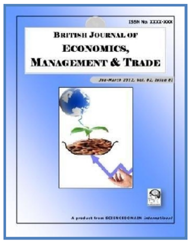

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Generalized Nash Equilibria of Nonmonetized Noncooperative Games on Lattices

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Authors' contributions

This work was carried out in collaboration between authors JL and SP. Author JL initialed the study. Authors JL and SP jointed together in writing the first draft of the manuscript and providing the proofs of all results. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we examine noncooperative games with both of the collection of strategies and the ranges of the utilities for the players are lattices, which are said to be nonmonetized. Then we extend the concept of Nash equilibria of noncooperative games to generalized Nash equilibria of nonmonetized noncooperative games. By applying some fixed point theorems in posets and by using the order preserving property of mappings, we prove an existence theorem of generalized Nash equilibria for nonmonetized noncooperative games.

Keywords: Lattice; Banach lattice; nonmonetized noncooperative game; generalized Nash equilibrium.

1. INTRODUCTION

In 1980, Giannessi [1] studied the optimization problems of mappings with ranges in a finite dimensional vector space equipped with a partial order induced by a cone. As significant consequences in this work, the concepts of vector optimization problems and vector variational inequalities were introduced in finite dimensional vector spaces. Since then, many authors have made many contributions on the vector optimization problems and their

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applications. As a generalization, the vector optimization concepts have been extended to Banach spaces by Huang and Fang [2], in 2005.

For most games in game theory, all utilities of the players are real valued functions; that is, the range of the utilities is a subset of real numbers (For example, see [3-10]). We think that there are a lot of literatures in game theory and in its related fields where the utilities of the players are not real valued functions as in social choice theory, voting games, implementation theory, …, the outcomes of some games may not be monetized; that is, the utilities of the players may not be represented by real valued functions. So the Nash equilibrium problems of such games, as optimization problems, turn to be vector optimization problems. Based on this motivation, Carl and Heikkilä [11], and Li [12-13] examined some noncooperative games with strategies as subsets of Banach spaces and with the ranges of the utility functions in some Banach lattices. Such games are said to be nonmonetized noncooperative. Since every Banach lattice is a set of objects equipped with some metric topology and algebraic structure, so the proofs of the existence of the generalized Nash equilibria of nonmonetized noncooperative games in [9] are under the same idea in the traditional games theory and by applying similar fixed point theorems in topological vector spaces.

More generally in the set of all possible outcomes of a game, there may be neither a topological structure nor an algebraic structure. But there is a special partial order on it−a lattice order. This lattice order in the outcome space describes the players' preference levels about the outcomes. So, for some pairs of outcomes in such a game (It may not for every pair of outcomes), through this lattice order, the players can determine which one is more preferable than other one. Similarly to the set of outcomes, for every player, in the set of his or her strategies, there is neither a topological structure nor an algebraic structure. But there is a lattice order.

The above revealed characteristics of some games lead us, in this paper, to examine the extensions of the concepts of games and Nash equilibrium. In Section 3, we introduce the nonmonetized noncooperative games in which both of the strategy sets and the outcome space are lattices. We also define the generalized Nash equilibrium for such extended games. Since the lack of the topological structures on the sets of strategies and on the outcome space, for the considered games, we cannot apply the fixed point theorems and the traditional techniques for the proof of the existence of Nash equilibria as in the topological vector spaces. To overcome this difficulty, in Section 2, we recall some fixed point theorems on partially ordered sets (posets), which will be applied to prove the existence of generalized Nash equilibria of nonmonetized noncooperative games in Section 3.

2. PRELIMINARIES

In this section, we recall some concepts and properties of posets and lattices. For the details, the readers are referred to [11], [14-15].

Let $P = (P, \succcurlyeq)$ be a poset; and let A be a subset of P. An element u of P is called an *upper bound* of the subset A if $x \preccurlyeq u$ for each $x \in A$. If $u \in A$, then u is called the *greatest element* of A, and denote $u = \max A$. A *lower bound* of A and the *smallest element* $\min A$ of $A$are defined similarly, replacing $x \preccurlyeq u$ above by $u \preccurlyeq x$. If the set of all upper bounds of A has the smallest element, we call it a *supremum* of A and denote it by $\sup A$ or A. An *infimum* of A, $\inf A = \wedge A$, is defined similarly. An element y is called a *maximal element* of A if $y \in A$, and if $z \in A$, and $y \preccurlyeq z$ imply that $y = z$. Similarly, a *minimal element* of A can be defined.

A poset $P = (P, \succcurlyeq)$ is called a *lattice* if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in P$. Denote $\inf\{x, y\} = x \wedge y$ and $\sup\{x, y\} = x \vee y$. A subset C of a poset $P = (P, \succcurlyeq)$ is called a *chain* if $x \preccurlyeq y$ or $y \preccurlyeq x$ for all $x, y \in C$.

