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# FIXED POINTS AND ALTERNATIVE PRINCIPLES

Sehie Park and Hoonjoo Kim

**Abstract.** In a recent paper, M. Balaj [B] established an alternative principle. The principle was applied to a matching theorem of Ky Fan type, an analytic alternative, a minimax inequality, and existence of solutions of a vector equilibrium theorem. Based on the first author's fixed point theorems, in the present paper, we obtain generalizations of the main result of Balaj [B] and their applications.

## 1. Introduction

In a recent paper, M. Balaj [B] established an alternative principle of the following type: *If $X$ and $Y$ are convex subsets of two locally convex Hausdorff topological vector spaces and $F$, $S : X \multimap Y$ are two multimaps satisfying certain conditions, then either $F(x_0) = \emptyset$ for some $x_0 \in X$ or $\bigcap_{x \in X} S(x) \neq \emptyset$.* This principle was applied to a matching theorem of Ky Fan type, an analytic alternative, a minimax inequality, and existence of solutions of a vector equilibrium theorem.

Actually, Balaj based on a fixed point theorem of Lassonde [L] for compositions of Kakutani maps. Quite long time ago, the first author obtained several fixed point theorems [P1-7] generalizing many previously known ones including Lassonde's.

Based on the first author's fixed point theorems, in the present paper, we obtain generalizations of the main result of Balaj [B] and their applications.

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## 2. Known fixed point theorems

Multimaps are called simply maps sometimes. For a multimap $T : X \multimap Y$, three maps $T^c : X \multimap Y$ the complement of $T$, $T^- : Y \multimap X$ the (lower) inverse of $T$ and $T^* : Y \multimap X$, the dual of $T$ are defined by $T^c(x) = Y \setminus T(x)$, $T^-(y) = \{x \in X \mid y \in T(x)\}$ and $T^*(y) := X \setminus T^-(y)$ resp.

For topological spaces $X$ and $Y$, a map $T : X \multimap Y$ is said to be: *upper semicontinuous* (u.s.c.) if for any closed set $B \subset Y$, the set $T^-(B) = \{x \in X \,|\, T(x) \cap B \neq \emptyset\}$ is closed in $X$; *compact* if $T(X)$ is contained in a compact subset of $Y$; *closed* if its graph is a closed subset of $X \times Y$.

The following is well-known:

**Lemma 1.** (i) *A composition of u.s.c. maps with comapct values is u.s.c. with comapct values.*

(ii) *If $Y$ is compact and $T : X \multimap Y$ is closed, then $T$ is u.s.c. with compact values.*

(iii) *If $T$ is u.s.c. with compact values, then $T(K)$ is compact whenever $K \subset X$ is compact.*

A map $T : X \multimap Y$ is called a *Kakutani map* whenever $X$ is a topological space, $Y$ a convex space (in the sense of Lassonde) and $T$ is u.s.c. with nonempty compact convex values. Let $\mathbb{K}(X, Y)$ be the class of all Kakutani maps $T : X \multimap Y$ and $\mathbb{K}_c(X, Y)$ the class of all maps $T : X \multimap Y$ those are finite compositions of Kakutani maps.

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over the rational field vanish. A u.s.c. map is said to be *acyclic* if it has compact acyclic values. Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $T : X \multimap Y$ and $\mathbb{V}_c(X, Y)$ the class of all maps $T : X \multimap Y$ those are finite compositions of acyclic maps.

A *polytope* $P$ in a t.v.s. $E$ is a homeomorphic image of a simplex $\Delta_n$.

For topological spaces $X$ and $Y$, we define the *"better" admissible class* $\mathfrak{B}$ of maps from $X$ into $Y$ as follows [P3,4,7]:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any natural number $n \in \mathbb{N}$, any continuous function $\phi : \Delta_n \to X$, and any continuous function $p : F\phi(\Delta_n) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi} \phi(\Delta_n) \subset X \overset{F}{\multimap} F\phi(\Delta_n) \xrightarrow{p} \Delta_n$$

has a fixed point.

