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Remarks on a Generalized KKM Theorem Without Convex Hull

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Abstract

In a recent paper of Z. Yang and Y. J. Pu [21], a KKM type theorem without convex hull was proved and applied to some existence theorems of solutions for (vector) Ky Fan minimax inequality, Ky Fan section theorem, variational relation problems, n-person noncooperative game, and n-person noncooperative multiobjective game. In the present paper, we show that the KKM type theorem and all results in [21] are consequences of already known results for ϕ_A-spaces and can be improved by eliminating some redundant restrictions.

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1 Introduction

In our recent study on the KKM theory originated from the celebrated Knaster–Kuratowski–Mazurkiewicz theorem in 1929, we introduced abstract convex spaces and a particular type of them, ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$; see [14] and the references therein. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on abstract convex spaces. Therefore, our abstract convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces in [8].

Motivated and inspired by the earlier research works on the KKM theory, in a recent paper [21], Yang and Pu introduced a new KKM map and proved a generalized KKM theorem without convex hull. As applications, some existence theorems of solutions for (vector) Ky Fan minimax inequality, Ky Fan section theorem, variational relation problems, n-person noncooperative game, and n-person noncooperative multiobjective game are obtained in [21] under some redundant restrictions. These results might misguide the readers.

In the present paper, in order to prevent such misdirect, we show that the KKM type theorem and all results in [21] are consequences of already known results for ϕ_A-spaces and can be improved by eliminating some redundant restrictions.

2 Preliminaries

A multimap $F : X \to 2^Y$ is a function from a set X into the power set of Y and $F^- : Y \to 2^X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. Multimaps are also called simply maps.

Let

$$\Delta_n := \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0\}$$

be the n-simplex.

The following is the origin of the KKM theory; see [6,7].

The KKM Theorem. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \to 2^{\Delta_n}$ be a KKM map* (*that is,* $\operatorname{co} A \subset G(A)$ *for each* $A \subset D$) *with closed* [*resp., open*] *values. Then* $\bigcap_{z \in D} G(z) \neq \emptyset$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [9-11,13-19]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\operatorname{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\operatorname{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\operatorname{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. Typical examples of abstract convex spaces are convexity spaces, convex spaces, H-spaces, G-convex spaces, ϕ_A-spaces, KKM spaces, partial KKM spaces and numerous examples of them; see [14] and references therein. We need the following:

Definition. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n has the vertices $\{e_i\}_{i=0}^n$ and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. For details, see the references in [12].

Definition. A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$.

Example. Known examples of ϕ_A-spaces are given in [8,14].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \to 2^E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Example. (1) Let X be a nonempty subset of a topological vector space E. A multimap $F : X \to 2^E$ is a KKM map if $\mathrm{co}\,\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ for each finite subset $\{x_1, \dots, x_n\} \subset X$.

(2) [12] For a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$, any map $T : D \to 2^X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \ \text{ for each } \ A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on some G-convex space $(X, D; \Gamma)$.

(3) [21] Let X be a nonempty subset of a (Hausdorff) topological vector space E. A multimap $F : X \to 2^E$ is called a generalized KKM mapping iff, for any finite $A := \{x_0, x_1, \dots, x_n\} \subset X$, there is a continuous function $\phi_n : \Delta_n \to E$ such that, for any $\lambda \in \Delta_n$, there exists $i \in J(\lambda)$ such that $\phi_n(\lambda) \in F(x_i)$, where

$$J(\lambda) := \{i \in \{0, 1, \dots, n\} \mid \lambda_i > 0\}.$$

Remark. In (3), note that $\phi_n(\Delta_J) \subset F(J) = \bigcup_{x_i \in J} F(x_i)$ for all $J \subset A$. Hence the triple $(E, X; \{\phi_A\}_{A \in \langle X \rangle})$ is a ϕ_A-space and $F : X \to 2^E$ is a KKM map in our sense.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

3 Main theorem

The following is a KKM theorem for ϕ_A-spaces in [8], whose proof is given here for the completeness:

Theorem 3.1 *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \to 2^X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1} G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM principle, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z\in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear.

Corollary 3.1 [21, Theorem 2.1] *Let X be nonempty and compact subset of a Hausdorff topological vector space E, a multimap $F : X \to 2^X$ be a closed-value generalized KKM map, and X have the fixed point property. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Note that the Hausdorffness of E and the fixed point property of X are redundant.

Corollary 3.2 [21, Corollary 2.1] *Let X be a nonempty, convex, and compact subset of a locally convex Hausdorff topological vector space E, and a multimap $F : X \to 2^X$ be a closed-value generalized KKM-map. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Note that the local convexity and Hausdorffness of E are redundant.

For a set D, a topological space E, and a multimap $G : D \to 2^E$, consider the following related four conditions:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [5]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [5], its authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

Recall that the following general KKM theorem in [16-19] generalizes Theorem 3.1:

Theorem 3.2 *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \to 2^E$ a map such that*

(1) $\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$]; and

(2) *there exists a nonempty compact subset K of E such that either*

(i) $\bigcap_{z\in M}\overline{G(z)}\subset K$ *for some* $M\in\langle D\rangle$*; or*

(ii) *for each* $N\in\langle D\rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D'\subset D$ *such that* $N\subset D'$ *and*

$$\overline{L_N}\cap\bigcap_{z\in D'}\overline{G(z)}\subset K.$$

Then we have $K\cap\ \bigcap_{z\in D}\overline{G(z)}\neq\emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K\cap\bigcap\{G(z)\mid z\in D\}\neq\emptyset$*;*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z)\mid z\in D\}\neq\emptyset$.