Definition 2.1. *A poset $P = (P, \succcurlyeq)$ is said to be*

1. strongly inductive whenever for every chain C in P, the supremum of C, denoted by supC or $\vee$C, exists in P;
2. strongly inversely inductive whenever for every chain C in P, the infimum of C, denoted by infC or $\wedge$C, exists in P;

As a consequence of Zorn's lemma, we have the following properties:

Remark 2.2. If a lattice $P = (P, \succcurlyeq)$ is strongly (inversely) inductive, then its greatest (smallest) element exists.

For any $z, w \in P$, we denote the order intervals below

$$[z) = \{x \in P: x \succcurlyeq z\}, (w] = \{x \in P: x \preccurlyeq w\} \; and \; [z,w] = [z) \cap (w] = \{x \in P: z \preccurlyeq x \preccurlyeq w\}.$$

Given posets $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$, we say that a mapping $F: X \to 2^U \backslash \{\emptyset\}$ is *order-increasing upward* if $x \preccurlyeq^X y$ in X and $z \in F(x)$ imply that $[z) \cap F(y)$ is nonempty; that is, if $x \preccurlyeq^X y$ in X and $z \in F(x)$, then there is $w \in F(y)$ such that $z \preccurlyeq^U w$. F is *order-increasing downward* if $x \preccurlyeq^X y$ in X and $w \in F(y)$ imply that $(w] \cap F(x)$ is nonempty. F is said to be *order-increasing* whenever F is both of order-increasing upward and downward.

As a special case, for single valued mappings, we have:

Given posets $(X, \succcurlyeq^X)$ and $(U, \succcurlyeq^U)$, a single valued mapping $F: X \to U$ is said to be *order-increasing* whenever $x \preccurlyeq^X y$ implies $F(x) \preccurlyeq^U F(x)$. An order-increasing mapping $F: X \to U$ is said to be *strictly order-increasing* whenever $x \prec^X y$ implies $F(x) \prec^U F(x)$.

A nonempty subset A of a subset Y of a poset $P = (P, \succcurlyeq)$ is said to be *order compact upward* in Y if for every chain C of Y that has a supremum in P the intersection $\cap\{[y) \cap A: y \in C\}$ is nonempty whenever $[y) \cap A$ is nonempty for every $y \in C$. If for every chain C of Y that has the infimum in P the intersection of all the sets $(y] \cap A$, $y \in C$ is nonempty whenever $(y] \cap A$, is nonempty for every $y \in C$, we say that A is *order compact downward* in Y. If both of these properties hold, we say that A is *order compactin Y*.

Let A be a subset of a poset $P = (P, \succcurlyeq)$. An element $c \in P$ is called a *sup-center* of A in P if $\sup\{c, x\}$ exists in P for each $x \in A$. If $\inf\{c, x\}$ exists in P for each $x \in A$, then c is called an *inf-center* of A in P. If c has both of these properties it is called an *order center* of A in P.

Let A be a nonempty subset of a poset $P = (P, \succcurlyeq)$. The set ocl(A) of all possible supremums and infimums of chains of A is called the *order closure of A*. If A = ocl(A), then A is *order closed*.

Now we recall a fixed point theorem on posets from [11]. It will be used in the proof of the existence of generalized Nash equilibria of nonmonetized noncooperative games in Section 3.

Theorem 2.12 [11]. *Let $P = (P,\succcurlyeq)$ be a poset. Assume that a set valued mappingF*: $P \to 2^P \setminus \{\varnothing\}$ *isorder-increasing*, *and that its values are order compact inF*(P). *If chains ofF*(P) *have supremums and infimums*, *and if* ocl($F(P)$) *has a sup-center or an inf-center inP*, *then F has minimal and maximal fixed points*.

3. NONMONETIZED NONCOOPERATIVE GAMES ON LATTICES

In this section, we examine the extensions of noncooperative games and the concept of Nash equilibria to nonmonetized noncooperative games and generalized Nash equilibria. The main theorem in this section is, by using a fixed point theorem in posets given by Carland Heikkilä [5], to prove the existence of generalized Nash equilibria of nonmonetized noncooperative games.

Definition 3.1.*Let n be a positive integer greater than* 1. *An n-person nonmonetized and noncooperative game is consisting of the following elements*:

1. *the set of n players*, *which is denoted byN* = {1, 2,…, n};
2. *For player i*, *hisstrategy set* $(S_i, \succcurlyeq_i)$ *is a lattice*,for i= 1, 2,…,n. We write $S = S_1 \text{x} S_2 \text{x} \ldots \text{x} S_n$.
3. *the outcome space* $(U; \succcurlyeq^U)$ *that is a lattice*;
4. *the set of n utilities functions* (*payoffs*) $\{P_1, P_2, \ldots, P_n\}$, *where* P_i *is the utility function for player i that is a mapping from*$S_1 \text{x} S_2, \ldots, \text{x} S_n$*to the lattice* $(U; \succcurlyeq^U)$, *fori*= 1, 2,…, n. *We denote*$P = \{P_1, P_2, \ldots, P_n\}$.

This game is denoted by$\Gamma = (N, S, P, U)$.

