Research Article

Evolution of the Minimax Inequality of Ky Fan

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There are quite a few generalizations or applications of the 1984 minimax inequality of Ky Fan compared with his original 1972 minimax inequality. In a certain sense, the relationship between the 1984 inequality and several hundreds of known generalizations of the original 1972 inequality has not been recognized for a long period. Hence, it would be necessary to seek such relationship. In this paper, we give several generalizations of the 1984 inequality and some known applications in order to clarify the close relationship among them. Some new types of minimax inequalities are added.

1. Introduction

The KKM theory is originated from the Knaster-Kuratowski-Mazurkiewicz (KKM for short) theorem of 1929 [1]. Since then, it has been found a large number of results which are equivalent to the KKM theorem; see [2, 3]. Typical examples of the most remarkable and useful equivalent formulations are Ky Fan's KKM lemma of 1961 [4] and his minimax inequality of 1972 [5]. The inequality and its various generalizations are very useful tools in various fields of mathematical sciences.

Since 1961, Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone in the history of the KKM theory was erected by Fan in 1961 [4]. His 1961 KKM Lemma (or the Fan-KKM theorem) extended the KKM theorem to arbitrary topological vector spaces and had been applied to various problems in his subsequent papers [5–10].

Recall that, at the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961–1984 [4–10]. A number of intersection theorems and their applications to various equilibrium problems followed. In our previous review [11], we recalled Fan's contributions to the KKM theory based on his celebrated 1961 KKM lemma,

and introduced relatively recent applications of the lemma due to other authors in the twenty-first century.

Then, the KKM theory was extended to convex spaces by Lassonde in 1983 [12] and to *c*-spaces (or H-spaces) by Horvath in 1983–1993 [13–16] and others. Since 1993, the theory has been extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others; see [2]. Since 2006, the main theme of the theory has become abstract convex spaces in the sense of Park [17–30]. The basic theorems in the theory have numerous applications to various equilibrium problems in nonlinear analysis and other fields. In our previous review [30], we recalled our versions of general KKM type theorems for abstract convex spaces and introduced relatively recent applications of various generalized KKM type theorems due to other authors in the twenty-first century.

While we were studying on [11, 30], we noticed that there are quite a few generalizations and applications of the 1984 minimax inequality of Fan compared with his 1972 inequality. In a certain sense, the 1984 inequality is not connected with several hundreds of known generalizations of the original minimax inequality. Hence, it would be necessary to seek such relationship. In this paper, we introduce several generalizations of two minimax inequalities of Fan and some known direct applications in order to clarify the close relationship among them.

This paper is organized as follows. In Section 2, we introduce Fan's 1961 KKM lemma, 1972 minimax inequality, and 1984 minimax inequality with some applications of them. Section 3 deals with the evolution of the 1984 inequality. Actually, we give its generalizations to convex spaces of Lassonde, convex spaces with the coercivity due to Chang, H-spaces originated by Horvath, and G-convex spaces and ϕ_A-spaces of Park. In Section 4, we introduce recently obtained basic concepts or results of the KKM theory such as the abstract convex spaces, generalized KKM theorems and general minimax inequalities. We show that one of our general minimax inequalities subsumes previously obtained inequalities on various types of KKM spaces in Section 3. Section 5 deals with other generalizations of the Fan minimax inequality and vector minimax inequality. Finally, we add some related historical remarks in Section 6.

2. The Origin and Fan's Applications

The KKM theorem of Knaster, Kuratowski, and Mazurkiewicz of 1929 [1] was extended by Fan in 1961 [4] as follows.

The 1961 KKM Lemma of Fan. Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:

   (i) convex hull of any finite subset $\{x_1, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$,

   (ii) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the *Fan-KKM lemma*, the *Fan-KKM theorem*, or the *KKMF theorem*. Fan and his followers applied his KKM lemma to various problems in many fields in mathematics; see [2, 29].

Five decades after the birth of the lemma, the above original form is still adopted by many authors in each year. But, it was found quite a long time ago by Lassonde [12] that the Hausdorffness is redundant. Moreover, note that Y can be any convex subset of a topological vector space.

Recall that an extended real-valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower semicontinuous* (l.s.c.) if $\{x \in X : f(x) > r\}$ is open for each $r \in \overline{\mathbb{R}}$.

For a convex set X, a function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* if $\{x \in X : f(x) > r\}$ is convex for each $r \in \overline{\mathbb{R}}$.

Similarly, the upper semicontinuity (u.s.c.) and the quasiconvexity can be defined.

One of the most remarkable equivalent formulations of the KKM theorem is the *minimax inequality* established by Fan from his KKM lemma. The following is the original form given by Fan in 1972 [5].

