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REMARKS ON FIXED POINTS, MAXIMAL ELEMENTS, AND EQUILIBRIA OF ECONOMIES IN ABSTRACT CONVEX SPACES: REVISITED

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Abstract. In our previous work [4], KKM theorems or coincidence theorems on abstract convex spaces were applied to the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtained generalizations or improvements of a number of known equilibria results, especially, in a work of Ding and Wang [2] on the so-called FC-spaces. Since then Ding and Feng [1] obtained similar results to corresponding ones in [2]. Moreover, recently Du [3] obtained generalizations of results in [1, 2] to abstract convex spaces in the sense of Park. In the present paper, we compare the results in [4] and [3] and give comments on the results of [3].

1. Introduction

In our previous work [4] in 2008, KKM theorems or coincidence theorems on abstract convex spaces are applied to obtain the Fan-Browder type fixed point theorems, existence of maximal elements, existence of economic equilibria and some related results. Consequently, we obtained generalizations or improvements of a number of known equilibria results, especially, in a earlier work of Ding and Wang [2] on the so-called FC-spaces.

Later in 2010, Ding and Feng [1] established two fixed point theorems in noncompact FC-spaces, two existence theorems of maximal elements for $\mathcal{L}_F^*$-correspondences and $\mathcal{L}_F^*$-majorized correspondences. Moreover, by applying the existence theorems of maximal elements, they obtained some equilibrium existence theorems for one person game, qualitative games and noncompact abstract economies with $\mathcal{L}_F^*$-majorized correspondences in FC-spaces. Ding

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and Feng claimed that their results generalize some known results in recent literature.

More recently in 2012, Du [3] proved some new fixed point theorems for set-valued mappings in noncompact abstract convex spaces. Next, two existence theorems of maximal elements for classes of $\mathcal{A}_{C,\theta}$ mappings and $\mathcal{A}_{C,\theta}$-majorized mappings are obtained in [3]. As applications, Du established new equilibria existence theorems for qualitative games and generalized games. Du claimed that his theorems improve and generalize the most known results in recent literature. Note that Du extended the results in [1].

Apparently, the authors of [1] and [3] did not mention the earlier work [4]. We found that some basic results of them are consequences of corresponding ones in [4]. Especially, [1] maintained obsolete concept of FC-spaces and [3] applied the abstract convex space theory developed by the present author since 2006.

In the present paper, we give several remarks on the results in [1] and [3] comparing with [4]. All preliminary things can be found in [4] and more recent results on abstract convex spaces can be found in [5] and references therein.

In Section 2, definitions and some basic facts on abstract convex spaces are introduced. Section 3 deals with general forms of the Fan-Browder type fixed point theorems in abstract convex spaces. Section 4 deals with various existence theorems on maximal elements for $\mathcal{L}_F$-majorized correspondences in abstract convex spaces. Finally, in Section 5, a new equilibrium existence theorem for one person game with $\mathcal{L}_F$-majorized correspondences is obtained in abstract convex spaces. Moreover, we introduce some results in [3] which can be improved by following our methods in [4].

2. Abstract convex spaces

Recall the following in [5] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Recall that [5] contains incorrect statements such as (V), (VI), Theorem 4, (XVI), and (XVII). These can be easily corrected.

3. Fixed point theorems

First of all, Du [3] adopted obsolete terminology like as compactly open [resp., closed], compact interior (cint), compact closure (ccl), etc. As we stated quite often, these can be eliminated by replacing the topology of relevant spaces by its compactly generated extension.

The following is given in [4, Theorem 4.3]:

Theorem 3.1. *Let $(E, D; \Gamma)$ be a partial KKM space* [*resp., a KKM space*], *and $S : D \multimap E$, $T : E \multimap E$ maps. Suppose that*

(1) *S is open-valued* [*resp., closed-valued*];

(2) *for each $x \in E$, $\mathrm{co}_\Gamma S^-(x) \subset T^-(x)$;*

(3) *there exists a nonempty subset K of E such that $K \subset S(N)$ for some $N \in \langle D\rangle$; and*

(4) *either*

(i) *$E \setminus K \subset S(M)$ for some $M \in \langle D\rangle$; or*

(ii) *there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and $L_N \setminus K \subset S(M)$ for some $M \in \langle D'\rangle$.*

Then T has a fixed point $\overline{x} \in E$, that is, $\overline{x} \in T(\overline{x})$.