In a n-person nonmonetized noncooperative game $\Gamma = (N, S, P, U)$, when all the n players 1, 2, …, n simultaneously and independently choose their own strategies $x_1, x_2, \ldots, x_n$, respectively, where $x_i \in S_i$, for i= 1, 2, 3,…, n, then player $i$will receive his or her utility (payoff) $P_i(x_1, x_2, \ldots, x_n) \in U$. For the convenience, for any $x = (x_1, x_2, \ldots, x_n) \in S$, and for every given i= 1, 2,…, n, as usual, we denote

$x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $S_{-i} = S_1 \text{x} S_2 \text{x} \ldots \text{x} S_{i-1} \text{x} S_{i+1} \text{x} \ldots \text{x} S_n$.

Then $x_{-i} \in S_{-i}$ and x can be simply written as $x = (x_i, x_{-i})$. Moreover, we denote

$P_i(S_i, x_{-i}) = \{P_i(t_i, x_{-i}): t_i \in S_i\}$.

Now we extend the concept of Nash equilibria of noncooperative games to generalized Nash equilibria of nonmonetized noncooperative games.

Definition 3.2. *In an n-person nonmonetized noncooperative game*$\Gamma = (N, S, P, U)$, *a selection of strategies* $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in S_1 S_2 \ldots S_n$*is called a generalized Nash equilibrium of the game*, *if the following order inequality holds*

$P_i(x_i, \bar{x}_{-i}) \preccurlyeq^U P_i(\bar{x}_i, \bar{x}_{-i})$, *for all* $x_i \in S_i$ *and for every* i =1, 2, …, n.

Lemma 3.3. *Let* $(S_i, \succcurlyeq_i)$ *be a lattice, for every* i= 1, 2, …,n. *Let* $S = S_1 S_2 \dots S_n$ *be the Cartesian product space of* $S_1, S_2, \dots, S_n$ *and let* $\succcurlyeq^S$ *be the coordinate partial order on S induced by the lattice orders* $\succcurlyeq_i$; *that is, for any* $x, y \in S$ *with* $x = (x_1, x_2, \dots, x_n)$ *and* $y = (y_1, y_2, \dots, x_n)$,

$x \succcurlyeq^S y$ *if and only if* $x_i \succcurlyeq_i y_i$, *for all* i= 1, 2, …,n.

Then $(S, \succcurlyeq^S)$ *is a lattice. Furthermore, if every* $(S_i, \succcurlyeq_i)$ *is Dedekind complete, then* $(S, \succcurlyeq^S)$ *is also Dedekind complete.*

Proof. The proof is straightforward and is omitted here.

The following theorem is the main result of this paper. It provides some conditions for the existence of generalized Nash equilibria of nonmonetizednoncooperative games.

Theorem 3.4. *Let* $\Gamma = (N, S, P, U)$ *be an n-person nonmonetizednoncooperative game. If*,for every player i =1, 2,…, n, his or her strategy set $(S_i, \succcurlyeq_i)$ is a strongly inversely inductive lattice and his or her utility function P_i satisfies the following conditions:

1. $P_i: S \rightarrow U$ is (single valued) order increasing with respect to the product order $\succcurlyeq^S$;
2. for any fixed $x_{-i} \in S_{-i}$, $P_i(S_i, x_{-i})$ is an order bounded and Dedekind complete subset of $(U, \succcurlyeq^U)$;
3. for any fixed $x_{-i} \in S_{-i}$, and for any $u \in P_i(S_i, x_{-i})$, the inverse image$\{z_i \in S_i: P_i(z_i, x_{-i}) = u\}$ is an order bounded and Dedekind complete subset of S_i;
4. if $(w_i, x_{-i}) \preccurlyeq^S (w_i, y_{-i})$, then, for any $u \in P_i(S_i, x_{-i}) \cap (P_i(w_i, y_{-i})]$, there is $z_i \in S_i$ such that $z_i \preccurlyeq_i w_i$ and $P_i(z_i, x_{-i}) = u$.

Then this nonmonetizednoncooperative game Γ has a generalized Nash equilibrium. Furthermore, Γ has minimal and maximal (with respect to the product lattice order $\succcurlyeq^S$) generalized Nash equilibria.

Proof. Since $(S_i, \succcurlyeq_i)$ is a lattice, for i= 1, 2, …,n, then, from Lemma 3.3, $(S, \succcurlyeq^S)$ is also a lattice. For every fixed i =1, 2, …,n, define a mapping $T_i: S \rightarrow 2^{S_i} \backslash \{\emptyset\}$ as follows

$$T_i(x) = \{z_i \in S_i: P_i(z_i, x_{-i}) = \bigvee_{t_i \in S_i} P_i(t_i, x_{-i})\}, \text{ for all } x = (x_1, x_2, \dots, x_n) \in S. \quad (1)$$

