space.

(9) The extended long line L^* is a KKM space $(L^*, D;\Gamma)$ with the ordinal space $D := [0,\Omega]$. But L^* is not a G-convex space.

(10) Any topological semilattice $(X, \leq)$ with path-connected interval, introduced by Horvath and Llinares is an abstract convex space. A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.

(10) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$ is the intersection of all closed convex subsets of H that contain A. Kirk and Panyanak showed that the triple $(H \supset X;\Gamma)$ is a partial KKM space.

(11) According to Horvath, a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions. For Horvath's convex space $(X;\Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.

(12) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ \Longrightarrow \text{H-space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ \Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

3. General KKM theorems for abstract convex spaces

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems [38,42,43]:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) *G has closed [resp., open] values; and*

(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).*

Then $\{G(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Consider the following related four conditions for a map $G : D \multimap E$:

(a) $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z \in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [42,43]:

Theorem B. *Let $(E, D; \Gamma)$ be a partial KKM space [that is, $1_E \in \mathfrak{KC}(E, D, E)$] and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$]; and*

(2) *there exists a nonempty compact subset K of E such that either*

(i) *$\bigcap_{z \in M} \overline{G(z)} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

Theorem B can be extended to $F \in \mathfrak{KC}(E, D, Z)$ instead of $1_E \in \mathfrak{KC}(E, D, E))$ as the follows [42,43]:

Theorem C. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E, D, Z)$, *and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $K \supset \bigcap \{\overline{G(y)} \mid y \in M\}$ *for some* $M \in \langle D \rangle$*; or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

Since we can assume K is closed without loss of generality, the closure notations in the coercivity conditions (2) in Theorems 2 and 3 might be erased.

4. Applications of other types of the KKM theorem

In 1990's, we established the foundations of the KKM theory on generalized convex spaces (G-convex spaces); for the literature, see [34,40,41] and references therein. Motivated by our G-convex space theory or others, a large number of authors obtained applications of KKM type theorems other than Fan's. We recall such applications in this section. Most of such applications are based on particular forms of Theorem A–C. We will not check this matter.

(I) Summary of Ansari et al. [4] in 2000: "In this paper, a general version of the KKM theorem is derived by using the concept of generalized KKM mappings introduced by Chang and Zhang. By employing our general KKM theorem, we obtain a general minimax inequality which contains several existing ones as special cases. As applications of our general minimax inequality, we derive an existence result for saddle point problems in a general setting. We also establish several existence results for generalized variational inequalities and generalized quasi-variational inequalities."

The main tool of this paper, Theorem 2.1, was known. The other results are routine applications of Theorem 2.1 or variants of known results. Recall that the generalized KKM map of Chang and Zhang is simply a KKM map on a G-convex space.

(II) In 2000, Kirk et al. [26] established the KKM theory on hyperconvex metric spaces, which are particular type of c-spaces.

(III) Abstract of Kalmoun and Rihai [22] in 2001: "Slightly modifying the topological KKM Theorem of Park and Kim (1996), we obtain a new existence theorem for generalized vector equilibrium problems related to an admissible multifunction. We work here under the general framework of G-convex space which does not have any linear structure. Also, we give applications to greatest element, fixed point and vector saddle point problems. The results presented in this paper extend and unify many results in the literature by relaxing the compactness, the closedness and the convexity conditions."

(IV) Abstract of Balaj [5] in 2003: "We introduce a new concept of generalized Knaster-Kuratowski-Mazurkiewicz (KKM) family of sets and related to this we obtain fixed point theorems and sections results in homotopically trivial spaces."

This paper is motivated by Park [31] and Horvath [19].

(V) Abstract of Hou [20] in 2003: "We first prove a new fixed point theorem from which the Kakutani's fixed point theorem in locally convex topological vector spaces is immediately extended to H-spaces. Then, we establish a new existence theorem of equilibrium for generalized games in H-spaces, by applying our fixed-point theorem."

This paper is based on a KKM lemma of Horvath on H-spaces.

(VI) Abstract of Jin et al. [21] in 2005: "Some new generalized KKM type theorems which include S-G-L-KKM theorems and G-L-KKM theorems are proved. As applications, some new minimax inequalities are given."

Recall that L-convex spaces mean L-spaces due to Ben-El-Mechaiekh et al.; see [40].

