

RECENT APPLICATIONS OF THE FAN-KKM THEOREM

Sehie Park

*The National Academy of Sciences, ROK, Seoul 137-044; and
Department of Mathematical Sciences, Seoul National University,
Seoul 151-747, KOREA
shpark@math.snu.ac.kr; parkcha38@daum.net*

ABSTRACT. In this review, firstly, we recall Ky Fan's contributions to the KKM theory based on his celebrated 1961 KKM lemma (or the Fan-KKM theorem). Secondly, we introduce relatively recent applications of the Fan lemma due to other authors in the 21st century. Finally, some historical remarks on related works are added.

1. Introduction

The Knaster–Kuratowski–Mazurkiewicz (KKM for short) theorem in 1929 [28] is concerned with a particular type of multimaps, later called KKM maps. The KKM theory, first called so by the author in 1992 [42,43], is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961, Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone on the history of the KKM theory was erected by Fan [8]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

At the beginning, the basic theorems in the theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84 [8-14]. Since then a large number of intersection theorems and their applications to equilibrium problems followed. Then, the KKM theory has been extended to convex spaces by Lassonde in 1983, and to c -spaces (or H-spaces) by Horvath in 1984-93 and others. Since

2010 Mathematics Subject Classification. 47H04, 47H10, 47J20, 47N10, 49J53, 52A99, 54C60, 54H25, 58E35, 90C47, 91A13, 91B50.

Key words and phrases. KKM theorem, Fan's KKM lemma, KKMF theorem, convex space, H-space, G-convex space, abstract convex space, fixed point, equilibrium problem.

1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. From 2006, the main theme of the theory became abstract convex spaces in the sense of Park. Consequently, the basic theorems in the theory have many applications to various equilibrium problems in nonlinear analysis and other fields.

However, even in the 21st century, still Fan's 1961 KKM lemma is applied by many authors to various problems. Our main aim in this review is to recall Fan's contributions to the KKM theory and to review relatively recent applications of his lemma due to other authors in the 21st century.

In Section 2, we recall Fan's contribution to the KKM theory based on his celebrated 1961 KKM lemma (or the KKMF theorem). In Section 3, we discuss relatively recent applications of the Fan lemma due to other authors in the 21st century. Finally, Section 4 deals with some historical remarks on related works.

All references given by the form (year) can be found in [44] or the references therein.

2. The Origin and Fan's Applications

In this section, we will follow our [44,46].

Knaster, Kuratowski, and Mazurkiewicz in 1929 [28] obtained the so-called KKM theorem from the Sperner combinatorial lemma in 1928, and applied it to a simple proof of the Brouwer fixed point theorem. Later these three theorems are known to be mutually equivalent.

The KKM theorem was extended by Fan in 1961 as follows:

Lemma. [8] *Let X be an arbitrary set in a topological vector space Y . To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

- (i) *convex hull of any finite subset $\{x_1, \dots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
- (ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the Fan-KKM lemma or the Fan-KKM theorem or the KKMF theorem. Fan assumed the Hausdorffness of Y , which was known to be superfluous later.

Fan also obtained the geometric or section property of convex sets, which is equivalent to the preceding Lemma. Fan [8] applied this property to give a simple proof of the Tychonoff fixed point theorem and to prove two results generalizing the Pontrjagin-Iohvidov-Kreĭn theorem on existence of invariant subspaces of certain linear operators. Also, Fan [9] applied his KKM lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff

theorem. The main results of Fan [10] were extended by Ma in 1969 [38], who obtained a generalization of the Nash equilibrium theorem for infinite case.

Moreover, “a theorem concerning sets with convex sections” was applied to prove many results in 1966 [10].

On the other hand, Browder in 1968 [2] obtained an equivalent result to Fan’s geometric lemma [8] in the convenient form of a fixed point theorem which is known as the Fan-Browder fixed point theorem. Later this is also known to be equivalent to the Brouwer theorem. Browder [2] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding (1976) and Yannelis and Frabhar (1983), to the existence of maximal elements in mathematical economics.

In 1969, Fan [11] deduced best approximation theorems from his geometric lemma and applied them to generalizations of the Brouwer theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps.