Here $\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) = \vee P_i(S_i, x_{-i})$. From Condition 2 of this theorem, $P_i(S_i, x_{-i})$ is an order bounded and Dedekind complete subset of U with fixed $x_{-i} \in S_{-i}$. Then $\bigvee_{t_i \in S_i} P_i(t_i, x_{-i})$ exists and is in $P_i(S_i, x_{-i})$. More precisely, it is the greatest element of the range $P_i(S_i, x_{-i})$; that is,

$$\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) = \text{Max } P_i(S_i, x_{-i}).$$

So from (1), $T_i(x) \subseteq S_i$, which is the inverse image of the value $\bigvee_{t_i \in S_i} P_i(t_i, x_{-i})$. Then from condition 3 in the hypothesis, for every fixed $x = (x_1, x_2, \dots, x_n) \in S$ and for every i =1, 2, …, n, we have:

$T_i(x)$ is a nonempty order bounded and Dedekind complete subset of S_i. (2)

Define the product mapping T: $S \to 2^S \backslash \{\varnothing\}$ by

$T(x) = T_1(x)T_2(x) \ \ldots\ T_n(x)$, for all $x \in S$.

From (2) we obtain that $T(x) \neq \varnothing$, for every $x \in S$. Then we show that T: $S \to S$ is an order increasing set-valued mapping, with respect to the product lattice order $\succcurlyeq^S$. To this end, we need to show that T: $S \to S$ is both of order increasing upward and order increasing downward with respect to $\succcurlyeq^S$.

At first, we prove the order increasing upward property. For any given x, $y \in S$, with $x \preccurlyeq^S y$, and for any given $z \in T(x)$, we need to show that there is $w \in T(y)$ such that $z \preccurlyeq^S w$. We write $z = (z_1, z_2, \ldots, z_n)$. From the definition of the product order $\preccurlyeq^S$, the condition $x \preccurlyeq^S y$ implies $x_i \preccurlyeq_i y_i$, for all i= 1, 2, …,n. Then the definition of the product mapping T and the hypothesis $z \in T(x)$ imply $z_i \in T_i(x)$, for every i= 1, 2, …,n; that is,

$$P_i(z_i, x_{-i}) = \bigvee_{t_i \in S_i} P_i(t_i, x_{-i}), \text{ for every } i = 1, 2, \ldots, n.$$

For any fixed $t_i \in S_i$, the condition $x \preccurlyeq^S y$ clearly implies $(t_i, x_{-i}) \preccurlyeq^S (t_i, y_{-i})$. Then from Condition 1 of the hypothesis, we obtain

$$P_i(t_i, x_{-i}) \preccurlyeq^U P_i(t_i, y_{-i}), \text{ for every } t_i \in S_i.$$

It implies

$$\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) \preccurlyeq^U \bigvee_{t_i \in S_i} P_i(t_i, y_{-i}).$$

From Condition 2 of the hypothesis again, the range $P_i(S_i, y_{-i})$ is an order bounded and Dedekind complete subset of U. Then there is $a_i \in S_i$, such that

$$P_i(a_i, y_{-i}) = \bigvee_{t_i \in S_i} P_i(t_i, y_{-i});$$

that is, $a_i \in T_i(y)$, for i =1, 2, …, n. Since $z_i \in S_i$, the above equality assures that

$$P_i(z_i, y_{-i}) \preccurlyeq^U P_i(a_i, y_{-i}).$$

Since $(S_i, \succcurlyeq_i)$ is a lattice, $z_i \vee a_i$ exists in S_i. Let $w_i = z_i \vee a_i \in S_i$. Then w_i satisfies

$$w_i \succcurlyeq_i z_i \quad \text{and} \quad w_i \succcurlyeq_i a_i, \text{ for } i = 1, 2, 3, \ldots, n. \tag{3}$$

The second order inequality above implies $(w_i, y_{-i}) \succcurlyeq^S (a_i, y_{-i})$. Since $a_i \in T_i(y)$, the order-increasing property of P_i: $S \to U$ implies

$$P_i(w_i, y_{-i}) \succcurlyeq^U P_i(a_i, y_{-i}) = \bigvee_{t_i \in S_i} (t_i, y_{-i}).$$

Since $w_i = z_i \vee a_i \in S_i$, then from the above order inequality, we have $P_i(w_i, y_{-i}) = \bigvee_{t_i \in S_i} (t_i, y_{-i})$, which is,

$w_i \in T_i(y)$, for i =1, 2, …, n. (4)

Let $w = (w_1, w_2, \ldots, w_n)$. By combining (3) and (4), we obtain $z \preccurlyeq^S w$ and $w \in T(y)$. It proves that T is an order increasing upward mapping on S.