The Fan Minimax Inequality. Let X be a compact convex set in a Hausdorff topological vector space. Let f be a real-valued function defined on $X \times X$ such that

   (a) for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X;

   (b) for each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.

Then, the minimax inequality

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x) \tag{1}$$

holds.

In [10], Fan applied his inequality to the following:

 (1) a variational inequality (extending Hartman-Stampacchia (1966) and Browder (1967));

 (2) a geometric formulation of the inequality (equivalent to the Fan-Browder fixed point theorem (1968));

 (3) separation properties of u.d.c. multimaps, coincidence and fixed point theorems;

 (4) properties of sets with convex sections (from which the Sion minimax theorem (1958), the equilibrium theorem of Nash (1951), and a variant of a theorem of Debrunner and Flor (1964) on extension of monotone sets easily follow);

 (5) a fundamental existence theorem in potential theory.

The inequality became a crucial tool in proving many existence problems in various fields of mathematical sciences, for example, nonlinear analysis, especially in fixed point theory, variational inequality problems, various equilibrium theory, mathematical programming, partial differential equations, game theory, impulsive control, and mathematical economics; see [21, 26, 31] and the references therein.

Fan [10] noted that the following 1984 generalized minimax inequality extending Allen's variational inequality of 1977 [32] follows from his 1984 KKM theorem.

Theorem 1 (see [10]). *Let X be a nonempty convex set in a Hausdorff topological vector space. Let f be a real-valued function defined on $X \times X$ such that*

   (a) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X;*

   (b) *for each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X;*

   (c) *$f(x, x) \leq 0$ for all $x \in X$;*

   (d) *X has a nonempty compact convex subset X_0 such that the set $\{y \in X : f(x, y) \leq 0 \text{ for all } x \in X_0\}$ is compact.*

Then, there exists a point $\widehat{y} \in X$ such that $f(x, \widehat{y}) \leq 0$ for all $x \in X$.

The following proof was given by Fan [10].

Theorem 1 $\Rightarrow$ The Fan Minimax Inequality. Using condition (a), we see that in the case compact X, condition (d) is fulfilled by taking X_0 as any nonempty closed convex subset of X. Thus, Theorem 1 reduces to the minimax inequality.

In [10], Theorem 1 was applied to a best approximation theorem, coincidence and fixed point theorems, a matching

theorem for two closed covers of a convex set, another proof of the Brouwer fixed point theorem, and a generalization of Shapley's KKM theorem.

Recall that the original Nash equilibrium theorem was proved by the Brouwer or the Kakutani fixed point theorem; see [33, 34]. Later, Fan [7] proved it by applying his result on sets with convex sections. Nowadays, it is known to be one of the most important applications of the Fan minimax inequality; see [10, 27]. Note that, in a wide sense, the Brouwer theorem, the KKM theorem, the Kakutani theorem, the Nash theorem, Fan's theorem on sets with convex sections, the Fan inequality, the Fan-Browder fixed point theorem, and many others are mutually equivalent; see [2, 3].

3. Evolution of the Minimax Inequality of Ky Fan

The Fan minimax inequality has been followed by a large number of generalizations and applications in the KKM theory on convex subsets of topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, generalized convex spaces due to Park, and other types of spaces. Furthermore, many authors generalized the lower semicontinuity and quasiconcavity in the inequality or replaced them by other requirements. Therefore, even for convex spaces, it is necessary to establish proper forms of the Fan minimax inequality which unify as many particular cases as possible.

Multimaps are also simply called maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

In this section, we show several major generalizations of the Fan minimax inequality in the chronological order.

3.1. Lassonde's Theorem. The concept of convex sets in a topological vector space was extended to convex spaces by Lassonde in 1983 [12], and further to c-spaces by Horvath in 1983–1993 [13–16]. A number of other authors also extended the concept of convexity for various purposes.

Definition 2. Let X be a subset of a vector space and D a nonempty subset of X. One calls (X, D) a *convex space* if co $D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D \rangle$ (see Park [35]). For a convex space (X, D), a subset C of X is said to be D-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies co$A \subset C$.

If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [12].

A nonempty subset L of a convex space X is called a c-*compact* set [12] if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

Lassonde [12] presented a simple and unified treatment of a large variety of minimax and fixed point problems. He first noticed that the Hausdorffness in the 1961 Fan-KKM lemma is redundant. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem; the principal topics treated by him may be listed as follows:

Fixed point theory for multimaps;

Minimax equalities;

Extensions of monotone sets;

Variational inequalities;

Special best approximation problems.