(VII) Abstract of Zeng et al. [53] in 2006: "In this paper, a generalized version of the famous KKM theorem is obtained by using the concept of generalized KKM mappings introduced by Chang and Zhang. By employing our generalized KKM theorem, we obtain a generalized minimax inequality which includes several existing ones as special cases. Further, by applying our generalized minimax inequality we establish an existence result for the saddle-point problem under general setting. Finally, we also derive some existence results for generalized equilibrium problems and generalized variational inequalities."

Recall that the generalized KKM map of Chang and Zhang is simply a KKM map on a G-convex space.

(VIII) From Abstract of Du and Deng [11] in 2007: "Some existence theorems of maximal elements for generalized $\mathcal{L}_{\theta,F_c}$-correspondence and generalized $\mathcal{L}_{\theta,F_c}$-majorized mappings are obtained in topological spaces without convexity. As applications, we establish new equilibrium existence theorems for qualitative games and generalized games with infinite sets of players and generalized $\mathcal{L}_{\theta,F_c}$-majorized preference correspondences in topological spaces without convexity."

This paper is based on certain artificial generalization of the KKM theorem.

(IX) Abstract of Fakhar and Zafarani [12] in 2007: "We give some new generalized R-KKM theorems in the nonconvexity setting of topological spaces. As an application we answer a question posed by Isac et al. for the lower and upper bounds equilibrium problem in topological spaces."

(X) Abstract of Fang and Huang [15] in 2007: "We introduce a new concept of generalized L-KKM mapping and establish some new generalized L-KKM type theorems without any convexity structure in topological spaces. As an application, an existence theorem of equilibrium points for an abstract generalized vector equilibrium problem is proved in topological spaces."

This paper concerns with L-spaces particular to G-convex spaces.

(XI) Abstract of Gonzalez et al. [17] in 2007: "Using a KKM-type theorem for L-spaces and L^*-KKM multifunctions, we obtain some results on the existence of fixed points and Nash equilibria in compact L-spaces."

The authors were concerned with particular types of G-convex spaces.

(XII) Abstract of Hammami [18] in 2007: "We present a generalized FKKM theorem and it's application to the existence of solution for the variational inequalities using a generalized coercivity type condition for correspondences defined in L-space."

(XIII) Abstract of Tang et al. [48] in 2007: "By using basic KKM theorem, a new matching theorem and some minimax inequalities for set-valued mappings defined on the FC-spaces are proved under very weak assumptions."

There appeared also many papers on FC-spaces which can be ignored.

(XIV) Alimohammady et al. [1] in 2008 introduced the notion of minimal generalized convex spaces and obtained the open and closed versions of the KKM principle in this new setting. Apparently this is motivated by Park's works on G-convex spaces. Their method consists in just replacing the topological structure in the relevant results by the more general minimal structure. Since any minimal space can be made into a topological space, results on minimal G-convex spaces can be deduced from the theory on G-convex spaces.

Almost same time, Park [Nonlinear Funct. Anal. Appl. 13(2) (2008), 179–1913] introduced a new concept of abstract convex minimal spaces which generalizes the authors' minimal generalized convex spaces. Using the new concept, he deduced generalizations of the KKM principle, coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces from the KKM type theorems on abstract convex spaces.

(XV) Abstract of Amini-Harandi et al. [2] in 2008: "A best proximity pair for a set-valued map $F : A \multimap B$ with respect to a set-valued map $G : A \multimap A$ is defined, and a new existence theorem of best proximity pairs for continuous set-valued maps is proved in nonexpansive retract metric spaces. As an application, we derive a coincidence point theorem."

(XVI) Kulpa and Szymanski [28] in 2008 introduced a series of theorems called Infimum Principles and applied them to some classical results in KKM theory. Most of the results in this paper originate from the Theorem of Indexed Families (Theorem 3) and are based on the techniques of simplicial structures and CO families (equivalently, multimaps with nonempty convex values and open fibers). Here a simplicial space is a topological space having a certain collection of singular simplices. As applications, they derived fixed point theorems due to Schauder, Tikhonov, Kakutani, and Fan-Browder; the minimax theorem; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Fan minimax inequality. Finally, they suggested a way of extending their results to a wider class of topological spaces called L*-spaces.