Now we prove the order increasing downward property. For any given $x, y \in S$, with $x \preccurlyeq^S y$, and for any $w \in T(y)$, we need to show that there is $z \in T(x)$ such that $z \preccurlyeq^S w$. We write $w = (w_1, w_2, \ldots, w_n)$. The hypothesis $w \in T(y)$ implies $w_i \in T_i(y)$, for every i= 1, 2, …, n; that is, $P_i(w_i, y_{-i}) = \bigvee_{t_i \in S_i} (t_i, y_{-i})$. Similarly to the proof of the first part, for any fixed $t_i \in S_i$, we have $(t_i, x_{-i}) \preccurlyeq^S (t_i, y_{-i})$. It implies

$$\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) \preccurlyeq^U \bigvee_{t_i \in S_i} P_i(t_i, y_{-i}) = P_i(w_i, y_{-i}). \quad (5)$$

Notice that

$$\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) \in P_i(S_i, x_{-i}). \quad (6)$$

Then by combining (5) and (6), we obtain

$$\bigvee_{t_i \in S_i} P_i(t_i, x_{-i}) \in P_i(S_i, x_{-i}) \cap (P_i(w_i, y_{-i})]. \quad (7)$$

Since $(w_i, x_{-i}) \preccurlyeq^S (w_i, y_{-i})$, from (7) and Condition 4 in the hypothesis, there is $z_i \in S_i$ such that

$z_i \preccurlyeq_i w_i$ and $P_i(z_i, x_{-i}) = \bigvee_{t_i \in S_i} P_i(t_i, x_{-i})$.

That is,

$z_i \preccurlyeq_i w_i$ and $z_i \in T_i(x)$, for i =1, 2, …, n.

Let $z = (z_1, z_2, \ldots, z_n)$. Then, from the above order inequalities, we get $z \preccurlyeq^S w$ and $z \in T(x)$. It proves that T is an order increasing downward mapping on S. And hence T is an order increasing mapping on S.

Next we show that T: $S \to S$ has order compact values in $T(S)$. More precisely, we have to show that, for every $x \in S$, $T(x)$ is both of order compact upward and order compact downward in $T(S)$. At first we show the order compact upward property of the values of T. Take any chain C in $T(S)$ (that has a supremum in S), which satisfies that

$[y) \cap T(x) \neq \emptyset$, for every $y \in C$.

Denote $y = (y_1, y_2, \ldots, y_n)$. Then

$[y)\cap T(x) = ([y_1)\cap T_1(x), [y_2)\cap T_2(x),\ldots, [y_n)\cap T_n(x))$. (8)

Since C is a chain in $T(S)$ (Which has a supremum in S), then $\{y_i: y\in C\}$ is a chain in $T_i(S)$ (and it also has a supremum in S_i). The hypothesis $[y)\cap T(x) \neq\emptyset$ and (8) imply

$$[y_i)\cap T_i(x) \neq\emptyset, \text{ for every } i =1, 2, \ldots, n \text{ and for every } y\in C. \quad (9)$$

Inequality (9) implies that, for every fixed i =1, 2, …,n and for every fixed $y\in C$, there is $\mu(y_i) \in S_i$ satisfying

$$\mu(y_i) \in T_i(x) \subseteq S_i \quad \text{and} \quad y_i \preccurlyeq_i \mu(y_i). \quad (10)$$

(Where μ, defined on CN, is a function of two variables y and i satisfying μ:$\{y\}N\to S$,for every fixed $y\in C$ and μ: $C\{i\} \to S_i$, for every fixed $i\in N$).So for every fixed i,when y varies in the chain C, from (10), we obtain a subset $\{\mu(y_i): y\in C\} \subseteq T_i(x) \subseteq S_i$, which is a chain in $T_i(x)$. Since $T_i(x)$ is an order bounded Dedekind complete subset in S_i, so $\{\mu(y_i): y\in C\}$ is also order bounded in $T_i(x)$, and hence $\vee\{\mu(y_i): y\in C\}$ exists in $T_i(x)$. It is denoted by

$$q_i = \vee\{\mu(y_i): y\in C\} \in T_i(x) \subseteq S_i, \text{ for every given } i =1, 2, \ldots, n. \quad (11)$$

(Note: q_i only depends on i, and it does not depend on y in C). By combining (10) and (11), for every fixed i =1, 2, …,n, we have

$y_i \preccurlyeq_i q_i$ and $q_i\in T_i(x)$, for all $y\in C$.

It implies

$$q_i\in \cap\{[y_i)\cap T_i(x): y\in C\}, \text{ for every fixed } i =1, 2, \ldots, n. \quad (12)$$

Let $q = (q_1, q_2, \ldots,q_n)$. From (8) and (12), we obtain $q\in\cap\{[y)\cap T(x): y\in C\}$. That is,

$\cap\{[y)\cap T(x): y\in C\} \neq\emptyset$.

Hence, $T(x)$ is order compact upward in $T(S)$. Very similarly to the above proof, we can show that, for every $x\in S$, $T(x)$ is order compact downward in $T(S)$. Hence $T(x)$ is order compact in $T(S)$.

Then we show that every chain of $T(S)$ has supremum and (or) infimum. Suppose that

$C = \left\{z^\lambda = \left(z_1^\lambda, z_2^\lambda, \ldots, z_n^\lambda\right): \lambda \in \Lambda\right\} \subseteq T(S)$

is a chain in $T(S)$ with index set Λ. Take a fixed $\gamma\in\Lambda$; that is, a fixed $z^\gamma\in C$. Then $z^\gamma\in T(S)$ and hencethere is $x^\gamma\in S$ such that $z^\gamma\in T(x^\gamma)$. Denote $x^\gamma = \left(x_1^\gamma, x_2^\gamma, \ldots, x_n^\gamma\right)$. From the definition of T, we have

$z_i^\gamma \in T_i(x^\gamma)$, for i =1, 2, …,n.