Moreover, Fan in 1972 [12] established a minimax inequality from the KKM theorem and applied it to many problems; see [46].

Furthermore, Fan in 1979 and 1984 [13,14] introduced a KKM theorem with a coercivity (or compactness) condition for noncompact convex sets as follows:

Theorem. [14] *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y , then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

From this, Fan extended many of known results to noncompact cases; see [46].

The concept of convex sets in a topological vector space is extended to convex spaces by Lassonde (1983), and further to c -spaces by Horvath (1983-91). A number of other authors also extended the concept of convexity for various purposes. Note that Lassonde first noticed that the Hausdorffness in the KKM theorem is redundant.

Definition. Let X be a subset of a vector space and D a nonempty subset of X . We call (X, D) a *convex space* if $\text{co } D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$; see Park (1994). If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde.

Lassonde (1983) presented a simple and unified treatment of a large variety of minimax and fixed point problems. He first noticed that Hausdorffness in the Fan lemma

is redundant. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem.

3. Recent Applications of the Fan-KKM Theorem

In this section, we introduce relatively recent applications of the Fan lemma (the KKMF theorem) due to other authors in the 21st century:

(I) In 2000, Lee and Lee [31] studied the existence of solutions to the vector variational-type inequalities for set-valued mappings on Hausdorff topological vector spaces using Fan's geometrical lemma, which is equivalent to the KKMF theorem.

(II) In 2000, Chadli et al. [5] applied the KKMF theorem to various equilibrium problems. The Hausdorffness is assumed in the KKMF theorem.

(III) Li in 2001 [32] used the technique of KKM map and a KKMF theorem to study the existence of eigenvectors for some maps on normed linear spaces.

(IV) In 2003, by applying the KKMF theorem, Kristály and Varga [29] proved two set-valued versions of the Fan minimax inequality, and applied them to fixed point theorems, a variational inclusion problem, a differential inclusion problem, and others.

(V) Abstract of Khanh and Luu [26] in 2004: "For vector quasivariational inequalities involving multifunctions in topological vector spaces, an existence result is obtained without a monotonicity assumption and with a convergence assumption weaker than semicontinuity. A new type of quasivariational inequality is proposed. Applications to quasicomplementarity problems and traffic network equilibria are considered. In particular, definitions of weak and strong Wardrop equilibria are introduced for the case of multivalued cost functions."

They are based on the KKMF theorem.

(VI) Li [33] in 2004 studied the existence of the solution of the variational inequality $\langle Tx - \xi, y - x \rangle \geq 0$ by applying the generalized projection operator $\pi_K : B^* \rightarrow B$, where B is a Banach space with dual space B^* and by using the KKMF theorem.

(VII) Abstract of Fakhar and Zafarani [7] in 2005: "Existence results for quasimonotone vector equilibrium problems and quasimonotone vector variational inequalities are obtained starting from an existence result for a scalar equilibrium problem involving two quasimonotone bifunctions."

This paper is based on the KKMF theorem.

(VIII) Abstract of Khanh and Luu [27] in 2005: "Some existence results for vector quasivariational inequalities with multifunctions in Banach spaces are derived by employing the KKMF theorem. In particular, we generalize a result by Lin, Yang and

Yao [36], and avoid monotonicity assumptions. We also consider a new quasivariational inequality problem and propose notions of weak and strong equilibria while applying the results to traffic network problems.”

(IX) Abstract of Hai and Khanh [23] in 2007: “A general quasiequilibrium problem is proposed including, among others, equilibrium problems, implicit variational inequalities, and quasivariational inequalities involving multifunctions. Sufficient conditions for the existence of solutions with and without relaxed pseudomonotonicity are established. Even semicontinuity may not be imposed.”

Fan’s 1984 KKM theorem was applied.

(X) Abstract of Hai and Khanh [24] in 2007: “We propose general variational inclusion problems which are slightly different from corresponding problems considered in several recent papers in the literature and show that they are advantageous. Sufficient conditions for the solution existence are established. As applications we derive consequences for several special cases of variational inclusion problems, quasioptimization problems, equilibrium problems and implicit variational inequalities . . .”