The following [12, Proposition I.1.4] extends the 1984 inequality of Fan.

Theorem 3 (see [12]). *Let X be a convex space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a convex l.s.c. function satisfying the following:*

  (i) *for each fixed $x \in X$, $y \mapsto f(x, y)$ is l.s.c. on (compact subsets of) X;*

  (ii) *for each fixed $y \in X$, $x \mapsto f(x, y)$ is concave on X;*

  (iii) *for all $x \in X$, $f(x, x) \leq 0$.*

Furthermore, assume that the following "coercivity" condition holds:

  (iv) *there are a compact set $K \subset X$ and a c-compact set $L \subset X$ such that for each $y \in X \setminus K$, there is an $x \in L$ with $f(x, y) + \varphi(y) > \varphi(x)$.*

Then, there exists $y_0 \in X$ such that $f(x, y_0) + \varphi(y_0) \leq \varphi(x)$ for all $x \in X$.

Consider that the following extends Theorem 3.

Theorem 3$'$. *In Theorem 3, assume that*

  (ii)$'$ *for each fixed $y \in X$, $x \mapsto f(x, y)$ is quasiconcave on X instead of* (ii).

Theorem 3$'$ $\Rightarrow$ Theorem 1. Note that (a)–(c) of Theorem 1 imply (i), (ii)$'$, and (iii) of Theorem 3$'$, respectively. Now, we show that (d) implies (iv). Let $K := \{y \in X : f(x, y) \leq 0$ for all $x \in X_0\}$. Then for all $y \in X \setminus K$, there exists $x \in X_0$ such that $f(x, y) > 0$. Since X_0 is a compact convex subset of a Hausdorff topological vector space, it is a c-compact set. Then, condition (iv) holds with $L := X_0$ and $\varphi \equiv 0$. Therefore, by Theorem 3$'$, there exists a point $\widehat{y} \in X$ such that $f(x, \widehat{y}) \leq 0$ for all $x \in X$.

Remark 4. (1) As shown in the above proof, the Hausdorffness in the 1984 inequality is essential. However, it is redundant in the original 1972 minimax inequality.

(2) The compactly closed (resp., open) sets adopted by Lassonde and many followers can be replaced simply by closed (resp., open) sets; see [22, 36]. Similarly, "l.s.c. on compact subsets of X" in (i) can be simply "l.s.c."

3.2. Chang Type Coercivity. In 1989, Chang [37] obtained a KKM theorem with a coercivity condition which eliminated the concept of c-compact sets. From a Fan-Browder type fixed point theorem equivalent to her KKM theorem, we noticed that the following holds [38].

Theorem 5 (see [38]). *Let X be a convex space, and $p, q : X \times X \to \mathbb{R} \cup \{+\infty\}$ and $h : X \to \mathbb{R} \cup \{+\infty\}$ be functions satisfying the following:*

(i) $q(x, y) \leq p(x, y)$ for $(x, y) \in X \times X$ and $p(x, x) \leq 0$ for all $x \in X$;

(ii) for each $y \in X$, $\{x \in X : p(x, y) + h(y) > h(x)\}$ is convex or empty;

(iii) for each $x \in X$, $\{y \in X : q(x, y) + h(y) > h(x)\}$ is open;

(iv) there exist a nonempty compact subset K of X and, for each finite subset N of X, a compact convex subset L_N of X containing N such that $y \in L_N \setminus K$ implies $q(x, y) + h(y) > h(x)$ for some $x \in L_N$.

Then, there exists a point $y_0 \in K$ such that

$$q(x, y_0) + h(y_0) \leq h(x) \quad \forall x \in X. \tag{2}$$

Moreover, the set of all such solutions y_0 is a compact subset of K.

Actually, Theorem 5 is equivalent to the following.

Theorem 5$'$. The case of $h \equiv 0$ in Theorem 5.

Theorem 5 $\Rightarrow$ Theorem 3. We apply Theorem 5 with $p(x, y) = q(x, y) = f(x, y)$ and $h(x) = \varphi(x)$. Then we obtain (i)–(iv) in Theorem 5 from Theorem 3 as follows.

(i) Clearly holds since $f(x, x) \leq 0$ for all $x \in X$.

(ii) For each $y \in X$,

$$
\begin{aligned}
\{x \in X : &f(x, y) + \varphi(y) > \varphi(x)\} \\
&= \{x \in X : f(x, y) - \varphi(x) > -\varphi(y)\}
\end{aligned}
\tag{3}
$$

is convex or empty since $x \mapsto f(x, y)$ is concave and $x \mapsto \varphi(x)$ is convex.