**Examples**. Subclasses of the better admissible class $\mathfrak{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with $R_\delta$ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or $m$ acyclic components, where $m$ is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of t.v.s.), admissible maps of Górniewicz (1976), $\sigma$-selectionable maps of Haddad and Lasry (1983), permissible maps of Dzedzej (1985), the Fan-Browder maps (codomains are convex sets), locally selectionable maps having convex values, the class $\mathbb{K}_c^+$ of Lassonde (1991), the class $\mathbb{V}_c^+$ of Park et al. (1994), and approximable maps of Ben-El-Mechaiekh and Idzik (1994), the admissible class $\mathfrak{A}_c^\kappa$ due to Park (1993). For the literature, see [P1-4].

Let $X$ be a convex subset of a vector space and $Y$ a nonempty set. If $S, T : X \multimap Y$ are two maps such that $T(\operatorname{co} A) \subset S(A)$ for each nonempty finite subset $A$ of $X$, then $S$ is said to be *KKM w.r.t. T*.

From now on, a *t.v.s.* means a Hausdorff topological vector space and a real locally convex Hausdorff topological vector space is abbreviated as an *l.c.s.*

Let $\mathcal{V}$ be the neighborhood system of the origin 0 of a t.v.s. $E$.

Recall that a nonempty subset $X$ of a t.v.s. $E$ is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset $K$ of $X$ and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$.

A compact subset $K$ of $X$ is said to be *Klee approximable into* $X$ if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope in $X$.

Recall that a nonempty subset $Y$ of a t.v.s. $E$ is said to be *almost convex* if for any neighborhood $V$ of the origin 0 of $E$ and for any finite set $\{y_1, \dots, y_n\}$ of $Y$, there exists a finite set $\{z_1, \dots, z_n\}$ of $Y$ such that $z_i - y_i \in V$ for all $i = 1, \dots, n$ and $\operatorname{co}\{z_1, \dots, z_n\} \subset Y$.

**Examples.** We give examples of Klee approximable sets in [P5,6]:

(1) If a subset $X$ of $E$ is admissible (in the sense of Klee), then every compact subset $K$ of $X$ is Klee approximable into $E$.

(2) Any polytope in a subset $X$ of a t.v.s. is Klee approximable into $X$.

(3) Any compact subset $K$ of a convex subset $X$ in an l.c.s. is Klee approximable into $X$.

(4) Any compact subset $K$ of a convex and locally convex subset $X$ of a t.v.s. is Klee approximable into $X$.

(5) Any compact subset $K$ of an admissible convex subset $X$ of a t.v.s. is Klee approximable into $X$.

(6) Let $X$ be an almost convex dense subset of an admissible subset $Y$ of a t.v.s. $E$. Then every compact subset $K$ of $Y$ is Klee approximable into $X$.

Note that (6) $\Leftarrow$ (5) $\Leftarrow$ (4) $\Leftarrow$ (3).

In 2004 [P5], the first author obtained the following fixed point theorem:

**Theorem 1.** *Let $X$ be a subset of a t.v.s. $E$ and $F \in \mathfrak{B}(X, X)$ a compact closed multimap. If $F(X)$ is Klee approximable into $X$, then $F$ has a fixed point.*

The following are recently obtained in 2007 [P6]:

**Corollary 1.1.** *Let $X$ be an almost convex admissible subset of a t.v.s. $E$ and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then $F$ has a fixed point.*

**Corollary 1.2.** *Let $X$ be an almost convex subset of an l.c.s. $E$ and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then $F$ has a fixed point.*

One of the simplest known examples of Corollary 1.2 is that *every compact continuous selfmap on an almost convex subset of a Euclidean space has a fixed point.* This generalizes the Brouwer fixed point theorem.

**Corollary 1.3.** [P4] *Let $X$ be an admissible convex subset of a t.v.s. $E$ and $F \in \mathfrak{B}(X, X)$ a compact closed map. Then $F$ has a fixed point.*

## 3. The Balaj type alternative principles

In this section, we apply our fixed point theorems to several alternative principles.

**Theorem 2.** *Let $X$ be an almost convex admissible subset of a t.v.s. and $Y$ a topological space, one of them compact. Let $F, S : X \multimap Y$ be two maps satisfying the following conditions:*

(1) *for each $x \in X$, $F(x) \subset S(x)$;*

(2) *$F$ and $S^*$ are empty-valued or acyclic maps;*

*Then at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $F(x_0) = \emptyset$.