Particular forms of Theorem 3.1.

1. [3, Theorem 3.1] *Let $(E, D; \Gamma)$ be a partial KKM space, and K be a nonempty compact subset of E. Suppose that $F : E \to 2^D$, $G : E \to 2^E$ be mappings such that*

(i) *$F(x) \subset G(x)$ for each $x \in E$;*

(ii) *for each $y \in D$, $F^-(y)$ is compactly open in E and for each $x \in K$, $F(x) \neq \emptyset$;*

(iii) *for each $N \in \langle D\rangle$, there exists a compact abstract convex subset L_N*

$$L_N \setminus K \subset \bigcup\{\mathrm{cint}\, G^-(y) \mid y \in L_N\}.$$

Then, there exists a point $\hat{x} \in E$, such that $\hat{x} \in \operatorname{co} G(\hat{x})$.

Note that it should be assumed $E \supset D$ here.

2. [3, Theorem 3.1] generalizes Theorem 3.1 of Ding and Feng [1] and Theorem 3.1 of Ding and Wang [2] from FC-spaces to abstract convex spaces, and the coercivity condition (iii) is weaker than the condition (3) in Theorem 3.1 of Ding and Wang [2].

The following is given in [4, Theorem 4.4]:

Theorem 3.2. *Let $(X \supset D; \Gamma)$ be a partial KKM space, $F : D \multimap X$, $G : X \multimap X$, and K a nonempty compact subset of X such that*

(1) *for each $x \in X$, $F^-(x) \subset G^-(x)$ and $G^-(x)$ is Γ-convex;*

(2) *for each $z \in D$, $F(z)$ is open in X;*

(3) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ containing N such that*

$$\bigcap_{z \in D'} \operatorname{cl}_{L_N} (L_N \setminus G(z)) \subset K;$$

(4) *for each $x \in K$, $F^-(x) \neq \emptyset$.*

Then there exists $\hat{y} \in X$ such that $\hat{y} \in G(\hat{y})$.

Particular forms of Theorem 3.2.

1. [3, Corollary 3.3] *Let $(E, D; \Gamma)$ be a partial KKM space, and K be a nonempty compact subset of E. Suppose that $F : E \to 2^D$, $G : E \to 2^E$ be mappings such that*

(i) *$F(x) \subset G(x)$ for each $x \in E$,*

(ii) *for each $y \in D$, $F^-(y)$ is compactly open in E and for each $x \in K$, $F(x) \neq \emptyset$,*

(iii) *for each $N \in \langle D \rangle$, there exists a compact abstract convex subset L_N of E containing N such that for each $x \in L_N \setminus K$, there exists a point $\overline{y} \in L_N$ such that $x \notin \operatorname{cl}_X[X \setminus G^-(\overline{y})]$.*

Then, there exists a point $\hat{x} \in E$, such that $\hat{x} \in \operatorname{co}_\Gamma G(\hat{x})$.

This is an incorrect form of Theorem 3.2 and X should be E.

4. Maximal elements in abstract convex spaces

We introduce the following in [4], which generalizes [2, Definition 2.4]:

Definition 4.1. Let X be a topological space, $(E \supset Y; \Gamma)$ an abstract convex space, $\theta : X \to E$ a function, and $\phi : X \multimap Y$ a map. Then

(1) ϕ is said to be of *class* $\mathcal{L}_{\theta,F}$ if

(a) for each $x \in X$, $\operatorname{co}_\Gamma \phi(x) \subset Y$ and $\theta(x) \notin \operatorname{co}_\Gamma \phi(x)$ for each $x \in X$,

(b) there exists a map $\psi : X \multimap Y$ such that $\psi(x) \subset \phi(x)$ for each $x \in X$ and $\psi^-(y)$ is open in X for each $y \in Y$, and

(c) $\{x \in X \mid \phi(x) \neq \emptyset\} = \{x \in X \mid \psi(x) \neq \emptyset\}$;

(2) (ϕ_x, ψ_x, N_x) is called a *$\mathcal{L}_{\theta,F}$-majorant* of ϕ at x if $\phi_x, \psi_x : X \multimap Y$ and N_x is an open neighborhood of x in X such that

(a) for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \operatorname{co}_\Gamma \phi_x(z)$,

