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COMMENTS ON THE FWC-SPACES OF H. LU AND J. ZHANG

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Abstract. Recently, Lu and Zhang [CAMWA 64 (2012) 570–588] introduced the concepts of FWC-spaces (short form of finite weakly convex spaces) as a unified form of many known modifications of G-convex spaces, and the better admissible class of multimaps on them. In this paper, we show that their FWC-spaces and their better admissible classes are inadequately defined and that their results can not be true.

1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz obtained an intersection theorem (simply, the KKM theorem), which is concerned with a particular type of multimaps called KKM maps later. The KKM theory is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Consequently, at the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84. Then, the KKM theory had been extended to convex spaces by Lassonde in 1983, and to c-spaces (or H-spaces) by Horvath in 1983-93 and others. Since 1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others; see [1].

While G-convex spaces were investigated by a large number of authors, the concept has been challenged by several authors who aimed to obtain more general concepts. In fact, a number of modifications or imitations of G-convex spaces followed; for example, L-spaces, spaces having property

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(H), FC-spaces, M-spaces, another L-spaces, simplicial spaces, L*-spaces and others. It is known in 2007-2008 [2-7] that all of such examples belong to the class of ϕ_A-spaces.

From 2006, all of the above mentioned classes of spaces are unified to that of abstract convex spaces [8-10], and the KKM theory tends to the research of such new spaces.

In 2012, Lu and Zhang [11] introduced the concepts of FWC-spaces (short form of finite weakly convex spaces) as a unified form of many known modifications of G-convex spaces, and the better admissible class of multimaps on them. They claimed a version of the Fan type section theorem in FWC-spaces without any linear and convex structure under much weaker assumptions, and next as its applications, some new coincidence theorems and minimax inequalities in FWC-spaces.

Our aim in this paper is to show that their FWC-spaces and their better admissible classes are inadequately defined and that their results can not be true.

In Section 2, we state that the artificial terminology in [11] can be eliminated. Section 3 deals with the definition of FWC-spaces. In Section 4, we give examples showing that their better admissible class of multimaps on their FWC-spaces is invalid. Section 5 deals with some historical remarks related to the KKM theory.

2. Artificial Terminology

In [11], its authors adopted the concepts of

compact closure (ccl),
compact interior (cint),
transfer compactly closed-valued (resp., open-valued) multimap, and
λ-transfer compactly lower (resp., upper) semicontinuous function.

These are not practical and hard to construct proper examples. Moreover, it is already known that "compact" or "compactly" in such terminology can be invalidated by adopting the compactly generated extension instead of the original topology of relevant spaces; see [12, 13].

3. FWC-spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. In 2012 [11], the following is derived:

Definition ([11, Def. 2.3])**.** A triple $(Y, D; \varphi_N)$ is said to be a finite weakly convex space (shortly, an FWC-space) if Y, D are two nonempty sets and for each $N = \{u_0, \ldots, u_n\} \in \langle D\rangle$ where some elements in N may be same, there exists a set-valued mapping $\varphi_N : \Delta_n \to 2^Y$ with nonempty values. When $D \subset Y$, the space is denoted by $(Y \supset D; \varphi_N)$. In case $Y = D$, let $(Y; \varphi_N) := (Y, Y; \varphi_N)$.

Its authors stated: "It is worthwhile noticing that Y and D in Definition 2.3 do not possess any linear, convex and topological structure and so the set-valued mapping φ_N has no continuity requirement. Even Y is a topological space, it is easy to see that convex subsets of topological vector spaces, Lassonde's convex spaces, H-spaces introduced by Horvath, G-convex spaces introduced by Park and Kim, L-convex spaces introduced by Ben-El-Mechaiekh et al., G-H-spaces introduced by Verma, pseudo H-spaces introduced by Lai et al., GFC-spaces due to Khanh et al., FC-spaces due to Ding, and many other topological spaces with abstract convex structure are all particular forms of FWC-spaces." For the references, see [11]. Here L-spaces are carelessly called L-convex spaces as many people does.

Recall that FC-spaces $(Y;\varphi_N)$ due to Ding is a particular form of FWC-spaces for topological spaces Y and continuous φ_N. According to the definitions of FC-spaces and FWC-spaces, for each $N = \{u_0,\dots,u_n\} \in \langle D\rangle$ where some elements in N may be same, there should be an infinite number of maps $\varphi_N : \Delta_n \to 2^Y$.

Note that all of the preceding examples of WFC-spaces are known to be ϕ_A-spaces as follows:

Definition. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

This is first introduced in 2007 and studied in [2-10]. Note that ϕ_A-spaces are correct form of WFC-spaces and belong to the class of KKM spaces (that is, abstract convex spaces satisfying abstract form of the KKM theorem and its open-valued version); see Section 5.