(b) $\bigcap_{x \in X} S(x) \neq \emptyset$.

**Proof.** Suppose that assertions (a) and (b) are both false. Note that $\bigcap_{x \in X} S(x) = \emptyset$ is equivalent to the fact that the map $S^*$ has nonempty values. Hence both $F$ and $S^*$ have nonempty values. Hence, by (2), $F \in \mathbb{V}(X, Y)$ and $S^* \in \mathbb{V}(Y, X)$. Let us consider the map $H = S^* \circ F \in \mathbb{V}_c(X, X)$. $H$ is a u.s.c. map with compact values, so $H$ is closed. Since $\mathbb{V}_c \subset \mathfrak{B}$, Corollary 1.1 is applicable as soon as we prove that the map $H$ is compact. Clearly this happens if $X$ is compact. When $Y$ is compact, since $S^*$ is a u.s.c. map with compact values, $S^*(Y)$ is compact by Lemma 1(iii). Since $H(X) \subset S^*(Y)$, $H$ is a compact map. By Corollary 1.1, $H$ has a fixed point. This implies that there exist $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in F(x_0) \subset S(x_0)$ and $x_0 \in S^*(y_0)$; a contradiction. □

The following are variants of Theorem 2:

**Theorem 3.** *Let* $X$ *be an admissible convex subset of a t.v.s. and* $Y$ *a convex space, one of them compact. Let* $F, S : X \multimap Y$ *be two maps satisfying the following conditions:*

(1) *for each* $x \in X$, $F(x) \subset S(x)$;

(2) $F$ *and* $S^*$ *are empty-valued or Kakutani maps;*

*Then at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $F(x_0) = \emptyset$.

(b) $\bigcap_{x \in X} S(x) \neq \emptyset$.

**Proof.** As in the proof of Theorem 2, we can define $H = S^* \circ F \in \mathbb{K}_c(X, X)$, where the intermediate space $Y$ is a convex space in the sense of Lassonde. Then we can apply Corollary 1.3. □

**Lemma 2.** [B] *Let* $X$ *be a topological space and* $Y$ *a nonempty convex set in an l.c.s.* $E$. *Let* $T : X \multimap Y$ *be a u.s.c. map with nonempty values such that* $\overline{\text{co}}\, T(x)$ *is compact for each* $x \in X$. *Then the map* $\overline{\text{co}}\, T$ *is u.s.c.*

**Theorem 4.** *Let* $X$ *be an almost convex admissible subset of a t.v.s. and* $Y$ *a nonempty convex subset in an l.c.s., one of them compact. Let* $F, G, S : X \multimap Y$ *be three maps satisfying the following conditions:*

(1) *for each* $x \in X$, $\text{co}F(x) \subset G(x) \subset S(x)$;

(2) $F$ *is empty-valued or a u.s.c. map;*

(3) $G$ *has compact values;*

(4) $S^*$ *is empty-valued or an acyclic map;*

*Then at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $F(x_0) = \emptyset$.

(b) $\bigcap_{x \in X} S(x) \neq \emptyset$.

**Proof.** Suppose that assertions (a) and (b) are both false. Then $F$ and $S^*$ have both nonempty values. By (1) and (3), $\overline{\text{co}}\, F(x) \subset G(x) \subset Y$ and $\overline{\text{co}}\, F(x)$ is a nonempty compact set, for each $x \in X$. By Lemma 2, the map $\overline{\text{co}}\, F$ is u.s.c., hence $\overline{\text{co}}\, F \in \mathbb{K}(X, Y)$. As in the proof of Theorem 2, we can define $H = S^* \circ \overline{\text{co}} F \in \mathbb{V}_c(X, X)$. Then we can apply Corollary 1.1. □

From now on, we consider the Balaj type alternative principles:

**Theorem 5.** *Let* $X$ *be an admissible convex subset of a t.v.s. and* $Y$ *a nonempty convex subset in an l.c.s., one of them compact. Let* $F, G, S, T : X \multimap Y$ *be four maps satisfying the following conditions:*

(1) *for each* $x \in X$, $\text{co}\, F(x) \subset G(x) \subset T(x)$*;*

(2) *for each* $y \in Y$, $\text{co}\, S^*(y) \subset T^*(y)$*;*

(3) $F$ *and* $S^*$ *are u.s.c.;*

(4) $G$ *and* $T^*$ *have compact values.*

*Then at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $F(x_0) = \emptyset$.