(iii) For each $x \in X$, $\{y \in X : f(x, y) + \varphi(y) > \varphi(x)\}$ is open since $y \mapsto f(x, y)$ is l.s.c. and $y \mapsto \varphi(y)$ is l.s.c.

(iv) There exist a nonempty compact subset K of X and, for each finite subset N of X, a compact convex subset L_N of X containing $L \cup N$ (since L is c-compact) such that $y \in L_N \setminus K \subset X \setminus K$ implies $f(x, y) + \varphi(y) > \varphi(x)$ for some $x \in L \subset L_N$.

Therefore, all of the requirements of Theorem 5 are satisfied. Therefore, the conclusion of Theorem 3 follows from Theorem 5.

3.3. For H-Spaces. In this subsection, we follow [39].

Definition 6. A triple $(X, D; \Gamma)$ is called an H-*space* if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

For an H-space $(X, D; \Gamma)$, a subset C of X is said to be H-*convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

A multimap $F : D \multimap X$ is said to be H-KKM if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$. A subset L of X is called an H-*subspace* of

$(X, D; \Gamma)$ if $L \cap D \neq \emptyset$, and for every $A \in \langle L \cap D \rangle$, $\Gamma_A \cap L$ is contractible.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [13–16] or an H-space by Bardaro and Ceppitelli [40]. Horvath noted that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces and are examples of compact c-spaces without having the fixed point property.

In the frame of H-spaces, we obtained several generalized minimax inequalities. The following is one of them in [39, Theorem 7].

Theorem 7 (see [39]). Let $(X, D; \Gamma)$ be an H-space, let $\phi : D \times X \to \overline{\mathbb{R}}$ and $\psi : X \times X \to \overline{\mathbb{R}}$ be functions, and let $\gamma \in \overline{\mathbb{R}}$ such that

(1) $\phi(x, y) \leq \psi(x, y)$ for all $(x, y) \in D \times X$ and $\psi(x, x) \leq \gamma$ for all $x \in X$,

(2) for each $x \in D$, $\{y \in X : \phi(x, y) \leq \gamma\}$ is (compactly) closed,

(3) for each $y \in X$, $\{x \in X : \psi(x, y) > \gamma\}$ is H-convex.

Suppose that there exists a nonempty compact subset K of X such that either

(i) there exists an $M \in \langle D \rangle$ such that

$$\{y \in X : \phi(x, y) \leq \gamma \ \forall x \in M\} \subset K, \tag{4}$$

(ii) or for each $N \in \langle D \rangle$, there exists a compact H-subspace L_N of X containing N such that for each $y \in L_N \setminus K$ there exists an $x \in L_N \cap D$ satisfying $\phi(x, y) > \gamma$.

Then, there exists a $\widehat{y} \in K$ such that

$$\sup_{x \in D} \phi(x, \widehat{y}) \leq \gamma. \tag{5}$$

Theorem 7 $\Rightarrow$ Theorem 5$'$. Any convex space is an H-space.

Note that Theorem 7 extends all of Theorems 1–5, 3$'$, and 5$'$ and that Theorem 7 with the coercivity condition (i) generalizes the original minimax inequality of Fan.

3.4. For G-Convex Spaces. Since 1996 the following became one of the main themes of the KKM theory [41, 42].

Definition 8. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \text{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$.

For a G-convex space $(X, D; \Gamma)$, a subset C of X is said to be G-*convex* with respect some $D' \subset D$ if for each $N \in \langle D' \rangle$, we have $\Gamma_N \subset C$.

A map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$.

There are lots of examples of G-convex spaces; see [3, 41] and the references therein. So, the KKM theory was extended to the study of KKM maps on G-convex spaces.

The following minimax inequality for G-convex spaces originates from [41, 42].

Theorem 9. *Let* $(X, D; \Gamma)$ *be a G-convex space, let* $\phi : D \times X \to \overline{\mathbb{R}}$ *and* $\psi : X \times X \to \overline{\mathbb{R}}$ *be functions, and let* $\gamma \in \overline{\mathbb{R}}$ *such that*

(1) $\psi(x, x) \leq \gamma$ *for all* $x \in X$,

(2) *for each* $z \in D$, $\{y \in X : \phi(z, y) \leq \gamma\}$ *is closed,*

(3) *for each* $y \in X$, $N \in \langle \{z \in D : \phi(x, y) > \gamma\} \rangle$ *implies* $\Gamma_N \subset \{x \in X : \psi(x, y) > \gamma\}$.