(XI) Abstract of Mitrović [39] in 2007: “In this paper, we prove the existence of a solution to the simultaneous nonlinear inequality problem. As applications, we derive the results on the simultaneous approximations, variational inequalities and saddle points.”

This paper is based on the KKMF theorem.

(XII) Niculaescu and Rovent [41] in 2007 introduced the concepts of the weighted M_p -mean for pairs of positive reals and M_p -concave real functions on nonempty compact convex subsets of a topological vector space. This includes concave functions and quasi-concave functions. The aim of this paper is to prove a nonsymmetric extension and a variant (for M_p -convex functions) of the Ky Fan minimax inequality based on Fan’s KKM lemma. As applications, the Nash equilibrium existence theorem is generalized to the existence of a g -equilibrium and a new proof of the Sion minimax theorem is obtained.

(XIII) Abstract of Farajzadeh et al. [19] in 2008: “We first define upper sign continuity for a set-valued mapping and then we consider two types of generalized vector equilibrium problems in topological vector spaces and provide sufficient conditions under which the solution sets are nonempty and compact. Finally, we give an application of our main results. The paper generalizes and improves results obtained by Fang and Huang in 2005 [16].”

They are based on a particular form of the 1984 KKM Theorem of Fan. Moreover, their Lemma 2.10 is based on Dobrowolski’s incorrect theorem; see Park [45].

(XIV) Abstract of Liu et al. [37] in 2008: “This paper is devoted to study a new class of generalized vector quasi-equilibrium problems with set-valued mappings. By

means of the KKMF theorem and lower semicontinuity with respect to cone order of the set-valued mapping, we obtain an existence result for this class of generalized vector quasi-equilibrium problems with set-valued mappings. . . .”

(XV) Abstract of Farajzadeh et al. [20] in 2009: “In this work, we consider a generalized nonlinear variational-like inequality problem, in topological vector spaces, and, by using the KKM technique, we prove an existence theorem. Our result extends a theorem of Ahmad and Irfan [1].”

They are based on a particular form of the 1984 KKM Theorem of Fan.

(XVI) Abstract of Li and Li [34] in 2009: “This paper deals with three classes of generalized vector quasi-equilibrium problems with or without compact assumptions. Using the well-known Fan-KKM theorems, their existence theorems for them are established. Some examples are given to illustrate our results.”

(XVII) Abstract of Khan [25] in 2010: “In this paper, we introduce and study a generalized class of vector implicit quasi complementarity problem and the corresponding vector implicit quasi variational inequality problem. By using Fan-KKM theorem, we derive existence of solutions of generalized vector implicit quasi variational inequalities without any monotonicity assumption and establish the equivalence between those problems in Banach spaces.”

(XVIII) Abstract of Mitrović and Merkle [40] in 2010: “We prove the existence of a solution to the generalized vector equilibrium problem with bounds. We show that several known theorems from the literature can be considered as particular cases of our results, and we provide examples of applications related to best approximations in normed spaces and variational inequalities.”

They are based on the 1984 Theorem of Fan.

(XIX) Abstract of Ceng and Huang [3] in 2010: “In this paper we study the solvability of the generalized vector variational inequality problem, the GVVI problem, with a variable ordering relation in reflexive Banach spaces. The existence results of strong solutions of GVVIs for monotone multifunctions are established with the use of the KKM-Fan theorem. . . .”

(XX) Abstract of Farajzadeh et al. [18] in 2010: “This paper deals with some existence theorems for generalized vector variational-like inequalities with set-valued mappings in topological vector spaces. . . .”

Using the KKMF theorem and the Kakutani-Fan-Glicksberg fixed-point theorem, we establish some existence results for these generalized variational-like inequalities. The results presented in this paper generalize and improve some results of Fang and Huang [15,16].”

(XXI) Abstract of Lin and Chen [35] in 2011: “We study the weak solutions and strong solutions of set equilibrium problems in real Hausdorff topological vector space settings. Several new results of existence for the weak solutions and strong solutions of set equilibrium problems are derived. . . .”

(XXII) Abstract of Golshan and Farajzadeh [22] in 2011: “In this paper, we introduce and study the generalized implicit vector variational inequality problems with set valued mappings in topological vector spaces. We establish existence theorems for the solution set of these problems to be nonempty compact and convex. Our results extend the results by Fang and Huang [15].”

