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Evolution of the 1984 KKM theorem of Ky Fan

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Abstract

Recently we established the Knaster, Kuratowski, and Mazurkiewicz (KKM) theory on abstract convex spaces. In our research, we noticed that there are quite a few generalizations and applications of the 1984 KKM theorem of Ky Fan compared with his celebrated 1961 KKM lemma. In a certain sense, the relationship between the 1984 theorem and hundreds of known generalizations of the original KKM theorem has not been recognized for a long period. There would be some reasons to explain this fact. Instead, in this paper, we give several generalizations of the 1984 theorem and some known applications in order to reveal the close relationship among them.

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1 Introduction

One of the earliest equivalent formulations of the Brouwer fixed point theorem of 1912 is the theorem of Knaster, Kuratowski, and Mazurkiewicz (the KKM theorem for short) of 1929 [1], which was concerned with a particular type of multimaps, later called KKM maps. The KKM theory (first called by the author in 1992 [2–4]) is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961 Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone in the history of the KKM theory was erected by Fan in 1961 [5]. His 1961 KKM lemma (or the Fan-KKM theorem or the KKMF theorem) extended the KKM theorem to arbitrary topological vector spaces and was applied to various problems in his subsequent papers [6–9]. Moreover, his lemma was extended in 1979 and 1984 [10, 11] to the 1984 KKM theorem with a new coercivity (or compactness) condition for noncompact convex sets with new applications.

Consequently, at the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Fan in 1961-84 [5–11]. A number of intersection theorems and their applications to equilibrium problems followed. In our previous review [12], we recalled Ky Fan's contributions to the KKM theory based on his celebrated 1961 KKM lemma and introduced relatively recent applications of the lemma due to other authors in the twenty-first century.

Then, the KKM theory was extended to convex spaces by Lassonde in 1983 [13], and to *c*-spaces (or H-spaces) by Horvath in 1983-93 [14–17] and others. Since 1993 the theory has been extended to generalized convex (*G*-convex) spaces in a sequence of papers of the present author and others. Since 2006 the main theme of the theory has become abstract convex spaces in the sense of Park. The basic theorems in the theory have many applications to various equilibrium problems in nonlinear analysis and other fields. In our another review [18], we recalled our versions of general KKM type theorems for abstract convex spaces and introduced relatively recent applications of various generalized KKM type theorems due to other authors in the twenty-first century.

While we were studying [12, 18], we noticed that there are quite a few generalizations and applications of the 1984 KKM theorem of Ky Fan. In a certain sense, nobody noticed the relation between the theorem and hundreds of known generalizations of the original KKM theorem. There would be some reasons to explain this fact. Instead, in this paper, we introduce several generalizations of the 1984 theorem and some known direct applications in order to reveal the close relationship among them.

In Section 2 of this paper, we introduce the recent concepts of abstract convex spaces and partial KKM spaces and the modern versions of some most general KKM type theorems. Section 3 recalls Ky Fan's contributions to the KKM theory based on his celebrated 1961 KKM lemma and his 1984 KKM theorem. Section 4 deals with the evolution of the 1984 theorem, and we deduce several important generalizations. Section 5 is devoted to the discussion on relatively recent applications of the 1984 theorem due to other authors in the twenty-first century. Finally, Section 6 deals with some remarks on current studies of the KKM theory.

2 Abstract convex spaces and general KKM theorems

Multimaps are also simply called maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following [19–21]:

Definition An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* (resp., a $\mathfrak{KO}$-*map*) if, for any closed-valued (resp., open-valued) KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ (resp., $F \in \mathfrak{KO}(E, D, Z)$).

Definition The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle respectively.

In our recent works [19–21], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed that many important results therein are related to the partial KKM principle.

Recall the following well-known diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space
$\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

For a long time, the present author tried to unify hundreds of generalizations of the KKM type theorems and, finally, obtained Theorems A-C below in [22–24]. The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems.

Theorem A *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ (resp., $1_E \in \mathfrak{KO}(E, D, E)$), and $G : D \multimap E$ be a multimap satisfying*

(1) G has closed (resp., open) values; and

(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D\rangle$ (that is, G is a KKM map).