Suppose that there exists a nonempty compact subset K *of* X *such that either*

(i) *there exists an* $M \in \langle D \rangle$ *such that*

$$\{y \in X : \phi(z, y) \leq \gamma \ \forall z \in M\} \subset K, \quad (6)$$

(ii) *or for each* $N \in \langle D \rangle$, *there exists a compact G-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$, *and for each* $y \in L_N \setminus K$ *there exists a* $z \in D'$ *satisfying* $\phi(z, y) > \gamma$.

Then, there exists a $\widehat{y} \in K$ *such that*

$$\sup_{z \in D} \phi(z, \widehat{y}) \leq \gamma. \quad (7)$$

Theorem 9 ⇒ Theorem 7. Any H-space is a G-convex space.

3.5. For ϕ_A-*Spaces.* Since 2007, the following became one of the main themes of the KKM theory [17, 43, 44].

Definition 10. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$\left(X, D; \{\phi_A\}_{A \in \langle D \rangle}\right) \text{ or simply } \left(X, D; \phi_A\right) \quad (8)$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (i.e., singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

For a ϕ_A-space $(X, D; \phi_A)$, a subset C of X is said to be ϕ_A-*convex relative to some* $D' \subset D$ if for each $A \in \langle D' \rangle$, we have $\phi_A(\Delta_{|A|-1}) \subset C$.

We define a KKM map $G : D \multimap X$ on a ϕ_A-space $(X, D; \phi_A)$ if, for each $N \in \langle D \rangle$ and $J \subset N$, we have

$$\phi_N \left(\Delta_{|J|-1} \right) \subset G(J), \quad (9)$$

where $\Delta_{|J|-1}$ is the face of $\Delta_{|N|-1}$ corresponding to J; see Park [17].

There are lots of examples of ϕ_A-spaces; see [3, 17, 43, 44] and the references therein. So, the KKM theory was extended to the study of KKM maps on ϕ_A-spaces.

The following minimax inequality for ϕ_A-spaces is new.

Theorem 11. *Let* $(X, D; \phi_A)$ *be a* ϕ_A-*space, let* $f : D \times X \to \overline{\mathbb{R}}$ *and* $g : X \times X \to \overline{\mathbb{R}}$ *be functions, and let* $\gamma \in \overline{\mathbb{R}}$ *such that*

(1) $g(x, x) \leq \gamma$ *for all* $x \in X$,

(2) *for each* $z \in D$, $\{y \in X : f(z, y) \leq \gamma\}$ *is closed,*

(3) *for each* $y \in X$ *and* $A \in \langle \{z \in D : f(z, y) > \gamma\} \rangle$, *one has* $\phi_A(\Delta_{|A|-1}) \subset \{x \in X : g(x, y) > \gamma\}$.

Suppose that there exists a nonempty compact subset K *of* X *such that either*

(i) *there exists an* $M \in \langle D \rangle$ *such that*

$$\{y \in X : \phi(z, y) \leq \gamma \ \forall z \in M\} \subset K, \quad (10)$$

(ii) *or for each* $N \in \langle D \rangle$, *there exist a compact* ϕ_A-*convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and for each* $y \in L_N \setminus K$ *there exists a* $z \in D'$ *satisfying* $f(z, y) > \gamma$.

Then, there exists a $\widehat{y} \in K$ *such that*

$$\sup_{z \in D} f(z, \widehat{y}) \leq \gamma. \quad (11)$$

Theorem 11 ⇒ Theorem 9. Any G-convex space is a ϕ_A-space such that $\phi_A(\Delta_{|A|-1}) \subset \Gamma_A$ for each $A \in \langle D \rangle$. Therefore, by putting $f := \phi$, $g := \psi$, and $\phi_A(\Delta_{|A|-1}) := \Gamma_A$, Theorem 11 reduces to Theorem 9.

4. Abstract Convex Spaces and General KKM Theorems

Recall the following in [3, 18, 19].

Definition 12. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_{\Gamma} D' := \bigcup \left\{ \Gamma_A : A \in \left\langle D' \right\rangle \right\} \subset E. \quad (12)$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_{\Gamma} D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 13. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \forall A \in \langle D \rangle, \quad (13)$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* (resp., a $\mathfrak{KO}$-*map*) if, for any closed-valued (resp., open-valued) KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ (resp., $F \in \mathfrak{KO}(E, D, Z)$).

Definition 14. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{RC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *(partial) KKM space* if it satisfies the (partial) KKM principle, respectively.

In our recent works [3, 18, 19], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed that many important results therein are related to the partial KKM principle.

Example 15. We give known examples of partial KKM spaces; see [3] and the references therein.

