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REMARKS ON AN EXAMPLE OF THE GFC-SPACES OF KHANH, LONG AND QUAN

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Abstract. Earlier we found that our ϕ_A-spaces can be made into G-convex spaces in several ways and that GFC-spaces due to Khanh et al. are all ϕ_A-spaces. Recently, they [JOTA 151: 552–572 (2011)] gave an example of a GFC-space which is a 'trivial' G-convex space. In this paper, we show that a GFC-space can be made into a nontrivial G-convex space $(X, D; \Gamma)$ iff it has a nontrivial KKM map $G : D \multimap X$. Consequently, their example has only a trivial KKM map and is not adequate to show that GFC-spaces properly extend G-convex spaces.

1. Introduction

In the KKM theory, generalized convex (simply, G-convex) spaces first appeared in 1993 and have been investigated by a large number of authors. Moreover, a number of modifications or imitations of G-convex spaces appeared; for example, L-spaces, spaces having property (H), pseudo H-spaces, M-spaces, G-H-spaces, another L-spaces, FC-spaces, simplicial spaces, L*-spaces, and others. Moreover, the concept of G-convex spaces has been challenged by several authors who aimed to obtain more general concepts. It is known in 2007-2008 [1-6] that most of such examples belong to the class of ϕ_A-spaces, which can be made into G-convex spaces in several ways. See also [7, 8].

From 2006, all of the above mentioned classes of spaces are unified to that of abstract convex spaces [9-12], and the KKM theory tends to the research of such new type of spaces.

On the other hand, since 2009, Khanh et al. [13-19] introduced GFC-spaces which are essentially same to ϕ_A-spaces, and asserted that GFC-spaces encompass G-convex spaces, FC-spaces, and many recent existing spaces with generalized convexity structures. But, based on [1-7], we stated

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several times that ϕ_A-spaces can be made into G-convex spaces in several ways; for example, see [7, 12]. However, Khanh et al. in 2011 [19] claimed that the notion of a GFC-space is properly more general than that of a G-convex space by giving an example.

Our aim in this paper is to maintain our claim that GFC-spaces can be made into G-convex spaces in several ways and these two classes of spaces are essentially same to that of ϕ_A-spaces; see Section 3. Moreover, a recent example of a ϕ_A-space or a GFC-space due to Khanh et al. [19] that can be made into a trivial G-convex space has only trivial KKM map. Hence the example is no use as the authors of [19] said for trivial G-convex spaces. Therefore the example is not adequate to show that GFC-spaces properly contain G-convex spaces.

2. G-convex spaces

Recall the following; see [8] and the references therein:

Definition. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ is the set of all nonempty finite subsets of D, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Definition. Let $(E, D; \Gamma)$ be a G-convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Note that the concept of a KKM map with respect to F has a long history originated from Park in 1992.

Recall the following in [1-8]:

Definition. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Proposition 2.1 ([7]). *Any ϕ_A-space $(X, D; \{\phi_A\})$ can be made into a G-convex space $(X, D; \Gamma)$ in several ways.*

Definition. For a ϕ_A-space $(X, D; \{\phi_A\})$, any multimap $G : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset G(J) \text{ for each } A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is called a KKM map.

Proposition 2.2. *A KKM map on a ϕ_A-space is also a KKM map on a new G-convex space $(X, D; \Gamma)$.*

In [1-7], we listed a number of examples of ϕ_A-spaces. In the next section, we discuss more details on ϕ_A-spaces.

3. GFC-spaces

Recall that most of modifications or variants of G-convex spaces $(X, D; \Gamma)$ are based on the replacement of $\Gamma(N)$ by the corresponding $\phi_N(\Delta_n)$; see [1-7].

The following form of ϕ_A-spaces $(X, D; \{\phi_A\})$ gives one of such examples:

Definition ([13, 15])**.** Let X be a topological space, A be a nonempty set and Φ be a family of continuous mappings $\varphi : \Delta_n \to X,\ n \in \mathbb{N}$. Then a triple (X, A, Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) iff, for each finite subset $N = \{a_0, a_1, \dots, a_n\}$ of A, there is $\varphi_N : \Delta_n \to X$ of the family Φ. (Later, we also use $(X, A, \{\varphi_N\})$ to denote (X, A, Φ).)

The authors of [19] stated as follows:

"The class of GFC-spaces contains a large number of spaces with various kinds of generalized convexity structures such as FC-spaces, G-convex spaces. Recall that a G-convex space of Park is a triple (X, A, Γ), where X and A are as (in the above) Definition and $\Gamma : \langle A \rangle \multimap X$ is such that, for each $N \in \langle A \rangle$ with cardinality $|N| = n + 1$, there exists a continuous map $\varphi_N : \Delta_n \to \Gamma(A)$ such that, for each $M \in \langle N \rangle,\ \varphi_N(\Delta_M) \subset \Gamma(M)$. A G-convex space (X, A, Γ) is called trivial iff, for all $N \in \langle A \rangle,\ \Gamma(N) = X$. Of course, any above-mentioned space can be made into a trivial G-convex space, but a trivial G-convex space has no use. In [12] it is asserted that any GFC-space is a (nontrivial) G-convex space, i.e., the latter is more general. However, in fact, the notion of a GFC-space is properly more general than that of a G-convex space as shown by an example."

