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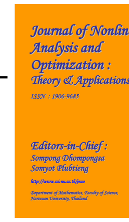


# A GENESIS OF GENERAL KKM THEOREMS FOR ABSTRACT CONVEX SPACES: REVISITED

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**ABSTRACT.** In our previous work [4], we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In this paper, we show that these three theorems are mutually equivalent. Actually, by adopting a method of making new abstract convex spaces from old, we give a direct proof of Theorem C from Theorem B.

**KEYWORDS** : Abstract convex space; (partial) KKM principle; (partial) KKM space.
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## 1. INTRODUCTION

The celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 is concerned with certain types of multimaps later called the KKM maps. The KKM theory, first named by the author, is the study of applications of equivalent formulations or generalizations of the KKM theorem. Actually the KKM theorem has several hundred generalizations in the literature.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory; see [1, 3].

In our previous work [2], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in [4], resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors. In [4], we recalled some historically important previous particular versions of our KKM type theorems in order to give a short history on each of them. Moreover, further remarks on related works were given in [5, 6].

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Note that the proofs of Theorems B and C in [4] are based on Theorem A, which is an abstract form of the original KKM Theorem. Some other proofs of Theorem C are also given in [5].

This paper is a continuation of [4]. In this paper, by adopting a method of making new abstract convex spaces from old, we give a direct proof of Theorem C from Theorem B. Consequently, Theorems A, B, and C are mutually equivalent.

## 2. ABSTRACT CONVEX SPACES

For the concepts of abstract convex spaces and KKM spaces, the reader may consult with the references in [3–6].

**Definition 2.1.** An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space $E$, a nonempty set $D$, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of $D$.

For any $D' \subset D$, the $\Gamma$-*convex hull* of $D'$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset $X$ of $E$ is called a $\Gamma$-*convex subset* of $(E, D; \Gamma)$ relative to $D'$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

**Definition 2.2.** Let $(E, D; \Gamma)$ be an abstract convex space and $Z$ a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then $G$ is called a *KKM map* with respect to $F$. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map $1_E$.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to $F$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

**Definition 2.3.** The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

We had the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ $\phi_A$-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

## 3. GENERAL KKM THEOREMS A, B, AND C

In [4], we gave standard forms of the KKM type theorems as follows:

**Theorem A.** *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) *$G$ has closed [resp., open] values; and*

(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D\rangle$ (that is, $G$ is a KKM map).*

*Then* $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

*Further, if*

(3) $\bigcap_{y\in M}\overline{G(y)}$ *is compact for some* $M\in\langle D\rangle$,

*then we have*

$$\bigcap_{y\in D}\overline{G(y)}\neq\emptyset.$$

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM principle.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space $Z$:

(a) $\bigcap_{y\in D}\overline{G(y)}\neq\emptyset$ implies $\bigcap_{y\in D}G(y)\neq\emptyset$.

(b) $\bigcap_{y\in D}\overline{G(y)}=\overline{\bigcap_{y\in D}G(y)}$ ($G$ is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D}\overline{G(y)}=\bigcap_{y\in D}G(y)$ ($G$ is *transfer closed-valued*).

(d) $G$ is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

**Theorem B.** *Let* $(E, D; \Gamma)$ *be a partial KKM space* [*that is,* $1_E\in\mathfrak{KC}(E,D,E)$] *and* $G : D\multimap E$ *a map such that*

(1) $\overline{G}$ *is a KKM map* [*that is,* $\Gamma_A\subset\overline{G}(A)$ *for all* $A\in\langle D\rangle$]*; and*

(2) *there exists a nonempty compact subset* $K$ *of* $E$ *such that either*

(i) $\bigcap\{\overline{G(y)}\mid y\in M\}\subset K$ *for some* $M\in\langle D\rangle$*; or*

(ii) *for each* $N\in\langle D\rangle$*, there exists a compact* $\Gamma$*-convex subset* $L_N$ *of* $E$ *relative to some* $D'\subset D$ *such that* $N\subset D'$ *and*

$$\overline{L_N}\cap\bigcap_{y\in D'}\overline{G(y)}\subset K.$$

*Then we have* $K\cap\bigcap_{y\in D}\overline{G(y)}\neq\emptyset$.

*Furthermore,*

$(\alpha)$ *if* $G$ *is transfer closed-valued, then* $K\cap\bigcap\{G(y)\mid y\in D\}\neq\emptyset$*;*

$(\beta)$ *if* $G$ *is intersectionally closed-valued, then* $\bigcap\{G(y)\mid y\in D\}\neq\emptyset$.

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A; see [4, 5].

Theorem B can be extended for $F\in\mathfrak{KC}(E,D,Z)$ instead of $1_E\in\mathfrak{KC}(E,D,E)$) as the following in [4, 5]:

**Theorem C.** *Let* $(E, D; \Gamma)$ *be an abstract convex space,* $Z$ *a topological space,* $F\in\mathfrak{KC}(E,D,Z)$*, and* $G : D\multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* $F$*; and*

(2) *there exists a nonempty compact subset* $K$ *of* $Z$ *such that either*

(i) $\bigcap\{\overline{G(y)}\mid y\in M\}\subset K$ *for some* $M\in\langle D\rangle$*; or*

(ii) *for each* $N\in\langle D\rangle$*, there exists a* $\Gamma$*-convex subset* $L_N$ *of* $E$ *relative to some* $D'\subset D$ *such that* $N\subset D'$*,* $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)}\cap\bigcap_{y\in D'}\overline{G(y)}\subset K.$$

*Then we have*

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

*Furthermore,*

$(\alpha)$ *if* $G$ *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*

$(\beta)$ *if* $G$ *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

**Remark 3.1.** 1. Taking $\overline{K}$ instead of $K$, the closure notation in (i) and (ii) can be erased.

2. It is clear from Theorem C $\Rightarrow$ Theorem B $\Rightarrow$ Theorem A(closed case). In our previous proofs of Theorems B and C are based on Theorem A.