The main theorem is obtained by applying KKMF theorem assuming the Hausdorffness.

(XXIII) Abstract of Farajzadeh [17] in 2011: “In this paper, we introduce and consider a new class of vector mixed quasi-variational inequality and vector complementarity problem in a topological vector space. We show that under certain conditions the solution set of the vector mixed quasi-complementarity problem equals to the set of the vector mixed quasi-variational inequalities. Using the Ky Fan KKM lemma, we study the existence of a solution of the vector mixed quasi-variational inequalities and vector mixed quasi complementarity problems. Moreover we discuss on some of our assumptions. Our results extend those of Farajzadeh et al. [21] to the vector case.”

Lemma 3.1 seems to be incorrect.

(XXIV) Abstract of László [30] in 2011: “In this paper, we introduce a new class of operators. We present some fundamental properties of the operators belonging to this class and, as applications, we establish some existence results of the solutions for several general variational inequalities involving elements belonging to this class.”

Existences of the solutions of general variational inequalities are based on the KKMF theorem assuming the Hausdorffness.

(XXV) Abstract of Costea et al. [6] in 2012: “The aim of this paper is to establish existence results for some variationallike inequality problems involving set-valued maps, in reflexive and nonreflexive Banach spaces. When the set K , in which we seek solutions, is compact and convex, we do not impose any monotonicity assumptions on the set-valued map A in the inequality problems. In the case when K is only bounded, closed, and convex, certain monotonicity assumptions are needed . . . We also provide sufficient conditions for the existence of solutions in the case when K is unbounded, closed, and convex.”

Ky Fan’s KKM lemma assuming the Hausdorffness is used to establish the existence of at least one solution for a certain inequality problem.

(XXVI) Abstract of Ceng and Yao [4] in 2012: “In this paper, utilizing the properties of the generalized f -projection operator and the well-known KKM and Kakutani-Fan-Glicksberg theorems, under quite mild assumptions, we derive some new existence theorems for the generalized set-valued mixed variational inequality and the generalized set-valued mixed quasi-variational inequality in reflexive and smooth Banach spaces, respectively. The results presented in this paper can be viewed as the supplement, improvement and extension of recent results in Wu and Huang [49].”

The KKMF theorem was applied.

(XXVII) Abstract of Tang and Huang [48] in 2012: “This paper is devoted to the existence of solutions for the variational- hemivariational inequalities in reflexive Banach spaces. Using the notion of the stable φ -quasimonotonicity and the properties of Clarke’s generalized directional derivative and Clarke’s generalized gradient, some existence results of solutions are proved when the constrained set is nonempty, bounded (or unbounded), closed and convex. Moreover, a sufficient condition to the boundedness of the solution set and a necessary and sufficient condition to the existence of solutions are also derived.”

Fan’s KKM lemma assuming the Hausdorffness is used.

4. Comments and Historical Remarks

Recall that the main theme of applications introduced in Section 3 can be summarized as follows:

- Vector variational-type inequalities
- Various quasi-equilibrium problems
- Eigenvector problems
- Set-valued minimax inequality
- Fixed point theorems
- Generalizations of Nash equilibrium theorem
- Variational inclusion problem
- Simultaneous nonlinear inequalities problem
- Differential inclusion problem
- (Vector mixed) quasi-variational inequality
- (Vector mixed) quasi-complementarity problem
- Traffic network problem
- Quasi-monotone vector equilibrium problem
- Generalized vector equilibrium problem
- Generalized (implicit) vector variational-like inequality
- Set equilibrium problem
- Set-valued mixed (quasi-)variational inequalities
- Variational-hemivariational inequalities

In 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [47] and the references therein. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its “open” version. Now the KKM theory becomes the study of spaces satisfying the partial KKM principle.

For abstract convex spaces, the following diagram is known:

$$\begin{aligned} \text{Simplex} &\implies \text{Convex subset of a t.v.s.} \implies \text{Convex space} \implies \text{H-space} \\ &\implies \text{G-convex space} \implies \phi_A\text{-space} \implies \text{KKM space} \\ &\implies \text{Partial KKM space} \implies \text{Abstract convex space} \end{aligned}$$

In our previous work [46], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and several equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [P6] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

In view of such generalizations of the KKM theory, many results in the works mentioned in Section 3 can be stated in more general situations without assuming the Hausdorffness of convex subsets.