(b) $\bigcap_{x \in X} S(x) \neq \emptyset$.

**Proof.** Suppose that assertions (a) and (b) are false. Then $F$ and $S^*$ have nonempty values. By (1), (4) and Lemma 2, $\overline{\text{co}}\, F \in \mathbb{K}(X, Y)$. Moreover, by (2) and (4), $\overline{\text{co}}\, S^*(y) \subset T^*(y)$ so $\overline{\text{co}}\, S^*(y)$ is compact for each $y \in Y$. By Lemma 2, the map $\overline{\text{co}}\, S^*$ is u.s.c., hence $\overline{\text{co}}\, S^* \in \mathbb{K}(Y, X)$. Let us consider the map $H = \overline{\text{co}}\, S^* \circ \overline{\text{co}}\, F \in \mathbb{K}_c(X, X)$. Since $K_c \subset \mathfrak{B}$, and $H$ is closed and compact, $H$ has a fixed point by Corollary 1.3. This implies that there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$y_0 \in \overline{\text{co}}\, F(x_0) \subset G(x_0) \subset T(x_0) \quad \text{and} \quad x_0 \in \overline{\text{co}}\, S^*(y_0) \subset T^*(y_0);$$

which is a contradiction. □

**Corollary 5.1.** [B, Theorem 3.1] *Let* $X$ *and* $Y$ *be nonempty convex sets in an l.c.s., and let* $X$ *or* $Y$ *be compact. Let* $F, G, S, T : X \multimap Y$ *be four maps satisfying the following conditions:*

(i) *for each* $x \in X$, $\text{co}\, F(x) \subset G(x) \subset T(x)$*;*

(ii) $S$ *is a KKM map w.r.t.* $T$*;*

(iii) $F$ *and* $S^*$ *are u.s.c.;*

(iv) $G$ *and* $T^*$ *have compact values.*

*Then at least one of the following assertions holds:*

(a) *There exists* $x_0 \in X$ *such that* $F(x_0) = \emptyset$.

(b) $\bigcap_{x\in X} S(x) \neq \emptyset$.

**Proof.** Comparing the requirements of Theorem 5 and Corollary 5.1, it suffices to show that

(2) for each $y \in Y$, $\operatorname{co} S^*(y) \subset T^*(y)$.

In fact, let $x \in \operatorname{co} S^*(y)$. Then there exists a finite set $A \subset S^*(y)$ such that $x \in \operatorname{co} A$. From $A \subset S^*(y)$ it follows at once, $y \notin S(A)$ and taking into account (ii), $y \notin T(\operatorname{co} A)$. Particularly $y \notin T(x)$, that is, $x \in T^*(y)$. Thus, (2) holds. Then we have the conclusion. □

As it is mentioned in [B], the compactness condition, for at least one of the sets $X$ and $Y$, is essential in Corollary 5.1. But the example in [B] showing this fact must be corrected as follows:

Let $X = Y = \mathbb{R}$ and $F, G, S, T : \mathbb{R} \multimap \mathbb{R}$ be defined by $F(x) = \{x+1, x+2\}$, $G(x) = [x+1, x+2]$, $T(x) = \mathbb{R}\backslash[x-1, x]$, and $S(x) = \mathbb{R}\backslash\{x\}$ for all $x \in \mathbb{R}$.

In [B], Balaj applied his Theorems 3.1 to the following:

(1) a matching theorem of Ky Fan type,

(2) an analytic alternative,

(3) a minimax inequality, and

(4) existence of solutions of a vector equilibrium theorem.

Since we obtained improved versions of [B, Theorem 3.1], the above applications also can be improved. From Theorem 4, we also obtain a matching theorem of Ky Fan type, an analytic alternative, a minimax inequality as follows.