(b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $\mathrm{co}_\Gamma\, \phi_x(z) \subset Y$,
(c) for each $y \in Y$, $\psi_x^-(y)$ is open in X;
(3) ϕ is said to be $\mathcal{L}_{\theta,F}$-*majorized* if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathcal{L}_{\theta,F}$-majorant (ϕ_x, ψ_x, N_x) of ϕ at x such that for any nonempty finite subset A of the set $\{x \in X \mid \phi(x) \neq \emptyset\}$,

$$\left\{z \in \bigcap_{x\in A} N_x \mid \bigcap_{x\in A} \mathrm{co}_\Gamma \phi_x(z) \neq \emptyset\right\} = \left\{z \in \bigcap_{x\in A} N_x \mid \bigcap_{x\in A} \mathrm{co}_\Gamma \psi_x(z) \neq \emptyset\right\}.$$

It is clear that every map of class $\mathcal{L}_{\theta,F}$ is $\mathcal{L}_{\theta,F}$-majorized. The above definitions generalize the corresponding ones for FC-spaces in [2], which was stated there to generalize the ones due to Ding and Tan; Ding, Kim and Tan; Tan and Yuan; Ding and Tarafdar; and Tulcea.

In the paper [4], we restricted ourselves to the case $X = Y$ is an abstract convex space and $\theta = 1_X$ and write $\mathcal{L}_F$ instead of $\mathcal{L}_{\theta,F}$.

The following definitions are given in [3, Definitions 2.4 and 2.5]:

Definition 4.2 ([3])**.** Let X be a topological space, and Y be a nonempty subset of an abstract convex space. Let $\theta : X \to Y$ be a single-valued mapping. Then, the mappings $\psi, \phi : X \to 2^Y$ are said to be an $\mathcal{A}_{C,\theta}$-pair if
(a) for each $x \in X$, $\theta(x) \notin \mathrm{co}(\phi(x)) \subset Y$ and $\psi(x) \subset \phi(x)$,
(b) the mapping $\psi^- : Y \to 2^X$ is compactly open valued on Y.

Definition 4.3 ([3])**.** Let X be a topological space, and Y be a nonempty subset of an abstract convex space. Let $\theta : X \to Y$ be a single-valued mapping, and $P : X \to 2^Y$ be a set-valued mapping. Then,
(i) P is said to be of class $\mathcal{A}_{C,\theta}$ if there exists an $\mathcal{A}_{C,\theta}$-pair such that
(a) $\mathrm{dom}\, P \subset \mathrm{dom}\, \psi$,
(b) for each $x \in X$, $P(x) \subset \phi(x)$.
(ii) $(\psi_x, \phi_x; N_x)$ is said to be an $\mathcal{A}_{C,\theta}$-majorant of P at $x \in X$ if N_x is an open neighborhood of x in X and the mapping $\psi_x, \phi_x : X \to 2^Y$ such that
(a) for each $z \in N_x$, $P(z) \subset \phi_x(z)$, $\theta(z) \notin \mathrm{co}_\Gamma(\phi_x(z))$ and $\{z \in N_x : P(z) \neq \emptyset\} \subset \{z \in N_x : \psi_x(z) \neq \emptyset\}$,
(b) for each $z \in X$, $\psi_x(z) \subset \phi_x(z)$ and $\mathrm{co}_\Gamma(\phi_x(z)) \subset Y$,
(c) the mapping $\psi_x^- : Y \to 2^X$ is compactly open valued on Y.
(iii) P is said to be an $\mathcal{A}_{C,\theta}$-majorized mapping if for each $x \in X$ with $P(x) \neq \emptyset$, there exists an $\mathcal{A}_{C,\theta}$-majorant $(\psi_x, \phi_x; N_x)$ of P at x, and for any nonempty finite subset $A \in \langle \mathrm{dom}\, P\rangle$, the set $\{z \in \bigcap_{x\in A} N_x : P(z) \neq \emptyset\} \subset \{z \in \bigcap_{x\in A} N_x : \psi_x(z) \neq \emptyset\}$.

Note that Definitions 4.1 and 4.3 are almost same.

In [3, Remark 2.6], Du noted that his notions of the mapping P being of class $\mathcal{A}_{C,\theta}$ [resp., $\mathcal{A}_{C,\theta}$-majorized] improve notions of mapping of class $\mathcal{L}^*_{\theta,F}$ [resp., $\mathcal{L}^*_{\theta,F}$-majorized], respectively introduced by Ding and Feng [1] from FC-space to abstract convex space, which in turn generalize the corresponding notions in Ding and Wang [2], Chowdhury et al., Yuan, Ding et al., and Ding.