REFERENCES

- [1] R. Ahmad, S.S. Irfan, *On the generalized nonlinear variational-like inequality problems*, Appl. Math. Lett. **19** (2006), 294–297.
- [2] F. E. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.
- [3] L.-C. Ceng, S. Huang, *Existence theorems for generalized vector variational inequalities with a variable ordering relation*, J. Glob. Optim. **46** (2010), 521–535.
- [4] L.-C. Ceng, J.-C. Yao, *Existence theorems for generalized set-valued mixed (quasi-)variational inequalities in Banach spaces*, J. Glob. Optim. DOI 10.1007/s10898-011-9811-1.
- [5] O. Chadli, Z. Chbani, H. Riahi, *Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities*, J. Optim. Theory Appl. **105**(2) (2000), 299–323.
- [6] N. Costea, D. A. Ion, C. Lupu, *Variational-like inequality problems involving set-valued maps and generalized monotonicity*, J. Optim. Theory Appl. DOI 10.1007/s10957-012-0047-0.
- [7] M. Fakhar, J. Zafarani, *Equilibrium problems in the quasimonotone case*, J. Optim. Th. Appl. **126**(1) (2005), 125–136.
- [8] K. Fan, *A generalization of Tychonoff’s fixed point theorem*, Math. Ann. **142** (1961), 305–310.
- [9] K. Fan, *Sur un théorème minimax*, C.R. Acad. Sci. Paris Sér. I. Math. **259** (1964), 3925–3928.

- [10] K. Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.
- [11] K. Fan, *Extensions of two fixed point theorems of F.E. Browder*, Math. Z. **112** (1969), 234–240.
- [12] K. Fan, *A minimax inequality and applications*, Inequalities III (O. Shisha, ed.), 103–113, Academic Press, New York, 1972.
- [13] K. Fan, *Fixed-point and related theorems for non-compact convex sets*, Game Theory and Related Topics (O. Moeschlin and D. Pallaschke, eds.), 151–156, North-Holland, Amsterdam, 1979.
- [14] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.
- [15] Y.-P. Fang, N.-J. Huang, *Existence results for systems of strong implicit vector variational inequalities*, Acta Math. Hung. **103** (2004), 265–277.
- [16] Y.-P. Fang, N.-J. Huang, *Existence results for generalized implicit vector variational inequalities with multivalued mappings*, Indian J. Pure Appl. Math. **36**(11) (2005), 629–640.
- [17] A. Farajzadeh, *On the vector mixed quasi-variational inequality problems*, J. Nonlinear Anal. Optim. **2**(1) (2011), 171–178.
- [18] A. P. Farajzadeh, A. Amini-Harandi, K. R. Kazmi, *Existence of solutions to generalized vector variational-like inequalities*, J. Optim. Theory Appl. **146** (2010), 95–104.
- [19] A. P. Farajzadeh, A. Amini-Harandi, D. O’Regan, *Existence results for generalized vector equilibrium problems with multivalued mappings via KKM Theory*, Abst. Appl. Anal. vol. 2008, Article ID 968478, 8 pages doi:10.1155/2008/968478.
- [20] A. P. Farajzadeh, A. Amini-Harandi, D. O’Regan, R. P. Agarwal, *New kinds of generalized variational-like inequality problems in topological vector spaces*, Appl. Math. Letters **22** (2009), 1126–1129.
- [21] A. P. Farajzadeh, M. A. Noor, S. Zainab, *Mixed quasi complementarity problems in topological vector spaces*, J. Global. Optim. **45** (2009), 229–235.
- [22] H. M. Golshan, A. Farajzadeh, *On generalized variational inequality problems*, J. Nonlinear Anal. Optim. **2**(1) (2011), 1–9.
- [23] N. X. Hai, P. Q. Khanh, *Existence of solutions to general quasiequilibrium problems and applications*, J. Optim. Theory Appl. **133** (2007), 317–327.
- [24] N. X. Hai, P. Q. Khanh, *The solution existence of general variational inclusion problems*, J. Math. Anal. Appl. **328** (2007), 1268–1277.
- [25] S. A. Khan, *Generalized vector implicit quasi complementarity problems*, J. Glob. Optim. DOI 10.1007/s10898-010-9557-1.
- [26] P. Q. Khanh, L. M. Luu, *On the existence of solutions to vector quasivariational inequalities and quasicomplementarity problems with applications to traffic network equilibria*, J. Optim. Theory Appl. **123**(3) (2004), 533–548.
- [27] P. Q. Khanh, L. M. Luu, *Some existence results for vector quasivariational inequalities involving multifunctions and applications to traffic equilibrium problems*, J. Global Optim. **32** (2005), 551–568.
- [28] B. Knaster, K. Kuratowski, S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für n -Dimensionale Simplexe*, Fund. Math. **14** (1929), 132–137.
- [29] A. Kristály, C. Varga, *Set-valued versions of Ky Fan’s inequality with application to variational inclusion theory*, J. Math. Anal. Appl. **282** (2003), 8–20.
- [30] S. László, *Some existence results of solutions for general variational inequalities*, J. Optim. Theory Appl. **150** (2011), 425–443.
- [31] B.-S. Lee, S.-J. Lee, *Vector variational-type inequalities for set-valued mappings*, Appl. Math. Lett. **13** (2000), 57–62.
- [32] J. Li, *Some eigenvector theorems proved by a Fan-KKM theorem*, J. Math. Anal. Appl. **263** (2001), 738–747.
- [33] J. Li, *On the existence of solutions of variational inequalities in Banach spaces*, J. Math. Anal. Appl. **295** (2004), 115–126.
- [34] X. B. Li, S. J. Li, *Existence of solutions for generalized vector quasi-equilibrium problems*, Optimization Lett. **4**(1) (2009), 17–28.