## 4. Applications

The following is Ky Fan type matching theorem [F];

**Theorem 6.** *Let $X$ be an almost convex admissible subset of a t.v.s. and $Y$ a nonempty convex subset in an l.c.s., one of them compact. Let $\{B_i : i \in I\}$ be a family of closed subsets of $X$ such that $\bigcup_{i\in I} B_i = X$ and $\{x_i : i \in I\}$ be a family (indexed also by $I$) of points of $X$ and the set $\{i \in I : x \notin B_i\}$ is finite for each $x \in X$. Let $F, G, S : X \multimap Y$ be three maps satisfying the following conditions:*

(1) *for each $x \in X$, $S(x) \subset G(x)$;*

(2) *$F$ is a u.s.c. map;*

(3) *$G^c$ has compact values;*

(4) *$S(X) = Y$;*

(5) $S^-$ *is an acyclic map;*
*Then there exists a nonempty subset $J$ of $I$ such that*

$$\text{co}F(\{x_i : i \in J\}) \cap G(\cap\{B_i : i \in J\}) \neq \emptyset.$$

**Proof.** For each $x \in X$, let $I(x) = \{i \in I : x \in B_i\}$. Then $I(x) \neq \emptyset$ for each $x \in X$, since $\{B_i : i \in I\}$ covers $X$. Define $H : X \multimap X$ by

$$H(x) = \{x_i : i \in I(x)\} \text{ for each } x \in X.$$

Then $H$ has nonempty values. For each $x \in X$, let $U(x) = X \setminus \bigcup\{B_i : i \notin I(x)\}$. Then $U(x)$ is an open neighborhood of $x$ and, if $z \in U(x)$, then $H(z) \subset H(x)$. This shows $H$ is u.s.c.. Define the maps $F_1, G_1, S_1 : X \multimap Y$ by

$$F_1 = F \circ H, \;\; G_1 = G^c, \;\; S_1 = S^c.$$

Note that $S_1^* = S^-$. From the hypothesis it readily follows that the maps $F_1, G_1, S_1$ satisfy the conditions (2), (3), (4) in Theorem 4 and $G_1(x) \subset S_1(x)$ for each $x \in X$. Since $F_1$ has nonempty values and by (4), $\bigcap_{x\in X} S_1(x) = \emptyset$, the conclusion of Theorem 4 does not hold. Consequently for some $x_0 \in X$, $\text{co}F_1(x_0) \not\subset G_1(x_0)$ or equivalently,

$$\text{co}F(\{x_i : i \in I(x_0)\}) \cap G(x_0) \neq \emptyset.$$

Since $x_0 \in \bigcap\{B_i : i \in I(x_0)\}$, putting $J = I(x_0)$ we obtain that

$$\text{co}F(\{x_i : i \in J\}) \cap G(\cap\{B_i : i \in J\}) \neq \emptyset.$$

□

Let $X$ be a convex subset of a vector space, $Y$ a nonempty set and $f, g : X \times Y \to \overline{\mathbb{R}}$ two functions. The function $g$ is called $f$-*quasiconvex* in $x$ if for any finite subset $A$ of $X$, $g(x, y) \leq \max_{u\in A} f(u, y)$, for any $x \in \text{co}A$ and all $y \in Y$.

From Theorem 4, we obtain an analytic alternative and a minimax inequality.

**Theorem 7.** *Let $X$ be an almost convex admissible subset of a t.v.s. and $Y$ a nonempty convex subset in an l.c.s., both of them compact. Let $\alpha \leq \beta < \gamma$ be numbers and $f, g, s : X \times Y \to \overline{\mathbb{R}}$ three functions satisfying the following conditions:*

(1) *the sets $\{(x, y) \in X \times Y : f(x, y) \leq \alpha\}$ and $\{(x, y) \in X \times Y : s(x, y) \geq \gamma\}$ are closed in $X \times Y$;*

(2) *$s(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$;*

(3) *$g$ is $f$-quasiconvex in $y$;*

(4) *for each $y \in Y$, $\{x \in X : s(x, y) \geq \gamma\}$ is acyclic;*

(5) *for each* $x \in X$, *the set* $\{y \in Y : g(x,y) \leq \beta\}$ *is closed in* $Y$;
*Then at least one of the following assertions holds:*
(a) *There exists* $x_0 \in X$ *such that* $f(x_0, y) > \alpha$ *for all* $y \in Y$.
(b) *There exists* $y_0 \in Y$ *such that* $s(x, y_0) < \gamma$ *for all* $x \in X$.