Then $\{G(y)\}_{y\in D}$ has the finite intersection property.

Further, if

(3) $\bigcap_{y\in M} \overline{G(y)}$ is compact for some $M \in \langle D\rangle$, then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Consider the following related four conditions for a map $G : D \multimap Z$:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type. The following is given in [22–24]:

Theorem B *Let $(E, D; \Gamma)$ be a partial KKM space (that is, $1_E \in \mathfrak{KC}(E, D, E)$) and $G : D \multimap E$ be a map such that*

(1) $\overline{G}$ is a KKM map (that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$); and

(2) there exists a nonempty compact subset K of E such that either

  (i) $\bigcap_{y\in M} \overline{G(y)} \subset K$ *for some* $M \in \langle D\rangle$*; or*
  (ii) for each $N \in \langle D\rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*;*
(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$*.*

Remark Considering the closure $\overline{K}$ instead of the compact set K, we may assume K is closed without loss of generality. Then the closure notations in each coercivity condition (i) and (ii) can be eliminated. Moreover, for the applications of Theorem B, see the recent work [25] and the references therein.

Theorem B can be extended to $F \in \mathfrak{KC}(E,D,Z)$ instead of $1_E \in \mathfrak{KC}(E,D,E)$ as the following in [22–24] shows:

Theorem C *Let* $(E,D;\Gamma)$ *be an abstract convex space,* Z *be a topological space,* $F \in \mathfrak{KC}(E,D,Z)$*, and* $G : D \multimap Z$ *be a map such that*
 (1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*
 (2) there exists a nonempty compact subset K *of* Z *such that either*
  (i) $K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ *for some* $M \in \langle D\rangle$*; or*
  (ii) for each $N \in \langle D\rangle$*, there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*
(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$*.*

Remark We may assume K is closed without loss of generality. Then the closure notations in each coercivity condition can be eliminated. As far as the author knows, Theorem C contains several hundred generalizations of the KKM theorem that appeared in the existing literature; see [12, 18, 21–28] and the references therein.

3 The origin and Fan's applications

In this section, we will follow [4, 29] and others.

Knaster, Kuratowski, and Mazurkiewicz in 1929 [1] obtained the following KKM theorem from the Sperner combinatorial lemma in 1928 and applied it to a simple proof of the Brouwer fixed point theorem.

The KKM theorem ([1]) *Let A_i $(0 \leq i \leq n)$ be $n + 1$ closed subsets of an n-simplex $p_0p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0}p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0}p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n, 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

This was extended by Ky Fan in 1961 [5] as follows.

The 1961 KKM lemma ([5]) *Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) convex hull of any finite subset $\{x_1, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.

(ii) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the Fan-KKM lemma or the Fan-KKM theorem or the KKMF theorem. Fan assumed the Hausdorffness of Y, which was known to be superfluous later. Fan and his followers applied his KKM lemma to various problems in many fields in mathematics; see [4, 25, 29].

Later, Fan [10, 11] introduced a KKM theorem with a more general coercivity (or compactness) condition for noncompact convex sets as follows.

The 1984 KKM theorem ([11]) *In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

This was first introduced in 1979 [10] without proof and was proved in 1984 [11] *via* an equivalent matching theorem for open covers of convex sets. We give a proof of the following for completeness.

The 1984 theorem $\Rightarrow$ The 1961 lemma Suppose $F(x_0)$ is compact for some $x_0 \in X$. Then $X_0 := \{x_0\}$ satisfies the requirement of the 1984 theorem. □

From the 1984 theorem or some equivalents, Fan [10, 11] extended many of known results to noncompact cases. We list some main results as follows:

Generalizations of the KKM theorem for noncompact cases;

Geometric formulations;

Fixed point and coincidence theorems;

Generalized minimax inequality (extends Allen's variational inequality (1977));
A matching theorem for open (closed) covers of convex sets;
The 1978 model of the Sperner lemma;
Another matching theorem for closed covers of convex sets;
A generalization of Shapley's KKM theorem (Shapley, 1973);
Results on sets with convex sections;
A new proof of the Brouwer theorem.