4. Better admissible maps

In [11], the following is given:

Definition ([11, Def. 2.4])**.** Let X be a topological space and $(Y, D; \varphi_N)$ be an FWC-space. The class $\tilde{\mathfrak{B}}(Y, D, X)$ of better admissible mappings is defined as follows: a set-valued mapping $T : Y \to 2^X$ belongs to $\tilde{\mathfrak{B}}(Y, D, X)$ if and only if for any $N = \{u_0,\dots,u_n\} \in \langle D\rangle$ and for any continuous mapping $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$, the composition $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point. When $Y = D$, we shall write $\tilde{\mathfrak{B}}(Y, X)$ instead of $\tilde{\mathfrak{B}}(Y, D, X)$.

Its authors stated: "Since Y and D in Definition 2.4 are two nonempty sets which do not possess any linear, convex and topological structure, the class $\tilde{\mathfrak{B}}(Y, D, X)$ unifies and extends many important classes of mappings, for example, the class $\mathfrak{A}^\kappa_c(Y, X)$, the class $A(Y, X)$ and the class $\mathfrak{B}(Y, X)$. (For the references, see [11].)"

Note that these examples are for topological spaces Y, not for mere set Y. So the authors of [11] did not prepare any proper example of their better admissible class.

We give an example for the class:

Exmaple 1. Let X be a topological space, $(Y, D; \varphi_N)$ an FWC-space, and $T : Y \to 2^X$ a constant map such that $T(y) = A \in 2^X$ for all $y \in Y$. Then $T(\varphi_N(\Delta_n)) = A$. For any continuous mapping $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$, we have $\psi(A) \subset \Delta_n$. For any $a \in \psi(A)$, we have $T(\varphi_N(a)) = A$ and hence

$$a \in \psi(A) = \psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N(a).$$

Therefore $T \in \tilde{\mathfrak{B}}(Y, D, X)$.

Remark. There are examples showing that the definitions of FWC-spaces and the above better admissible classes are not adequate.

Exmaple 2. Let X be a topological space, $(Y, D; \varphi_N)$ an FWC-space, and $T : Y \to X$ a single-valued non-constant map. Choose two different points $p,\ q \in Y$ such that $T(p) \neq T(q)$. Choose two different points $a,\ b \in \Delta_n$. Since $\varphi_N : \Delta_n \to 2^Y$ can be arbitrarily chosen, suppose that

$$\varphi_N(a) = \{p\},\ \varphi_N(b) = \{q\},\ \textit{and}\ \varphi_N(x) = \{p, q\}\ \ \textit{if}\ \ x \notin \{a, b\}$$

and hence $\varphi_N(\Delta_n) = \{p, q\} \subset Y$.

Then $T(\varphi_N(\Delta_n)) = \{T(p), T(q)\}$. Let $\psi : T(\varphi_N(\Delta_n)) \to \Delta_n$ be a continuous map such that $\psi(T(p)) = b,\ \psi(T(q)) = a$. Then $\psi \circ T|_{\varphi_N(\Delta_n)} \circ \varphi_N$ is fixed point free.

In fact, suppose $x \in \psi \circ T \circ \varphi_N(x)$ where $x \in \{a, b\}$ = Range of ψ. Then

$$a \in \psi \circ T \circ \varphi_N(a) = \psi \circ T(\{p\}) = \{b\},\ b \in \psi \circ T \circ \varphi_N(b) = \psi \circ T(\{q\}) = \{a\},$$

which contradict $a \neq b$.

This shows $T \notin \tilde{\mathfrak{B}}(Y, D, X)$.

Remark. Even Y is a topological space and T is continuous (hence belongs to $\mathfrak{A}_c^\kappa(Y, X)$), this example shows that Definitions 2.3 and 2.4 of [11] are not well-defined. Therefore the other results in [11] can not be true.

The correct form of the better admissible maps due to the present author is defined as follows:

Definition. Let X and Y be topological spaces. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any natural $n \in \mathbb{N}$, any continuous function $\phi : \Delta_n \to X$, and any continuous function $p : F\phi(\Delta_n) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset X \overset{F}{\multimap} F\phi(\Delta_n) \xrightarrow{p} \Delta_n$$

has a fixed point.

Recently, the present author observed [18] that at least twenty seven papers of other authors are concerned with our better admissible maps.

5. Historical Remarks

The paper [11] does not reflect recent studies on the KKM theory.

In 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [8-10] and the references therein.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. Now the KKM theory becomes the study of spaces satisfying the partial KKM principle.

In our work [10], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [10] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

The origin of the better admissible class $\mathfrak{B}$ of multimaps is given in [14, 15] as a generalization of the admissible class $\mathfrak{A}_c^\kappa$ due to Park earlier. Later, for topological spaces X and Y, we defined the *"better" admissible class* $\mathfrak{B}$ of maps from X into Y [16-18]. A number of authors imitated our definition, sometimes incorrectly; see [18].

Finally, we believe that any reputed journal should clarify certain incorrect papers in their publications.

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