In [3], Du dealt mainly with either the case (I) $X = Y$, and X is an abstract convex space, and $\theta = 1_X$, or the case (II) $X = \prod_{i\in I} X_i$, and $\theta = \pi_i : X \to X_i$ is the projection of X onto X_i and X_i is an abstract convex space. In both cases (I) and (II), he wrote $\mathcal{A}_C$ in place of $\mathcal{A}_{C,\theta}$.

The following is given as [4, Theorem 6.1]:

Theorem 4.1. *Let $(X;\Gamma)$ be a partial KKM space, $G : X \multimap X$ be of class $\mathfrak{L}_F$ and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, $G(x)$ is Γ-convex;*

(2) *for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$\bigcap_{x\in L_N} \mathrm{cl}_{L_N}\left(L_N \setminus G^-(x)\right) \subset K.$$

Then there exists $\hat{y} \in K$ such that $G(\hat{y}) = \emptyset$.

Similar forms of Theorem 4.1.

1. [3, Theorem 4.3] *Let $(X;\Gamma)$ be a partial KKM space, and let K be a nonempty compact subset of X. Suppose that the mapping $P : X \to 2^X$ is of class $\mathcal{A}_C$ and satisfy*

(i) *for each $N \in \langle X\rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup\{\mathrm{cint}(P^-(y)) : y \in L_N\}.$$

Then, there exists a point $\hat{x} \in K$ such that $P(\hat{x}) = \emptyset$.

2. In [3, Remark 4.4], Du noted that [3, Theorem 4.3] generalizes most existence theorems of maximal elements in the literature, for example; see Theorem 4.1 of Ding and Feng [1], Theorem 5.2 of Ding and Wang [2], Theorems of Yuan, Chowdhury et al., Tan-Yuan, and so on.

Closely examining the proof of [2, Lemma 5.1] with replacing FC$(\cdot)$ by $\mathrm{co}_\Gamma(\cdot)$, it leads to the following generalization as in [4, Theorem 6.2]:

Theorem 4.2. *Let X be a regular topological space and $(E \supset Y;\Gamma)$ an abstract convex space. Let $\theta : X \to E$ and $P : X \multimap Y$ be $\mathfrak{L}_{\theta,F}$-majorized. If each open subset of X containing $B := \{x \in X \mid P(x) \neq \emptyset\}$ is paracompact, then there exists a map $\phi : X \multimap Y$ of class $\mathfrak{L}_{\theta,F}$ such that $P(x) \subset \phi(x)$ for all $x \in X$.*

Particular or similar gorms of Theorem 4.2.

1. [4, Remark] *For FC-spaces, Theorem 4.2 reduces to* [2, *Lemma* 5.1], *which was said in* [2] *to generalize results of Ding; Ding and Tan; Ding, Kim and Tan; Tan and Yuan; and Tulcea.*

2. [3, Lemma 4.1] *Let X be a regular topological space, and Y be a nonempty subset of an abstract convex space $(E;\Gamma)$. Let $\theta : X \to E$ and $P : X \to 2^Y$ be an $\mathcal{A}_{C,\theta}$-majorized mapping. If each open subset of X containing the set $B =$*dom P *is paracompact, then there exists a $\mathcal{A}_{C,\theta}$-pair $\psi, \phi : X \to 2^Y$ such that $P(x) \subset \phi(x)$ for each $x \in X$ and* dom $P \subset$ dom ψ.

3. [3, Remark 4.2] notes that [3, Lemma 4.1] generalizes Lemma 4.1 of Ding and Feng [1], Lemma 5.1 of Ding and Wang [2], results of Yuan, Chowdhury et al., and Tan-Yuan.

From Theorems 3.2 and 4.2, we deduce the following generalization [4, Theorem 6.3] of Theorem 4.1 on the existence of maximal elements of $\mathcal{L}_F$-majorized maps:

Theorem 4.3. *Let* $(X;\Gamma)$ *be a paracompact partial KKM space,* $P : X \multimap X$ *an* $\mathcal{L}_F$*-majorized map, and* K *a nonempty compact subset of* X*. Suppose that*

(1) *for each* $x \in X$*,* $P(x)$ *is* Γ*-convex;*

(2) *for each* $N \in \langle D\rangle$*, there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that*

$$\bigcap_{x\in L_N} \mathrm{cl}_{L_N}\left(L_N \setminus P^{-}(x)\right) \subset K.$$

Then there exists $\hat{y} \in K$ *such that* $P(\hat{y}) = \emptyset$*.*

As we noted in [4], for FC-spaces, Theorem 4.3 reduces to [2, Theorem 5.3], which was said in [2] to generalize results of Ding and Tan; Tan and Yuan; Ding, Ding and Tan; and Borglin and Keiding.