(1) Every ϕ_A-space is a KKM space. More precisely, for a ϕ_A-space $(X, D; \phi_A)$, the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D \rangle, |A| = n + 1$, is a KKM space. This KKM space may not be G-convex; see [25].

(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.

(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.

(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := \text{conv}_H(A)$ for each $A \in \langle X \rangle$, Kirk and Panyanak showed that the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle.

(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.

(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

(7) Recently, Kulpa and Szymanski [45] found some partial KKM spaces which are not KKM spaces.

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.}$$
$$\Longrightarrow \text{Lassonde type convex space}$$
$$\Longrightarrow \text{H-space}$$
$$\Longrightarrow \text{G-convex space}$$
$$\Longrightarrow \phi_A\text{-space} \tag{14}$$
$$\Longrightarrow \text{KKM space}$$
$$\Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems [20, 23, 24].

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{RC}(E, D, E)$(resp., $1_E \in \mathfrak{RD}(E, D, E)$), and $G : D \multimap E$ be a multimap satisfying the following:*

(1) *G has closed (resp., open) values,*

(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (i.e., G is a KKM map).*

Then $\{G(z)\}_{z \in D}$ has the finite intersection property.
Further, if

(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then one has

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset. \tag{15}$$

Since all of the spaces in Section 3 are ϕ_A-spaces and hence KKM spaces, Theorem A can be applied to them, that is, Theorem 1–11 for the case (3).

Consider the following related four conditions for a map $G : D \multimap E$.

(a) $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z \in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is *intersectionally closed valued*).

(c) $\bigcap_{z \in D} \overline{G(z)} = \bigcap_{z \in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed valued.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [20, 23, 24].

Theorem B. *Let $(E, D; \Gamma)$ be a partial KKM space (i.e., $1_E \in \mathfrak{RC}(E, D, E)$) and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map (i.e., $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$),*

(2) *there exists a nonempty compact subset K of E such that either*

(i) *$\bigcap_{z \in M} \overline{G(z)} \subset K$ for some $M \in \langle D \rangle$,*

(ii) *or for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K. \tag{16}$$

Then one has $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.
Furthermore,

(α) *if G is transfer closed valued, then $K \cap \bigcap\{G(z) : z \in D\} \neq \emptyset$;*

(β) *if G is intersectionally closed valued, then $\bigcap\{G(z) : z \in D\} \neq \emptyset$.*

Remark 16. (1) Every ϕ_A-space is a KKM space. Hence, Theorem B works for ϕ_A-spaces.

(2) We may assume that K is closed. Then, the closure notations in each coercivity condition can be eliminated.

From the preceding theorem, we obtained several prototypes of minimax inequalities in [29]. Here we give only one of them with its proof for completeness.

Theorem 17 ([29]). *Let* $(E, D; \Gamma)$ *be a partial KKM space, let* $f : D \times E \to \overline{\mathbb{R}}, g : E \times E \to \overline{\mathbb{R}}$ *and be extended real-valued functions, and let* $\alpha, \beta \in \overline{\mathbb{R}}$ *such that*

(0) *for each* $x \in E$, $g(x, x) \leq \beta$,

(1) *the map* $G : D \multimap E$ *defined by* $G(z) := \{y \in E : f(z, y) \leq \alpha\}$ *for* $z \in D$ *is intersectionally closed-valued (resp., transfer closed-valued),*

(2) *for each* $y \in E$, $\text{co}_\Gamma \{z \in D : f(z, y) > \alpha\} \subset \{x \in E : g(x, y) > \beta\}$,

(3) *the coercivity condition* (2) *of Theorem B holds.*

Then, (a) *there exists a* $y_0 \in E$ (*resp.,* $y_0 \in K$) *such that* $f(z, y_0) \leq \alpha$ *for all* $z \in D$; *and*
(b)*if* $\alpha = \beta := \sup_{x \in E} g(x, x)$, *then one has*

$$\inf_{y \in E} \sup_{z \in D} f(z, y) \leq \sup_{x \in E} g(x, x). \tag{17}$$

Lemma 18. *Under the hypothesis of Theorem 17, condition* (2) *implies that* $G : D \multimap X$ *is a KKM map.*

Proof. Suppose, on the contrary, that there exists a finite $N \subset D$ such that $\Gamma_N \not\subset G(N)$. Then, there exist a $y \in \Gamma_N$ such that $y \notin G(z)$ or $f(z, y) > \alpha$ for all $z \in N$. Hence, $N \subset \{z \in D : f(z, y) > \alpha\}$ and, by (2), we have $\Gamma_N \subset \{x \in X : g(x, y) > \beta\}$. Since $y \in \Gamma_N$, we have $g(y, y) > \beta$. This contradicts (1). $\square$

Proof of Theorem 17. By Lemma, G is a KKM map. Therefore, all the requirements of Theorem B are satisfied. Hence, (a) follows from Theorem B. Note that (b) is a simple consequence of (a). $\square$

Remark 19. (1) Comparing Theorem 17 with the original Fan minimax inequality, note that condition (1) generalizes the l.s.c. of $f(x, \cdot)$, (2) generalizes the quasiconcavity of $f(\cdot, y)$ whenever $E = D$ and $f = g$, and (3) generalizes the compactness of the underlying space.