Contrary to this claim, from the beginning, we stated that "any G-convex space is a ϕ_A-space. Any ϕ_A-space can be made into a G-convex space. Therefore, G-convex spaces and ϕ_A-spaces are essentially same" in [1-7]. Moreover, in [12, the last line of page 3], it is clearly said that "Every ϕ_A-space can be made into a G-convex space; see [7]. Recently ϕ_A-spaces are called GFC-spaces in [13] . . ."

Definition ([19])**.** Let (X, A, Φ) be a GFC-space and Y be a nonempty set. Let $T : X \multimap Y$, $F : A \multimap Y$ be two set-valued mappings. F is called a KKM mapping with respect to T, shortly, T-KKM mapping iff, for each $N = \{a_0, \ldots, a_n\} \subset A$ and $\{a_{i_0}, \ldots, a_{i_k}\} \subset N$, $T(\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} F(a_{i_j})$.

Note that in case $X = Y$ and T is the identity map on X, F is called a KKM map on a ϕ_A-space $(X, A; \{\phi_A\})$ in our sense.

Now, we have the following:

Proposition 3.1. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a G-convex space $(X, D; \Gamma)$ iff it has a KKM map $G : D \multimap X$.*

Proof. Suppose that $(X, D; \{\phi_A\})$ has a KKM map $G : D \multimap X$. Then we have

$$\phi_A(\Delta_J) \subset G(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Define a map $\Gamma : \langle D \rangle \multimap X$ by $\Gamma(J) := G(J) = \bigcup\{G(a) \mid a \in J\}$ for each $A \in \langle D \rangle$ and each $J \in \langle A \rangle$. Then we have

$$\phi_A(\Delta_J) \subset \Gamma(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Therefore $(X, D; \Gamma)$ is a G-convex space.

Conversely, suppose that $(X, D; \{\phi_A\})$ is a G-convex space $(X, D; \Gamma)$. By putting $G(z) := \Gamma(\{z\})$ for $z \in D$, we have a KKM map $G : D \multimap X$ as above. In fact, for each A with $|A| = n + 1$, we have a continuous function $\phi_A : \Delta_n \to G(A) =: \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset G(J) =: \Gamma(J)$. Hence $\phi_A(\Delta_J) \subset \Gamma(J) \subset G(J)$ for each $A \in \langle D \rangle$ and $J \in \langle A \rangle$. Therefore $G : D \multimap X$ is a KKM map on $(X, D; \{\phi_A\})$.

Let us call that a KKM map $G : D \multimap X$ is said to be 'trivial' if $G(z) = X$ for all $z \in D$. Then we have the following:

Proposition 3.2. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a trivial G-convex space $(X, D; \Gamma)$ iff it has a trivial KKM map $G : D \multimap X$.*

Proof. Suppose that $(X, D; \{\phi_A\})$ is a trivial G-convex space $(X, D; \Gamma)$, that is, $\Gamma_A = X$ for all $A \in \langle D \rangle$. Define $G : D \multimap X$ by $G(z) := X$ for all $z \in D$. Then $\Gamma_A = G(A)$ for all $A \in \langle D \rangle$. Moreover, for all $A \in \langle D \rangle$ and $J \in \langle A \rangle$, we have $\phi_A(\Delta_{|J|-1}) \subset X = G(J)$. Hence G is a trivial KKM map.

Conversely, suppose that there is a trivial KKM map $G : D \multimap X$. Then, for all $A \in \langle D \rangle$ and $J \in \langle A \rangle$, we have $\phi_A(\Delta_{|J|-1}) \subset G(J) = X$. Define $\Gamma : \langle D \rangle \multimap X$ by $\Gamma_A := X$ for all $A \in \langle D \rangle$. Then we have a trivial G-convex space $(X, D; \Gamma)$.

The following is a contrapositive of Proposition 3.2:

Proposition 3.3. *A ϕ_A-space $(X, D; \{\phi_A\})$ is a nontrivial G-convex space $(X, D; \Gamma)$ iff it has a nontrivial KKM map $G : D \multimap X$.*

Comments to the example in [19]. This is an example of a ϕ_A-space which is a 'trivial' G-convex space. Therefore, it can not have a nontrivial KKM map.

As expressed in [19], of course, any GFC-space without nontrivial KKM map has no use in the KKM theory. Consequently, in order to show that GFC-spaces properly generalize G-convex spaces, the authors of [19] should give an example of a GFC-space with a nontrivial KKM map which is not G-convex. This is impossible by Proposition 3.3.

4. Conclusion

All of the works [13-19] of Khanh et al. on GFC-spaces do not reflect recent studies on abstract convex spaces [8-12] and ϕ_A-spaces [1-8]. Note that GFC-spaces studied in [13-19] are all ϕ_A-spaces given in [1-7] more early. In [19], its authors gave an example of a GFC-space which is a 'trivial' G-convex space. In this paper, we show that the example of a GFC-space has only trivial KKM map and no use in the KKM theory as the authors of [19] said to a trivial G-convex space.

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