That is,

$$P_i(z_i^\gamma, x_{-i}^\gamma) = \bigvee_{t_i \in S_i} P_i(t_i, x_{-i}^\gamma). \tag{13}$$

For any $z^\lambda \in C$ with $z^\lambda \succcurlyeq^S z^\gamma$, the order inequality $z_i^\lambda \succcurlyeq_i z_i^\gamma$ holds, that implies $(z_i^\lambda, x_{-i}^\gamma) \succcurlyeq^S (z_i^\gamma, x_{-i}^\gamma)$, for i =1, 2, …,n. From the order increasing property of P_i, we have

$$P_i(z_i^\lambda, x_{-i}^\gamma) \succcurlyeq^U P_i(z_i^\gamma, x_{-i}^\gamma), \text{ for } i = 1, 2, \ldots, n. \tag{14}$$

Combining (13) and (14), and from $z_i^\lambda \in S_i$, we obtain

$$P_i(z_i^\lambda, x_{-i}^\gamma) = \bigvee_{t_i \in S_i} P_i(t_i, x_{-i}^\gamma), \text{ for every fixed } i = 1, 2, \ldots, n.$$

That is, $z_i^\lambda \in T_i(x^\gamma)$, for all $z^\lambda \succcurlyeq^S z^\gamma$ in C. So $\{z_i^\lambda : z^\lambda \succcurlyeq^S z^\gamma$ and $z^\lambda \in C\}$ is a subset of $T_i(x^\gamma)$. Notice that $T_i(x^\gamma)$ is an order bounded and Dedekind complete subset of S_i, then $\vee\{z_i^\lambda : z^{\lambda S} z^\gamma$ and $z^\lambda \in C\}$ exists in $T_i(x^\gamma)$. Denote

$$v_i = \vee\{z_i^\lambda : z^\lambda \succcurlyeq^S z^\gamma \text{ and } z^\lambda \in C\} \in T_i(x^\gamma). \tag{15}$$

Let $v = (v_1, v_2, \ldots, v_n)$. The first part of (15) implies $v = \vee\{z^\lambda \in C: z^\lambda \succcurlyeq^S z^\gamma\}$. Since C is a chain, it yields that

$$v = \vee\{z^\lambda \in C: z^\lambda \succcurlyeq^S z^\gamma\} = \vee\{z^\lambda \in C\}.$$

Hence v is the supremum of this chain C in $T(S)$. Furthermore, the second part of (15) implies that $v \in T(x^\gamma)$. So the supremum of this chain C in $T(S)$ is in $T(S)$.

On the other hand, for the given chain $C = \{z^\lambda = (z_1^\lambda, z_2^\lambda, \ldots, z_n^\lambda) : \lambda \in \Lambda\} \subseteq T(S)$, we recognize that $\{z_i^\lambda : z^\lambda \in C\}$ is a chain in S_i. Since S_i is strongly inversely inductive. It guarantees the existence of the infimum of the chain $\{z_i^\lambda : z^\lambda \in C\}$ in S_i. Then

$$b_i = \wedge\{z_i^\lambda : z^\lambda \in C\}.$$

It exists and is in S_i. Let $b = (b_1, b_2, \ldots, b_n)$. From the above definition, b is the infimum of this chain C in S. Hence for any arbitrary chain C in $T(S)$, both of sup$C$and inf$C$exist in S.

Finally, notice that $(S, \succcurlyeq^S)$ is a lattice, then for every given element $c \in S$, we havethat both of $c \vee x$ and $c \wedge x$ are in S, for every $x \in$ocl($T[S]$). So c is both of a sup-center and an inf-center of ocl($T[S]$) in S.

Then the mapping T: $S \to S$satisfies all conditions of the fixed point theorem 2.12 [5]. Hence T has a fixed point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ with $\bar{x} \in T(\bar{x})$. It implies $\bar{x}_i \in T_i(\bar{x})$; that is, $P_i(\bar{x}_i, \bar{x}_{-i}) = \vee_{t_i \in S_i} P_i(t_i, \bar{x}_{-i})$, for i =1, 2, …, n.

It is equivalent to

$P_i(t_i, \bar{x}_{-i}) \preccurlyeq^U P_i(\bar{x}_i, \bar{x}_{-i})$, for all $t_i \in S_i$,

for every i =1, 2, …, n. This shows that $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$ is a generalized Nash equilibrium of this game.

Furthermore, from Theorem 2.12 [11], the mapping$T$ has minimal and maximal fixed points. From the above argument, which are the minimal and maximal (with respect to the product lattice order$\succcurlyeq^S$) generalized Nash equilibria for this game. This completes the proof of the theorem.