As an application of Theorems 4.2 and 4.3, we have the following existence theorem on maximal elements:

Theorem 4.4 ([3, Theorem 4.5])**.** *Let* $(X;\Gamma)$ *be a paracompact partial KKM space, and let* K *be a nonempty compact subset of* X*. Let* $P : X \to 2^X$ *be an* $\mathcal{A}_C$*-majorized mapping and satisfy*

(i) *for each* $N \in \langle X\rangle$*, there exists a compact abstract convex subset* L_N *of* X *containing* N *such that*

$$L_N \setminus K \subset \bigcup\{\mathrm{cint}(P^{-}(y)) : y \in L_N\}.$$

Then, there exists a point $\hat{x} \in K$ *such that* $P(\hat{x}) = \emptyset$*.*

In [3, Remark 4.6], Du noted that Theorem 4.4 generalizes Theorem 5.3 of Ding and Wang [2], other results due to Chowdhury et al., Tan -Yuan, Borglin-Keiding, and Yannelis.

5. Equilibria of abstract economies in abstract convex spaces

We adopt the following definitions in [3].

Let I be a (finite or infinite) set of players. For each $i \in I$, let its strategy set X and $Y_i (i \in I)$ be nonempty set, and let $Y = \prod_{i\in I} Y_i$. $P_i : X \to 2^{Y-i}$ be the preference correspondence of i-th player. The collection $\Lambda = (X; Y_i; P_i)_{i\in I}$ will be called a *qualitative game.* A point $\hat{x} \in X$ is said to be an *equilibrium* of the qualitative game, if $P_i(\hat{x}) = \emptyset$ for each $i \in I$.

A *generalized game* is a quintuple family $\Lambda = (X; Y_i; A_i; B_i; P_i; \theta_i)_{i\in I}$, where X is a nonempty set, I is a (finite or infinite) set of players such that for each $i \in I$, Y_i is a nonempty set and $Y = \prod_{i\in I} Y_i$. $A_i, B_i : X \to$

2^{Y_i}, $\theta_i : X \to Y_i$ are the constraint correspondences, and $P_i : X \to 2^{Y_i}$ is the preference correspondence. An *equilibrium* of the generalized game Λ is a point $\hat{x} \in X$ such that for each $i \in I$, $\theta_i(\hat{x}) \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. If $\theta_i = \pi_i : X \to X_i$ is the projection of X onto X_i, then our definition of an equilibrium point coincides with the standard definition given by Chowdhury et al., and if, in addition, $A_i = B_i$ for each $i \in I$, our definition of an equilibrium point generalizes the standard definition, for example, Borglin and Keiding and Gale and Mas-Colell. For the references, see [3].

In [4], as an application of Theorem 4.1, we deduced the following equilibrium existence theorem for a one-person game:

Theorem 5.1 ([4, Theorem 6.4]). *Let $(X;\Gamma)$ be a partial KKM space, $A, B, P : X \multimap X$, and K a nonempty compact subset of X. Suppose that*

(1) *for each $x \in X$, $\mathrm{co}_\Gamma A(x) \subset \overline{B}(x)$;*

(2) *for each $y \in X$, $A^-(y)$ is open in X;*

(3) *$A \cap P$ is of class $\mathcal{L}_F$ and has Γ-convex values;*

(4) *for each $N \in \langle X \rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$\bigcap_{x \in L_N} \mathrm{cl}_{L_N} \big(L_N \setminus (A \cap P)^-(x)\big) \subset K;$$

(5) *for each $x \in K$, $A(x) \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{y}) \cap P(\hat{y}) = \emptyset$.

It is noted in [4] that, for FC-spaces, Theorem 5.1 reduces to [2, Theorem 6.1], which was claimed in [2] to improve and generalize results of Ding and Tarafdar; Ding and Tan; Tan and Yuan; Ding; and Ding and Tan.