Note that Theorem 3 is a consequence of a Fan type matching theorem. In a KKM space, we can deduce such classical theorems without using any Infimum Principles. In fact, such theorems are consequences of some equivalents of the KKM theorem on a simplicial space and hence are typical particular results of KKM theory for abstract convex spaces; see [41] and the references therein. Moreover, we note that simplicial spaces and L*-spaces are particular types of KKM spaces.

(XVII) Abstract of Wen [50] in 2008: "A new KKM theorem is established in L-convex spaces. As applications, a Ky Fan matching theorem for compactly open covers, a Fan-Browder coincidence theorem, a Fan-Browder fixed point theorem and a maximal element theorem are obtained in L-convex spaces. . . . Finally, the equilibrium existence theorems for abstract economies and qualitative games in L-convex spaces are yielded."

Here L-convex spaces are L-spaces in the sense of Ben-El-Mechaiekh et al. All results are already known in more general forms.

(XVIII) In 2009, Chbani et al. [7] obtained the existence of solutions of an initial value problem by means of a recent generalization of the famous KKMF theorem in 2000 [6].

(XIX) Abstract of Ansari et al. [3] in 2009: "In this paper, we consider vector variational inequality and vector F-complementarity problems in the setting of topological vector spaces. We extend the concept of upper sign continuity for vector-valued functions and provide some existence results for solutions of vector variational inequalities and vector F-complementarity problems. Moreover, the nonemptiness and compactness of solution sets of these problems are investigated under suitable assumptions. We use a version of Fan-KKM theorem and Dobrowolski's fixed point theorem to establish our results. . . ."

Later it was known that Dobrowolski's theorem had the incorrect proof.

(XX) From Abstract of Chbani et al. [7] in 2009: "In this paper we are interested in the existence of solutions of an initial value problem, by means of a recent generalization of the famous KKM-Fan's lemma in 2000 [6]."

This lemma means the Fan minimax inequality.

(XXI) Abstract of Khanh et al. [25] in 2009: "We define a generalized KKM mapping, called T-KKM mapping, and the corresponding generalized KKM property, which include many counterparts existing in the literature. KKM-type theorems, coincidence theorems and geometric section theorems are established to generalize recent known results."

Note that any GFC-space is a ϕ_A-space due to Park more early.

(XXII) Abstract of Tang et al. [47] in 2009: "We generalize Ky Fan's minimax inequality to vector-valued function with values in a topological vector space acting on the product of two other topological vector spaces which are connected by another function. In these results, the concavity or convexity on a function is transferred to another function. And a sufficient condition for the existence of solution for a variational inclusion is given."

This paper is based on Cheng's generalization [9] in 1997 of the Fan KKM lemma.

(XXIII) From Abstract of Lu [30] in 2009: "The main purpose of this paper is to generalize the KKMF theorem under the nonconvexity setting of topological space. Furthermore, as its applications, existence theorems for a saddle point problem and the Nash equilibrium problem for non-cooperative games are obtained in general topological spaces without any convexity structure and linear structure."

(XXIV) Abstract of Yang and Deng [51] in 2009: "The new concept of FC-hull for a nonempty subset in FC-spaces is introduced, . . . On this basis, new classes of L_c-correspondences and L_c-majorized correspondences without open lower sections are introduced in FC-spaces. Some existence theorems of maximal elements for L_c-correspondences and L_c-majorized correspondences are proved in FC-spaces. As applications, some equilibrium existence theorems for qualitative games and generalized games with infinite set of players and L_c-majorized correspondences are established in FC-spaces."

The authors keep the misconception that FC-spaces include G-convex spaces.

(XXV) Abstract of Khan [23] in 2010: "In this paper, we introduced the generalized vector variational inequality-type problem and the generalized vector complementarity-type problem in the setting of topological vector space. By utilizing a modified version of the Fan-KKM theorem, we investigated the nonemptiness and compactness of solution sets of these problems without the demipseudomonotonicity assumption. Further, we prove that solution sets of both the problems are equivalent to each other under some suitable conditions."

Theorem 2.1 is an incorrect theorem of Dobrowolski.

(XXVI) Abstract of Khanh and Quan [24] in 2010: "Applying generalized KKM-type theorems established in our previous paper [25], we prove the existence of solutions to a general variational inclusion problem, which contains most of the existing results of this type. As applications, we obtain minimax theorems in various settings and saddle-point theorems in particular. Examples are given to explain advantages of our results."