Remarks on Theorem 3.4:

1. The order increasing property (condition 1) describes that at higher order selection of strategies, every player will receive better utility.

2. Condition 2 indicates that for any given player, with any fixed strategies for other players, the set of his or her possible payoffs is an order bounded and Dedekind complete subset of(U, $\succcurlyeq^U$), which has both of the greatest (max) and the smallest (min) elements.

3. The order increasing property (condition 1) also implies that if $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a generalized Nash equilibrium of this game, then for any given player i, as other players select their strategies $\bar{x}_{-i}$, then the following stability of the utility for player i holds:

   $P_i(t_i, \bar{x}_{-i}) = P_i(\bar{x}_i, \bar{x}_{-i})$, for all $t_i \in S_i$ with $t_i \succcurlyeq_i \bar{x}_i$.

4. In the proof of the order compactness, the conditions that the chain C in $T(S)$ has a supremum or has an infmum in S are not used.

5. If all strategy sets S_1, S_2,…,S_n are complete lattices with the total or linear order, then Condition 4 in Theorem 3.4can be reduced by Condition 1; and therefore, it can be removed from the hypothesis.

6. If the outcome space (U, $\succcurlyeq^U$) is the real number set (○, ≥) with the ordinal order, then the Condition 2 can be replaced by:

   For any fixed $x_{-i} \in S_{-i}$, $P_i(S_i, x_{-i})$is a bounded and closed subset of○.

7. If the outcome space (U, $\succcurlyeq^U$) is the real number set (○, ≥) with the ordinal order, then this game automatically turns to be a quantum game. Hence the nonmonetizednoncooperative games are immediately generalized quantum noncooperative games, and the generalized Nash equilibria extended the Nash equilibria in quantum noncooperative games.

4. NONMONETIZED NONCOOPERATIVE GAMES ON BANACH LATTICES

In game theory, there are some very popular cases where the strategy sets are Banach lattices on which except the lattice orders, there are complete norms and algebraic structures. These can be considered as special cases of Definition 3.1. Based on the popularity and the importance of such games, it is worth to examine their properties in this section. For the reader's convenience, we recall some concepts and properties of Banach lattices here (For more details, see [13]).
A Banach space X equipped with a lattice order ^{X} is called a Banach lattice, which is written as (X; ^{X}), if the following properties hold:

1. $x\ ^X y$ implies $x + z\ ^X y + z$, for all $x, y, z \in X$.
2. $x\ ^X y$ implies $\alpha x\ ^X \alpha y$, for all $x, y \in X$ and$\alpha \geq 0$.
3. $|x|\ ^X |y|$ implies $||x|| \geq ||y||$, for every $x, y \in X$.

Where, as usual, the origin of X is denoted by 0 and $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x^+ \vee x^-$, for all $x \in X$, and $||\cdot||$ denotes the norm of the Banach space X.

Theorem 1.10 [11]. *Let P be a closed and bounded ball in a reflexive lattice ordered Banach spaceX, and assume that*$||x^+|| = ||\sup\{0, x\}|| \leq ||x||$*holds for all*$x \in X$. *Then every increasing mappingF*: $P \rightarrow 2^P$, *whose values are nonempty and weakly sequentially closed*, *has a fixed point*.

Let (X; ^{X}) be a Banach lattice. It is known that $|||x||| = ||x||$, for every $x \in X$. So from Property 3 of Banach lattices, it implies that, for all $x \in X$, $||x^+|| \leq |||x||| = ||x||$. Then the following theorem is an immediately consequence of Theorem 1.10 [11], which will be used in this section.

Theorem 4.1. *Let P be a closed and bounded ball in a reflexive Banach lattice* (X; ^{X}). *Then every increasing mapping F*: $P \rightarrow 2^P \backslash \{\emptyset\}$, *whose values are nonempty and weakly sequentially closed*, *has a fixed point*.

Similarly tothe results in Lemma 3.3,we have

Lemma 4.2. *Let n be a fixed positive integer greater than* 1. *For every*i= 1, 2, …,n, *let* (X_i, $\succcurlyeq_i$) *be a reflexive Banach lattice equipped with the lattice order*$\succcurlyeq_i$*and with norm*$||\cdot||_i$. *Let*$X = X_1 X_2 \dots X_n$*be the Cartesian product space of*$X_1, X_2, \dots, X_n$*and let*$\succcurlyeq^X$*be the coordinate partial order on X induced by the lattice orders* $\succcurlyeq_i$. *Define the maximum norm*$||\cdot||_X$*on*X *as follows*:

$||x||_X = \text{Max}\{||x_1||_1, ||x_2||_2, \dots, ||x_n||_n\}$, *for any*$x \in X$*with*$x = (x_1, x_2, \dots, x_n)$.

Then (X, $\succcurlyeq^X$) *is also a reflexive Banach lattice equipped with the lattice order*$\succcurlyeq^X$*and with norm*$\|\cdot\|_X$.