In [4], we gave Final Remark as follows: Until now we showed that Lemmas 3.1, 3.2, and 5.1, Theorems 3.1, 5.1, 5.2, 5.3, and 6.1 for FC-spces in [2] are generalized to our abstract convex spaces in better forms. Other results in [2] can also be improved by following our method in [4].

On the other hand, as an application of Theorem 4.3, Du [3] proved the following existence theorem of equilibrium points for one person game in abstract convex space.

Theorem 5.2 ([3, Theorem 5.2]). *$(X;\Gamma)$ be a partial KKM space and K be a closed and compact subset of X, and let $K = \mathrm{co}_\Gamma K$. Suppose that the mappings A, B, and $P : X \to 2^X$ satisfy*

(i) *for each $x \in X$, $\mathrm{co}_\Gamma A(x) \subset \overline{B}(x)$ and $A(x) \cap P(x) \subset K$,*

(ii) *the mapping $A^- : X \to 2^X$ is compactly open valued on X,*

(iii) *the mapping $A \cap P : X \to 2^X$ is of class $\mathcal{A}_C$ and $A(x) \neq \emptyset$ for each $x \in K$,*

(iv) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\mathrm{cint}((A \cap P)^-(y) : y \in L_N\}.$$

Then, there exists a point $\hat{x} \in K$ such that $\hat{x} \in \bar{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Du [3] noted that Theorem 5.2 improves and generalizes Theorem 5.1 of Ding and Feng [1], Theorem 6.1 of Ding and Wang [2], other results of Yuan, Chowdhury et al., and Tan-Yuan.

In [3], as an another application of Theorem 4.4, its author can obtain the following existence of equilibria for qualitative games.

Theorem 5.3 ([3, Theorem 5.4]). *Let $\Lambda = (X; X_i; P_i)_{i \in I}$ be a qualitative game. For each $i \in I$, suppose that the following conditions are satisfied*

(i) $(X_i; \Gamma_i)_{i \in I}$ *is a family of paracompact abstract convex space such that $(X; \Gamma)$ is a partial KKM space, and K is a nonempty compact subset of X,*

(ii) $P_i : X \to 2^{X_i}$ *is an $\mathcal{A}_C$-majorized mapping,*

(iii) $W_i = \{x \in X : P_i(x) \neq \emptyset\}$ *is open in X,*

(iv) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\text{cint}(P_i^-(\pi_i(y))) : y \in L_N\}.$$

Then, Λ has an equilibrium point in K.

In [3, Remark 5.5], its author noted that Theorem 5.3 improves and generalizes Theorem 5.2 of Ding and Feng [1], Theorem 6.2 of Ding and Wang [2], other results of Yuan, Chowdhury et al., and Tan-Yuan.

Moreover, applying Theorem 5.3, Du [3] proved the following equilibria existence theorem for a noncompact generalized game.

Theorem 5.4 ([3, Theorem 5.6]). *Let $\Lambda = (X; X_i; A_i; B_i; P_i; \pi)_{i \in I}$ be a generalized game. Let K be a compact and closed subset of X and $\text{co}_\Gamma(\pi_i(K)) = \pi_i(K)$. Suppose that for each $i \in I$,*

(i) $(X_i; \Gamma_i)$ *is a paracompact abstract convex space such that $(X; \Gamma)$ is a partial KKM space,*

(ii) *for each $x \in X$, $\text{co}_\Gamma(A_i(x)) \subset \bar{B}_i(x)$, $A_i(x) \cap P_i(x) \subset \pi_i(K)$ and $\text{dom}\, A_i = X$,*

(iii) *for each $y \in X_i$, $A_i^-(y)$ is compactly open in X,*

(iv) $W_i = \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ *is open in X,*

(v) $A_i \cap P_i : X \to 2^{X_i}$ *is an $\mathcal{A}_C$-majorized mapping,*

(vi) *for each $N \in \langle X \rangle$, there exists a compact abstract convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup \{\text{cint}(A_i \cap P_i)^-(y) : y \in L_N\}.$$

Then, Λ has an equilibrium point $\hat{x}$ in K.

In [3, Remark 5.7], its author noted that Theorem 5.4 generalized Theorem 5.3 of Ding and Feng [1], Theorem 6.3 of Ding and Wang [2], other results of Chowdhury et al. and Tan-Yuan.

Finally, recall that most of results in [3] can be improved by the method of [4], and that most of authors mentioned in this paper used to claim generalizations without giving any supporting examples or justifications.

References

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