(XXVII) Abstract of Yang and Deng [52] in 2010: "A new coincidence theorem for admissible set-valued mappings is proved in FC-spaces with a more general convexity structure. As applications, an abstract variational inequality, a KKM type theorem and a fixed point theorem are obtained."

This paper is based on the well-known coincidence theorem due to Park and Kim (1996) for admissible set-valued mappings in G-convex spaces.

(XXVIII) From Abstract of Tang et al. [49] in 2010: "We establish some new generalized KKM-type theorems based on weakly generalized KKM mapping without any convexity structure in topological spaces. As applications, some minimax inequalities and an existence theorem of equilibrium points for an abstract generalized vector equilibrium problem are proved in topological spaces."

(XXIX) Abstract of Farajzadeh and Zafarani [16] in 2010: "An existence result for the equilibrium problem is proved in a general topological vector space. As applications, existence results are derived for variational inequality problems, vector equilibrium problems and vector variational inequality problems. . . ."

This paper is based on a variant of the Fan KKM lemma.

(XXX) Abstract of Chebbi et al. [8] in 2011: "We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend Fan's matching theorem."

This paper is based on an extension of Fan's lemma to (quasi-) compactly closed KKM maps to an L-space. This extension is deduced from the classical KKM theorem which is equivalent to Fan's lemma.

(XXXI) Lin et al. [29] in 2011 proved a generalized KKM theorem, and applied it to a generalized equi-KKM theorem, common fixed point theorems for a family of multivalued maps, and the Kakutani-Fan-Glicksberg fixed point theorem. They also showed that an existence theorem of the common fixed point theorem is equivalent to the Kakutani-Fan-Glicksberg fixed point theorem.

(XXXII) From Abstract of Sankar Raj and Somasundaram [46] in 2011: "Let A, B be nonempty subset of a normed linear space X. We introduce a new class of multivalued mappings $\{T : A \multimap B\}$, called R-KKM mappings, which extends the notion of KKM mappings. First, we discuss some sufficient conditions for which the set $\bigcap\{T(x) : x \in A\}$ is nonempty. Using this nonempty intersection theorem, we attempt to prove a extended version of Fan-Browder multivalued fixed point theorem, in a normed linear space setting, by providing an existence of a best proximity point."

For R-KKM maps, see [39].

(XXXIII) Abstract of Colao et al. [10] in 2012: "An equilibrium theory is developed in Hadamard manifolds. The existence of equilibrium points for a bifunction is proved under suitable conditions, and applications to variational inequality, fixed point and Nash equilibrium problems are provided. The convergence of Picard iteration for firmly nonexpansive mappings along with the definition of resolvents for bifunctions in this setting is used to devise an algorithm to approximate equilibrium points."

Hyperbolic spaces due to Reich and Shafrir [46] are also particular cases of c-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic. See [46] and [36,41].

5. General comments

Recall that the main theme of all of the papers introduced in Section 4 can be summarized as follows:

Various forms of KKM-type nonempty-intersection theorems
Generalized (vector) equilibrium problem

Problems on maximal element, greatest element, fixed point, and coincidence point
Vector saddle point problem
Sections in homotopically trivial spaces
Equilibrium for generalized (or other types of) games
Vector minimax inequality
Generalized (vector) variational inequality
The lower and upper bounds equilibrium problem
The Nash equilibrium problem
Best proximity pairs
Equilibrium existence problems for abstract economies and qualitative games
Initial value problem
Equilibrium problem with monotone bifunctions
Vector complementarity problem
General variational inclusion problem
Generalized vector variational inequality-type problem
Generalized vector complementarity-type problem
Minimax theorems in various settings
Abstract generalized vector equilibrium problem
Equilibrium theory for Hadamard manifolds

We note the following comments:

(1) All of the papers introduced in Section 4 are concerned with G-convex spaces or variants of them. Hence they are abstract convex spaces satisfying the KKM principle.

(2) All of the papers in Section 4 are based on certain KKM type theorems which can be deduced from one of our basic KKM theorems A–C.

(3) Consequently, all results in the papers introduced in Section 4 can be stated in more general formulations based on more general KKM type theorems.

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