## 4. A WAY OF MAKING NEW ABSTRACT CONVEX SPACES FROM OLD

In this section we deduce Theorem C from Theorem B, based on a method of making new abstract convex spaces from old.

Consider the situation in Theorem C:

**Definition 4.1.** Let $(E, D; \Gamma)$ be an abstract convex space, $Z$ a topological space, and $F : E \multimap Z$ a map. Let $\Lambda_A := F(\Gamma_A)$ for each $A \in \langle D \rangle$. Then $(Z, D; \Lambda)$ is called the *abstract convex space induced by F*.

Let $Y \subset Z$ and $D' \subset D$ such that $\Lambda_B \subset Y$ for each $B \in \langle D' \rangle$. Then $Y$ is called a $\Lambda$-*convex subset* of $Z$ relative to $D'$, and $(Y, D'; \Lambda')$ a *subspace* of $(Z, D; \Lambda)$ whenever $\Lambda' = \Lambda|_{\langle D' \rangle}$.

**Proposition 4.2.** *A KKM map* $G : D \multimap Z$ *on an abstract convex space* $(E, D; \Gamma)$ *with respect to* $F : D \multimap Z$ *is simply a KKM map on the corresponding abstract convex space* $(Z, D; \Lambda)$ *induced by* $F$.

*Proof.* Simply note that $\Lambda_A := F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$. □

**Proposition 4.3.** *For an abstract convex space* $(E, D; \Gamma)$*, the corresponding abstract convex space* $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a partial KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z)$.

*The abstract convex space* $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z) \cap \mathfrak{KO}(E, D, Z)$.

*Proof.* $(Z, D; \Lambda)$ is a partial KKM space

$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ (that is, $\Lambda_A = F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$), it has the finite intersection property of map-values.

$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ with respect $F$, it has the finite intersection property of map-values.

$\Longleftrightarrow$ $F \in \mathfrak{KC}(E, D, Z)$.

Similarly, for open-valued KKM map $G : D \multimap Z$, it has the finite intersection property of map-values $\Longleftrightarrow$ $F \in \mathfrak{KO}(E, D, Z)$. □

The following is our main result in this paper:

**Proposition 4.4.** *Theorems B and C are equivalent.*

*Proof.* By putting $E = Z$ and $F = 1_E$, Theorem C reduces to Theorem B. Now we show that Theorem C follows from Theorem B as follows.

Let $(Z, D; \Lambda)$ be the abstract convex space induced by $F$ with $\Lambda_A := F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$. Then

(1) $\overline{G}$ is a KKM map on $(Z, D; \Lambda)$.

(2) Condition (i) implies Theorem B(i) with $\overline{F(E)} \cap K$ instead of $K$.

In fact, since $F(\Gamma_A) \subset \overline{G}(A)$ for each $A \in \langle D \rangle$, we have $F(\Gamma_{\{y\}}) \subset \overline{G}(y)$ for each $y \in D$ and so

$$K \supset \bigcap_{y \in M} \overline{G}(y) \supset \bigcap_{y \in M} F(\Gamma_{\{y\}}) \cap \bigcap_{y \in M} \overline{G}(y).$$

Hence

$$\overline{F(E)} \cap K \supset \bigcap \{\overline{G(y)} \mid y \in M\} \text{ for some } M \in \langle D \rangle.$$

(3) Condition (ii) implies Theorem B(ii) with $\overline{F(E)} \cap K$ instead of $K$ and with a compact $\Lambda$-convex subset $\overline{F(L_N)}$ of $Z$ instead of $L_N$.

In fact, this can be shown by the following two facts:

(a) $A \in \langle D' \rangle$ implies $\Gamma_A \subset L_N$ and hence $\Lambda_A = F(\Gamma_A) \subset F(L_N) \subset \overline{F(L_N)}$. So $\overline{F(L_N)}$ is compact and $\Lambda$-convex.

(b) $K \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}$ by assumption and clearly we have $\overline{F(E)} \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}$. Hence

$$\overline{F(E)} \cap K \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}.$$

Consequently, replacing $(E, D; \Gamma)$, $K$, $L_N$ in Theorem B by $(Z, D; \Lambda)$, $\overline{F(E)} \cap K$, $\overline{F(L_N)}$, respectively, all of the requirements of Theorem B are satisfied. Therefore we have the conclusion

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

The statements $(\alpha)$ and $(\beta)$ are routine. □

## 5. COMMENTS ON RELATED WORKS

In this section, we give some comments on our previous versions of generalized KKM theorems A, B, and C appeared in our previous works:

(1) Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [1].

(2) In 2008 and 2010 [1, 3], we established the basis of the KKM theory and gave particular forms of Theorems A and B. Note that [3] contains some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

(3) In 2009 [2], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in this paper, resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors.

(4) In 2011 [4], we established the basic KKM theorems A, B, and C, and recalled some historically important previous particular versions of these theorems in order to give a short history on each of them. Moreover, further comments on related works are given.

(5) In 2011 [5], we deduced Theorems B and C from Theorem A and added a new proof of Theorem C. Some corrections of the coercivity conditions appeared in previous versions of such KKM type theorems were given.

(6) In 2012 [6], we give several generalizations of the 1984 KKM theorem of Ky Fan and some known applications in order to recover the close relationship among them. Theorems A, B, and C are stated as the final ones in the evolution of the KKM theorem from 1929. It is stated that, as far as the author knows, Theorem C contains several hundred generalizations of the KKM theorem appeared in the existing literature; see the references therein.

(7) Further study on our new method of making abstract convex spaces from old will appear.

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