At a conference in 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows:
Potential theory;
Pontrjagin spaces or Bochner spaces in inner product spaces;
Operator ideals;
Weak compactness of subsets of locally convex topological vector spaces;
Function algebras;
Harmonic analysis;
Variational inequalities;
Free boundary value problems;
Convex analysis;
Mathematical economics;
Game theory;
Mathematical statistics.

We may add the following fields to this list: nonlinear functional analysis, approximation theory, optimization theory, fixed point theory, and some others; see Park [30].

4 Evolution of the 1984 KKM theorem of Ky Fan

Until now, nobody claimed any generalization of the 1984 KKM theorem of Ky Fan. In this section, we show several major generalizations of the theorem in the chronological order and that such generalizations are the consequences of Theorem B.

4.1 Lassonde's theorem

The concept of convex sets in a topological vector space was extended to convex spaces by Lassonde in 1983 [13] and further to *c*-spaces by Horvath in 1983-93 [14–17]. A number of other authors also extended the concept of convexity for various purposes.

Definition Let X be a subset of a vector space and D be a nonempty subset of X. We call (X, D) a *convex space* if $\mathrm{co}D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D\rangle$; see Park [3]. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [13].

A nonempty subset L of a convex space X is called a *c-compact* set [13] if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

Lassonde [13] presented a simple and unified treatment of a large variety of minimax and fixed point problems. He first noticed that the Hausdorffness in the 1961 lemma is redundant. More specifically, he gave several KKM type theorems for convex spaces (X, D) and proposed a systematic development of the method based on the KKM theorem; the principal topics treated by him may be listed as follows:

Fixed point theory for multimaps;
Minimax equalities;
Extensions of monotone sets;
Variational inequalities;
Special best approximation problems.

We begin with the following particular form of [13, Theorem I] for $X = Y$ in 1983, which extends the 1984 theorem of Fan:

Theorem 1 ([13]) *Let D be an arbitrary set in a convex space X and $F : D \multimap X$ be a multimap having the following properties:*

(i) for each $x \in D$, $F(x)$ is compactly closed in X;

(ii) F is a KKM map, that is, for any finite subset $\{x_1, \dots, x_n\}$ of D,

$$\mathrm{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i);$$

(iii) for some c-compact subset $L \subset X$, $\bigcap\{F(x) \mid x \in L \cap D\}$ is compact.

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

Theorem 1 ⇒ The 1984 theorem of Fan In the 1984 theorem, (Y, D) is a convex space with $D := X$. Note that, by adopting the compactly generated extension of the original topology of Y, $F : X \multimap Y$ becomes a closed-valued KKM map; see Park [30, 31]. Let L be the compact convex subset of a Hausdorff t.v.s. Y containing X_0. So L is a c-compact set; see [13]. Hence, $\bigcap\{F(x) \mid x \in L \cap D\}$ is compact as a closed subset of a compact set $\bigcap\{F(x) \mid x \in X_0\}$. Therefore, all of the requirements of Theorem 1 are satisfied. Now, the 1984 theorem follows from Theorem 1 of Lassonde. □

Remark As shown in the above proof, Hausdorffness is essential in the 1984 theorem. Note that it does not generalize the 1961 lemma without assuming Hausdorffness. Moreover, the compactly closed sets adopted by Lassonde and many followers can be replaced simply by closed sets; see [30, 31].

4.2 Chang's theorem

In 1989, Chang [32] obtained the following theorem which eliminated the concept of c-compact sets in Theorem 1.