Furthermore, *for every*i= 1, 2, …,n, *let*S_i*be a closed ball in*X_i*with center at its origin and with radius d*, *for some positive number*d. *Then* S= $S_1 S_2 \ldots S_n$*is the closed ball in X with center at the origin of X and with the same radius*d.

Proof. The proof is straight forward and is omitted here.

Theorem 4.3. For every i = 1, 2, …, n, let (X_i, $\succcurlyeq_i$) be a reflexive Banach lattice. Suppose that, in an n-person nonmonetizednoncooperative game Γ = (N, S, P, U), for every player i, his or her strategy set is a closed ball in (X_i, $\succcurlyeq_i$): $S_i=\{x_i \in X_i: \|x_i\|_i \leq d\}$, for some fixed d > 0. If, for every player i =1, 2, …, n, his or her utility fuction P_i satisfies conditions 1, 2, 4 in Theorem 3.4 and the
following condition:

3′. For any fixed $x_{-i} \in S_{-i}$, and for any u $\in P_i(S_i, x_{-i})$, the inverse image$\{z_i \in S_i: P_i(z_i, x_{-i}) = u\}$ is (topologically) closed, order bounded and Dedekind complete subset of S_i.

Then this nonmonetizednoncooperative game Γ has a generalized Nash equilibrium.

Proof. In fact, let *T*be the mapping defined in the proof of Theorem 3.4, where it has been proved that *T* is order increasing. Since every bounded closed subset of a reflexive Banach space is weakly compact, the topological closedness in Condition 3′ of this theorem implies that *T* has weakly sequentially closed values. Then this theorem immediately follows from Theorem 4.1.

5. CONCLUSION

We believe that the equilibrium problems with non-real valued utilities are important. That is because if the preference relation on an alternative set *U* for a gameis just a partial orders on *U*, which may not be a totally ordering, then, under such a preference relation, two elements in the alternative set *U* may not be comparable to tell which one is better than other one; and therefore, the preference relation cannot be represented by a real utility function. We consider the following example.

Apple-Pear Example (See [13]). Suppose that in a box there are 100 apples and 100 pears. Let *U* be the collection of all possible selections (subsets) of fruits from this box. Suppose that the taste of apples and the taste of pears for a decision maker are not comparable. Denote an arbitrary element in *U* by (x, y), where x, y are the numbers of apples and pears in this outcome, respectively. Then the non-comparable tastes preference relation $\succcurlyeq$ on *U* is defined by

(x_1, y_1) (x_2, y_2) if and only if $x_1 \geqslant x_2$ and $y_1 \geqslant y_2$, for all (x_1, y_1), (x_2, y_2) $\in U$.

This preference relation on *U* is not rational (complete). It is a partial order on *U*. Therefore, it is worth to investigating some games with the range of utilities equipped with a partially rational preference relation ordering.

For deeply understanding the concept of generalized Nash equilibrium,we provide a concrete example of nonmonetizednoncooperative gamesbelow.

Example 5.1 [12].(the extended prisoner's dilemma – complicated version). Two suspects, designated Suspect 1 and Suspect 2, are held in separate cells without any means of communicating with each other. There are two crimes (I and II) for which these suspects are being held. There is enough evidence to convict each of them of minor offenses related to crimes I and II, but not enough evidence to convict either of them of the principal crimes I or II unless one of them acts as an informant against the other ("finks") for major crime I or II.

If they both stay quiet for both crime I and crime II, then each will be convicted of both minor offenses, and will spend one year in prison for crime I and fined $10 for crime II. If only one finks for crime I and both stay quiet for crime II, then the informant will not be charged for crime I but will be fined $10 for crime II, and the informant will testify against the other suspect, who will be convicted of the principal offense for crime I resulting in a three-year prison sentence and also be fined $20 for crime II. If they both stay quiet for crime I and only one of them finks for crime II, then the informant will not be charged for crime II but will spend one year in prison for crime I, and the informant will testify against the other suspect, who will be convicted of the principal offense for crime II resulting in a $30 fine and also sentenced to two years in prison for crime I. If both suspects fink for both crimes I and II, then each will spend two years in prison for crime I and be fined $20 for crime II.

Every suspect has the following four possible strategies:

QQ, QF, FQ, FF.

The possible outcomes (payoffs) for this game can be described by the following table:

	QQ	QF	FQ	FF
QQ	(–1, –10), (–1, –10)	(–2, –30), (–1, 0)	(–3, –20), (0, –10)	(–3, –30), (0, 0)
QF	(–1, 0), (–2, –30)	(–2, –20), (–2, –20)	(–3, –10), (–1, –30)	(–3, –20), (–1, –20)
FQ	(0, –10), (–3, –20)	(–1, –30), (–3, –10)	(–2, –20), (–2, –20)	(–3, –30), (–2, –10)
FF	(0, 0), (–3, –30)	(–1, –20), (–3, –20)	(–2, –10), (–3, –30)	(–2, –20), (–2, –20)

We see that the action (strategy) profile (FF, FF) is a generalized Nash equilibrium of this game.

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Authors have declared that no competing interests exist.

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