(2) In condition (1), transfer closedness can be replaced by mere closedness. For a long period, instead of closedness, some authors adopted the so-called compactly closedness, transfer compactly closedness, or even by the finitely closedness when E is a convex subset of a t.v.s. Moreover, (1) holds whenever G is l.s.c. This also can be replaced by compactly l.s.c., transfer compactly l.s.c, or finitely l.s.c. This kind of terminology is not essential and generalizes nothing important.

Now we show the following.

Theorem 17 $\Rightarrow$ *Theorem 11.* Every ϕ_A-space is a partial KKM space. Put $(E, D; \Gamma) := (X, D; \phi_A)$ with $\Gamma_A := \phi_A(\Delta_{|A|-1})$ and $\alpha = \beta := \gamma$ in Theorem 17, then it reduces to Theorem 11.

Remark 20. (1) The above proof shows that Theorem 17 subsumes all of Theorems 1–11 and numerous variants or particular forms of them appeared in previous works.

(2) There are other theorems also containing Theorem 1–11; see [26, 29].

5. Other Types of Generalizations

This section deals with some of other generalizations of the Fan minimax inequality and vector minimax inequality.

5.1. Generalizations of Quasiconcavity. The compactness, convexity, lower semicontinuity, and quasiconcavity in the inequality are extended or modified by a large number of authors. For example, the quasicocavity is extended to γ-DQCV by Zhou and Chen in 1988. Further, Lin and Tian in 1993 [31, Theorem 3] defined γ-DQCV in slightly more general form:

Let Y be a convex subset of a Hausdorff t.v.s. E and let $\emptyset \neq X \subset Y$. A functional $\varphi(x, y) : X \times Y \to \overline{\mathbb{R}}$ is said to be γ-*diagonally quasi-concave* (γ-DQCV) in x if, for any finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_\lambda \in \text{co}\{x_1, \ldots, x_m\}$, we have $\min_{1 \leq j \leq m} \varphi(x_j, x_\lambda) \leq \gamma$.

Theorem 21 (see [31]). *Let* Y *be a nonempty convex subset of a Hausdorff t.v.s. E, let* $\emptyset \neq X \subset Y$, *and let* $\varphi : X \times Y \to \overline{\mathbb{R}}$ *be a functional such that*

(i) $(x, y) \mapsto \varphi(x, y)$ *is l.s.c. in* y,

(ii) $(x, y) \mapsto \varphi(x, y)$ *is* γ-*DQCV in* x,

(iii) *there exists a nonempty subset* C *of* X *such that* $\bigcap_{x \in C} \{y \in Y : \varphi(x, y) \leq \gamma\}$ *is compact and* C *is contained in a compact convex subset* B *of* Y.

Then there exists a point $y^* \in Y$ *such that* $\varphi(x, y^*) \leq \gamma$ *for all* $x \in X$.

Lin and Tian [31] proved Theorem 21 by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem. Moreover, they showed that Theorem 21 is equivalent to the Fan KKM lemma (where Hausdorffness is redundant). Note that Theorem 21 improves the 1984 inequality of Fan.

Moreover, the $\mathscr{C}$-quasiconcavity due to Hou in 2009 [46] unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) and the $\mathscr{C}$-concavity (weaker than concavity).

Here we introduce a new usage of ϕ_A-spaces and a new minimax inequality.

Definition 22 (see [46]). Let X be a topological space, and let $D, Y \subset X$. A real function $f : X \times Y \to \mathbb{R}$ is said to be $\mathscr{C}$-*quasiconcave* on D if, for any $N = \{x^0, x^1, \ldots, x^n\} \in \langle D \rangle$, there exists a continuous map $\phi_N : \Delta_n \to Y$ such that

$$\min \left\{ f\left(x^i, \phi_N(\lambda)\right) : i \in J \right\} \leq f\left(\phi_N(\lambda), \phi_N(\lambda)\right) \tag{18}$$

for all $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, where $J := \{i : \lambda_i \neq 0\}$.