Theorem 2 ([32]) *Let D be a nonempty subset of a convex space X and $F : D \multimap X$ be a multimap. Suppose that*

(i) for each $x \in D$, $F(x)$ is closed in X;

(ii) F is a KKM map;

(iii) there exist a nonempty compact subset K of X and, for each finite subset N of D, a compact convex subset L_N of X containing N such that

$$L_N \cap \bigcap\big\{F(x) \mid x \in L_N \cap D\big\} \subset K.$$

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

Theorem 2 ⇒ Theorem 1 Let $K := \bigcap\{F(x) \mid x \in L \cap D\}$ in Theorem 1. Then (iii) of Theorem 1 implies (iii) of Theorem 2. In fact, by (iii) of Theorem 1, since L is c-compact, for any $N \in \langle D\rangle$, there exists a compact convex subset L_N containing L and N such that $\bigcap\{F(x) \mid x \in L \cap D\}$ is compact. Then we have

$$L_N \cap \bigcap\{F(x) \mid x \in L_N \cap D\} \subset L_N \cap \bigcap\{F(x) \mid x \in L \cap D\} = L_N \cap K \subset K.$$

This is just (iii) of Theorem 2. Hence, Theorem 1 follows from Theorem 2. □

Note that Theorem 2 is due to Chang [32, Theorem 2.1] and improves Lassonde [13, Theorem III] who assumed the Hausdorffness of the underlying topological vector space and adopted a stronger condition than (iii).

4.3 For H-spaces

In this subsection, we follow [33].

Definition A triple $(X, D; \Gamma)$ is called an H-*space* (by Park in 1992) if X is a topological space, D is a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ is a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$.

For an H-space $(X, D; \Gamma)$, a subset C of X is said to be H-*convex* if for each $A \in \langle D\rangle$, $A \subset C$ implies $\Gamma_A \subset C$. A multimap $F : D \multimap X$ is said to be H-KKM if $\Gamma_A \subset F(A)$ for each $A \in \langle D\rangle$. A subset L of X is called an H-*subspace* of $(X, D; \Gamma)$ if $L \cap D \neq \emptyset$ and for every $A \in \langle L \cap D\rangle$, $\Gamma_A \cap L$ is contractible.

If $D = X$, we denote $(X; \Gamma)$ instead of $(X, X; \Gamma)$, which is called a c-space by Horvath [14–17] or an H-space by Bardaro and Ceppitelli [34]. It is notable that a torus, the Möbius band, or the Klein bottle can be regarded as c-spaces and are examples of compact H-spaces without having the fixed point property.

Theorem 3 ([33]) *Let $(X, D; \Gamma)$ be an H-space and $G : D \multimap X$ be an H-KKM multimap with closed values. Suppose that there exists a nonempty compact subset K of X such that either*

(i) $\bigcap\{G(x) \mid x \in M\} \subset K$ for some $M \in \langle D\rangle$; or
(ii) for each $N \in \langle D\rangle$, there exist a compact H-subspace L_N of X containing N such that

$$L_N \cap \bigcap\{G(x) \mid x \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

Theorem 3 ⇒ Theorem 2 Any convex space is an H-space. □

4.4 For *G*-convex spaces

Since 1996 the following has become one of the main themes of the KKM theory.

Definition A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\Delta_n = \text{co}\{e_i\}_{i=0}^n$ is the standard n-simplex, and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$.

For a G-convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is called a *KKM map* if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$. A subset C of X is said to be *G-convex* with respect to a subset $D' \subset D$ if for each $B \in \langle D' \rangle$, we have $\Gamma_B \subset C$.

Note that D is not necessarily a subset of X. However, for a G-convex space $(X \supset D; \Gamma)$, a subset C of X is said to be *G-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$.

There are lots of examples of G-convex spaces; see [21] and the references therein. So, since 1996 the KKM theory has been extended to the study of KKM maps on G-convex spaces based on the following.

Theorem 4 *Let $(X, D; \Gamma)$ be a G-convex space and $G : D \multimap X$ be a closed-valued KKM map such that there exists a nonempty compact subset K of X such that either*

(i) $\bigcap\{G(z) \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact G-convex subset L_N of X with respect to D' such that

$$L_N \cap \bigcap\{G(z) \mid x \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

The following KKM theorem for G-convex spaces originates from [26, 27] and simply follows from Theorem 4.

Corollary 4.1 *Let $(X \supset D; \Gamma)$ be a G-convex space and $G : D \multimap X$ be a closed-valued KKM map such that there exists a nonempty compact subset K of X satisfying either*

(i) $\bigcap\{G(z) \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that

$$L_N \cap \bigcap\{G(z) \mid z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Theorem 4 ⇒ Theorem 3 Since every H-space is a G-convex space, Theorem 3 follows from Corollary 4.1. □

Remark Similarly, for Theorems 1-3, the case (i) also holds, and generalizes the 1961 KKM lemma of Ky Fan.