- [35] Y.-C. Lin, H.-J. Chen, *Solving the set equilibrium problems*, Fixed Point Theory Appl. vol.2011, Article ID 945413, 13pp.
- [36] K. L. Lin, D. P. Yang, J. C. Yao, *Generalized vector variational inequalities*, J. Optim. Theory Appl. **92** (1997), 425–443.
- [37] Q.-M. Liu, L. Fan, G. Wang, *Generalized vector quasi-equilibrium problems with set-valued mappings*, Appl. Math. Lett. **21** (2008), 946–950.
- [38] T.-W. Ma, *On sets with convex sections*, J. Math. Anal. Appl. **27** (1969), 413–416.
- [39] Z. D. Mitrović, *The simultaneous nonlinear inequalities problem and applications*, J. Inequal. Pure Appl. Math. **8**(3) (2007), Article 84, 8pp.
- [40] Z. D. Mitrović, M. Merkle, *On a generalized vector equilibrium problem with bounds*, Appl. Math. Lett. **23** (2010), 783–787.
- [41] C. Niculescu, I. Rovent, *Fans inequality in the context of M_p -convexity*, Applied Analysis and Differential Equations, pp.267–274, World Sci. Publ., Hackensack, NJ, 2007.
- [42] S. Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, Ed.), pp.248–277, World Sci. Publ., River Edge, NJ, 1992.
- [43] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc. **31** (1994), 493–519.
- [44] S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 187–222.
- [45] S. Park, *Remarks on recent results in analytical fixed point theory*, Nonlinear Analysis and Convex Analysis (NACA 2005, Okinawa), pp.517–525, Yokohama Publ., Yokohama, 2007.
- [46] S. Park, *A brief history of the KKM theory*, RIMS Kôkyûroku, Kyoto Univ. **1643** (2009), 1–16.
- [47] S. Park, *The KKM principle in abstract convex spaces: Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.
- [48] G.-J. Tang, N.-J. Huang, *Existence theorems of the variational-hemivariational inequalities*, J. Glob. Optim. DOI 10.1007/s10898-012-9884-5.
- [49] K. Q. Wu, N.-J. Huang, *The generalized f -projection operator and set-valued variational inequalities in Banach spaces*, Nonlinear Anal. **71** (2009), 2481–2490.