Note that $(Y, D; \{\phi_N\}_{N \in \langle D \rangle})$ is a ϕ_A-space.

In [26], we obtained the following generalization of the Fan minimax inequality:

Theorem 23 (see [26]). *Let X be a topological space, D a nonempty subset of X, and $f, g : X \times X \to \mathbb{R}$. Assume that:*

(1) *$f \leq g$ on $X \times X$,*

(2) *there exist $x^1, \ldots, x^n \in D$ such that $K = \bigcap_{i=1}^n \overline{G(x^i)}$ is compact where $G(x) = \{z \in X : f(x, z) \leq \mu\}$ and $\mu = \sup_{y \in X} g(y, y)$;*

(3) *$g|_{D \times X}$ is $\mathscr{C}$-quasiconcave on D,*

(4) *for each $x \in D$, $\{y \in X : f(x, y) \leq \mu\}$ is intersectionally closed [resp., transfer closed].*

Then there exists $\widetilde{z} \in X$ (resp., $\widetilde{z} \in K$) such that

$$\sup_{x \in D} f(x, \widetilde{z}) \leq \sup_{y \in X} g(y, y) \tag{19}$$

holds.

Furthermore, S.-Y. Chang in 2010 extended the $\mathscr{C}$-quasiconcavity to 0-pair-concavity and obtained a new Fan type inequality; see Park [26].

5.2. Vector Ky Fan Minimax Inequality. As in [47, Section 3.2], we obtained a vector Ky Fan minimax inequality in [28].

Definition 24 (see [47]). Let X be a Hausdorff topological space, Z a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with Int $P \neq \emptyset$, and $D, Y \subset X$. A function $f : X \times Y \to Z$ is called $\mathscr{C}$-P-quasiconcave on A if and only if, for any finite subset $\{x_0, x_1, \ldots, x_n\}$ of D, there exists a continuous mapping $\phi_n : \Delta_n \to Y$ such that, for any $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, there exists $i \in J(\lambda)$ such that

$$f(\phi_n(\lambda), \phi_n(\lambda)) \in f(x_i, \phi_n(\lambda)) + P, \tag{20}$$

where $J(\lambda) := \{i \in \{0, 1, \ldots, n\} : \lambda_i > 0\}$.

In [47], the $\mathscr{C}$-P-quasiconcavity is called the generalized P-quasiconcave. In the above definition, note that $(Y, D; \{\phi_n\})$ is a ϕ_A-space.

Definition 25 (see [47]). Let X be a Hausdorff topological space, and let Z be a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with Int $P \neq \emptyset$. A vector-valued function $f : X \to Z$ is called P-continuous at $x_0 \in X$ if and only if, for any open neighborhood V of the zero element in Z, there exists an open neighborhood U of x_0 in X such that, for any $x \in U$, $f(x) \in f(x_0) + V + P$. In particular, f is called P-continuous on X if and only if f is P-continuous at every point of X.

Theorem 26 (see [28]). *Let X be a nonempty and compact subset of a Hausdorff topological vector space E, and let Z be a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with Int $P \neq \emptyset$. A mapping $f : X \times X \to Z$ satisfies the following:*

(i) *for any fixed $y \in X$, $x \mapsto f(x, y)$ is P-continuous;*

(ii) *for any fixed $x \in X$, $y \mapsto f(x, y)$ is $\mathscr{C}$-P-quasiconcave on X;*

(iii) *for any $x \in X$, $f(x, x) \notin$ Int P.*

Then there exists $x^* \in X$ such that $f(x^*, y) \notin$ Int P for any $y \in X$.

6. Comments and Historical Remarks

Recall that the main applications of recent generalized KKM theorems are as follows; see [11, 30]:

Vector variational-type inequalities,

Various quasiequilibrium problems,

Eigenvector problems,

Set-valued minimax inequality,

Fixed point theorems,

Generalizations of Nash equilibrium theorem,

Variational inclusion problem,

Simultaneous nonlinear inequalities problem,

Iferential inclusion problem,

(Vector mixed) quasivariational inequality,

(Vector mixed) quasicomplementarity problem,

Traffic network problem,

Quasi-monotone vector equilibrium problem,

Generalized vector equilibrium problem,

Generalized (implicit) vector variational-like inequality,

Set equilibrium problem,

Set-valued mixed (quasi)variational inequalities,

Variational-hemivariational inequalities.

In 2006–2009, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [3, 18, 19] and the references therein. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. Now the KKM theory becomes the study of spaces satisfying the partial KKM principle.

In our work [3], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. We add more than a dozen statements as their applications, including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [3] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

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