4.5 For ϕ_A-spaces

Since the concept of G-convex spaces first appeared in 1993 [26], a number of modifications or imitations of the concept due to other authors have followed. Most of such examples are unified by the following ϕ_A-spaces in 2007 [35].

Definition A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A*-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, a subset C of X is said to be ϕ_A*-convex* with respect to a subset $D' \subset D$ if for each $B \in \langle D'\rangle$, we have $\operatorname{Im}\phi_B := \phi_B(\Delta_{|B|-1}) \subset C$.

We collect some known facts on ϕ_A-spaces as follows in [24, 25, 36, 37].

Proposition

(1) Any ϕ_A-space $(X, D; \{\phi_A\})$ can be made into a G-convex space $(X, D; \Gamma)$ in several ways.

(2) For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \textit{for each } A \in \langle D\rangle \textit{ and } J \in \langle A\rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

(3) For a ϕ_A-space $(X, D; \{\phi_A\})$, the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D\rangle$ with $|A| = n + 1$ is a KKM space. This KKM space may not be G-convex.

Because of Proposition (3), Theorem 4 can be extended to ϕ_A-spaces as follows.

Theorem 5 *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a closed-valued map such that*

$$\operatorname{Im}\phi_A := \phi_A(\Delta_{|A|-1}) \subset G(A) \quad \textit{for each } A \in \langle D\rangle.$$

Suppose that there exists a nonempty compact subset K of X such that either

(i) $\bigcap\{G(z) \mid z \in M\} \subset K$ for some $M \in \langle D\rangle$; or

(ii) for each $N \in \langle D\rangle$, there exist $D' \subset D$ containing N and a compact ϕ_A-convex subset L_N of X with respect to D' such that

$$L_N \cap \bigcap\{G(z) \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Theorem 5 ⇒ Theorem 4 Recall that every G-convex space is a ϕ_A-space. □

4.6 For partial KKM spaces

As we have seen in Section 2, recently we introduced the concepts of KKM spaces and partial KKM spaces. Since all of the spaces in Sections 4.1-4.5 are ϕ_A-spaces, and hence KKM spaces, Theorems A-C can be applied to them.

Example We give known examples of partial KKM spaces; see [21] and the references therein:

(1) Every ϕ_A-space is a KKM space; see [26].
(2) A connected linearly ordered space $(X, \leq)$ can be made into a KKM space.
(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space.
(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := \mathrm{conv}_H(A)$ for each $A \in \langle X \rangle$, Kirk and Panyanak showed that the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle.
(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$.
(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space.

More recently, Kulpa and Szymanski [38] found some partial KKM spaces which are not KKM spaces.

As usual, for a partial KKM space $(X, D; \Gamma)$, a KKM map $G : D \multimap X$ is a map satisfying $\Gamma_A \subset G(A)$ for each $A \in \langle D \rangle$.

Theorem 6 *Let $(X, D; \Gamma)$ be a partial KKM space and $G : D \multimap X$ be a closed-valued KKM map such that there exists a nonempty compact subset K of X satisfying either*

(i) $\bigcap\{G(z) \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of X with respect to D' such that

$$L_N \cap \bigcap\{G(z) \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof Considering $\overline{K}$ instead of K, (i) and (ii) imply the requirements of Theorem B. Now, the conclusion follows from Theorem B. □

Theorem 6 ⇒ Theorem 5 Recall that every ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ is a KKM space $(X, D; \Gamma)$ with $\Gamma_A := \mathrm{Im}\, \phi_A$ for each $A \in \langle D \rangle$; see [25]. Moreover, $G : D \multimap X$ in Theorem 5 becomes a KKM map on $(X, D; \Gamma)$. Hence, this is also a partial KKM space and satisfies the requirements of Theorem 6. □

The KKM theorem, the 1961 lemma, and the 1984 theorem of Fan are all extended to Theorems 1-6. Moreover, all of them are included into Theorem B.