**Proof.** We define the maps $F, G, S : X \multimap Y$ by

$$F(x) = \{y \in Y : f(x,y) \leq \alpha\},$$
$$G(x) = \{y \in Y : g(x,y) \leq \beta\},$$
$$S(x) = \{y \in Y : s(x,y) < \gamma\}.$$

Since the graphs of $F$ and $S^*$ are closed and $X$ and $Y$ are compact, $F$ and $S^*$ are u.s.c. with compact values. Since $\beta < \gamma$ and $s(x,y) \leq g(x,y)$ for all $(x,y) \in X \times Y$, $G(x) \subset S(x)$ for all $x \in X$. By (5), $G$ has compact values. Let $x \in X$ and $y \in \mathrm{co}F(x)$. Then for some finite subset $A$ of $F(x)$, $y \in \mathrm{co}A$. By (3), $g(x,y) \leq \max_{u \in A} f(x,u) \leq \alpha \leq \beta$, so $\mathrm{co}F(x) \subset G(x)$ for all $x \in X$.

Therefore the maps $F, G, S$ satisfy all the requirements of Theorem 4, so at least one of the following assertions holds:

There exists $x_0 \in X$ such that $F(x_0) = \emptyset$, that is $f(x_0, y) > \alpha$ for all $y \in Y$.

There exists $y_0 \in Y$ such that $y_0 \in \bigcap_{x \in X} S(x)$, that is $s(x, y_0) < \gamma$ for all $x \in X$.

□

**Theorem 8.** *Let* $X$ *be an almost convex admissible subset of a t.v.s. and* $Y$ *a nonempty convex subset in an l.c.s., both of them compact. Let* $f, g, s : X \times Y \to \overline{\mathbb{R}}$ *be three functions satisfying conditions (2), (3) in Theorem 7 and the following conditions:*
(1′) $f$ *is l.s.c. and* $s$ *is u.s.c. on* $X \times Y$;
(4′) *for each* $y \in Y$ *and real number* $\gamma$, $\{x \in X : s(x,y) \geq \gamma\}$ *is acyclic;*
(5′) *for each* $x \in X$, $g(x,y)$ *is l.s.c. in* $y$;
*Then* $\inf_{y \in Y} \max_{x \in X} s(x,y) \leq \sup_{x \in X} \min_{y \in Y} f(x,y)$.

**Proof.** First let us observe that if $f$ is l.s.c. on $X \times Y$, then for each $x \in X$, $f(x, \cdot)$ is l.s.c. in $y$ and therefore its minimum $\min_{y \in Y} f(x,y)$ on the compact set $Y$ exists. Similarly $\sup_{x \in X} s(x,y)$ can be replaced by $\max_{x \in X} s(x,y)$.

Suppose the conclusion would be false and choose real numbers $\alpha, \beta, \gamma$ such that

$$\sup_{x \in X} \min_{y \in Y} f(x,y) < \alpha \leq \beta < \gamma < \inf_{y \in Y} \max_{x \in X} s(x,y).$$

Even the function $f, g, s$ satisfy all the requirements of Theorem 7, but

if (a) happens, then $\sup_{x\in X} \min_{y\in Y} f(x,y) \geq \min_{y\in Y} f(x_0,y) > \alpha$, a contradiction

if (b) happens, then $\inf_{y\in Y} \max_{x\in X} s(x,y) \leq \max_{x\in X} s(x,y_0) < \gamma$, a contradiction. □

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Sehie Park
The National Academy of Sciences, Republic of Korea,
Seoul 137–044; and

Department of Mathematical Sciences, Seoul National University,
Seoul 151–747, Korea.
E-mail: shpark@math.snu.ac.kr; parkcha38@daum.net

Hoonjoo Kim
Department of Mathematical Education, Sehan University,
YoungAm-Gun, Chunnam, 530–702, Korea.
E-mail:hoonjoo@ sehan.ac.kr