5 Recent applications of the 1984 KKM theorem of Ky Fan

In this section, we introduce relatively recent applications of the 1984 KKM theorem of Fan due to other authors in the twenty-first century.

(I) Abstract of Hai and Khanh [39] in 2007: 'A general quasiequilibrium problem is proposed including, among others, equilibrium problems, implicit variational inequalities, and quasivariational inequalities involving multifunctions. Sufficient conditions for the existence of solutions with and without relaxed pseudomonotonicity are established. Even semicontinuity may not be imposed.'

In [39], Fan's 1984 KKM theorem was applied. Moreover, the authors deduced the following 'modification' of the 1984 theorem from a result of Lin [40]:

Theorem ([39]) *Assume that V is a convex set in a Hausdorff topological vector space and $H : V \multimap V$ is a KKM map in V with closed values. Assume further that there exists a nonempty compact subset $D \subset V$ such that, for all finite subsets $M \subset V$, there is a compact convex subset L_M of V containing M such that*

$$L_M \setminus D \subset \bigcup_{x \in L_M} \{V \setminus H(x)\}.$$

Then $\bigcap_{x \in V} H(x) \neq \emptyset$.

This follows from Theorem 2 of Chang [32] earlier in 1989 without assuming Hausdorffness.

(II) Abstract of Farajzadeh *et al.* [41] in 2008: 'We first define upper sign continuity for a set-valued mapping and then we consider two types of generalized vector equilibrium problems in topological vector spaces and provide sufficient conditions under which the solution sets are nonempty and compact. Finally, we give an application of our main results. The paper generalizes and improves results obtained by Fang and Huang in 2005 [42].'

They are based on the following particular form of the 1984 theorem of Fan:

Lemma *Let K be a nonempty subset of a topological vector space X and $F : K \multimap X$ be a KKM map with closed values. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then $\bigcap_{x \in K} F(x) \neq \emptyset$.*

Moreover, their Lemma 2.10 is based on an incorrect theorem.

(III) Abstract of Farajzadeh *et al.* [43] in 2009: 'In this work, we consider a generalized nonlinear variational-like inequality problem in topological vector spaces, and, by using the KKM technique, we prove an existence theorem. Our result extends a theorem of Ahmad and Irfan [44].'

They are based on the preceding particular form of the 1984 theorem of Fan.

(IV) Abstract of Mitrović and Merkle [45] in 2010: 'We prove the existence of a solution to the generalized vector equilibrium problem with bounds. We show that several known theorems from the literature can be considered as particular cases of our results, and we provide examples of applications related to the best approximations in normed spaces and variational inequalities.'

They are based on the 1984 KKM theorem of Fan.

Remark The authors of the above four papers listed in this section could not recognize the relationship between the 1984 theorem and other generalized KKM theorems.

6 Remarks on current studies

Recall that the main applications of recent generalized KKM theorems are as follows (see [12, 18]):

Vector variational-type inequalities;

Various quasi-equilibrium problems;
Eigenvector problems;
Set-valued minimax inequality;
Fixed point theorems;
Generalizations of Nash equilibrium theorem;
Variational inclusion problem;
Simultaneous nonlinear inequalities problem;
Differential inclusion problem;
(Vector mixed) quasi-variational inequality;
(Vector mixed) quasi-complementarity problem;
Traffic network problem;
Quasi-monotone vector equilibrium problem;
Generalized vector equilibrium problem;
Generalized (implicit) vector variational-like inequality;
Set equilibrium problem;
Set-valued mixed (quasi-)variational inequalities;
Variational-hemivariational inequalities.

In 2006-09, we proposed new concepts of abstract convex spaces and the (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [19–21] and the references therein. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A partial KKM space is an abstract convex space satisfying the partial KKM principle. A KKM space is an abstract convex space satisfying the partial KKM principle and its 'open' version. Now, the KKM theory becomes the study of spaces satisfying the partial KKM principle.

In our work [21], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. We add more than a dozen statements as their applications, including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [21] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

Competing interests
The author declares that he has no competing interests.

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