4 Some Applications

4.1 Ky Fan Minimax Inequality

Here we give a new usage of ϕ_A-spaces: In [3], its author gave a necessary and sufficient condition for the existence of a pure-strategy Nash equilibrium for non-cooperative games in topological spaces. He adopted the following concept:

Definition. Let X be a topological space, and $D,Y\subset X$. A real function $f:X\times Y\to\mathbb{R}$ is said to be $\mathcal{C}$*-quasiconcave* on D if, for any $N=\{x^0,x^1,\dots,x^n\}\in\langle D\rangle$, there exists a continuous map $\phi_N:\Delta_n\to Y$ such that

$$\min\{f(x^i,\phi_N(\lambda))\mid i\in J\}\leq f(\phi_N(\lambda),\phi_N(\lambda))$$

for all $\lambda:=(\lambda_0,\lambda_1,\dots,\lambda_n)\in\Delta_n$, where $J:=\{i\mid\lambda_i\neq 0\}$.

Note that $(Y,D;\{\phi_N\}_{N\in\langle D\rangle})$ is a ϕ_A-space.

By Propositions 1 and 2 in [3], the $\mathcal{C}$-quasiconcavity unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) [1] and the $\mathcal{C}$-concavity (weaker than concavity) [4].

In [21], the $\mathcal{C}$-quasiconcavity is called generalized quasiconcavity.

In [20], we obtained the following generalization of the Fan minimax inequality:

Theorem 4.1 [20, Theorem 4.11] *Let X be a topological space, D a nonempty subset of X, and* $f,g:X\times X\to\mathbb{R}$*. Assume that:*

(1) $f\leq g$ *on* $X\times X$*;*

(2) *there exist* $x^1,\dots,x^n\in D$ *such that* $K=\bigcap_{i=1}^n\overline{G(x^i)}$ *is compact where* $G(x)=\{z\in X\mid f(x,z)\leq\mu\}$ *and* $\mu=\sup_{y\in X}g(y,y)$*;*

(3) $g|_{D\times X}$ *is* $\mathcal{C}$*-quasiconcave on* D*; and*

(4) *for each* $x\in D$, $\{y\in X\mid f(x,y)\leq\mu\}$ *is intersectionally closed [resp., transfer closed].*

Then there exists $\tilde{z}\in X$ *[resp.,* $\tilde{z}\in K$*] such that*

$$\sup_{x\in D}f(x,\tilde{z})\leq\sup_{y\in X}g(y,y)$$

holds.

From Theorem 4.1, we have the following useful result:

Corollary 4.1 [2, Corollary 3.1] *Let X be a compact topological space, D be a nonempty subset of X, and $g : X \times X \to R$. Suppose $g|_{D\times X}$ is $\mathcal{C}$-quasiconcave on D, and the function $y \mapsto g(x, y)$ is l.s.c. for each $x \in D$. Then*

$$\min_{y\in X} \sup_{x\in D} g(x, y) \le \sup_{y\in X} g(y, y).$$

Next an existence theorem of solution for Ky Fan minimax inequality was proved.

Corollary 4.2 [21, Theorem 3.1] *Let X be a nonempty and compact subset of a (Hausdorff) topological vector space E, and X have the fixed point property. A function $f : X \times X \to \mathbb{R}$ satisfies*

(i) *for any fixed $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous;*
(ii) *for any fixed $x \in X$, $y \mapsto f(x, y)$ is $\mathcal{C}$-quasiconcave on X;*
(iii) *for any $x \in X$, $f(x, x) \le 0$.*

Then there exists $x^* \in X$ such that $f(x^*, y) \le 0$ for any $y \in X$.

Note that Hausdorffness of E and the fixed point property of X are redundant.

4.2 Vector Ky Fan Minimax Inequality

As in [21, Section 3.2], we obtain a vector Ky Fan minimax inequality.

Definition [21, Definition 3.2] Let X be a Hausdorff topological space, Z be a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with $\operatorname{Int} P \neq \emptyset$, and $D, Y \subset X$. A function $f : X \times Y \to Z$ is called $\mathcal{C}$-P-quasiconcave on A iff, for any finite subset $\{x_0, x_1, \ldots, x_n\}$ of D, there exists a continuous mapping $\phi_n : \Delta_n \to Y$ such that, for any $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, there exists $i \in J(\lambda)$ such that

$$f(\phi_n(\lambda), \phi_n(\lambda)) \in f(x_i, \phi_n(\lambda)) + P,$$

where $J(\lambda) := \{i \in \{0, 1, \ldots, n\} \mid \lambda_i > 0\}$.

In [21], the $\mathcal{C}$-P-quasiconcavity is called the generalized P-quasiconcave. In the above definition, note that $(Y, D; \{\phi_n\})$ is a ϕ_A-space.

Definition [21, Definition 3.3] Let X be a Hausdorff topological space, Z be a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with $\operatorname{Int} P \neq \emptyset$. A vector-valued function $f : X \to Z$ is called P-continuous at $x_0 \in X$ iff, for any open neighborhood V of the zero element in Z, there exists an open neighborhood U of x_0 in X such that, for any $x \in U$, $f(x) \in f(x_0) + V + P$. In particular, f is called P-continuous on X iff f is P-continuous at every point of X.

Theorem 4.2 *Let X be a nonempty and compact subset of a Hausdorff topological vector space E, and Z be a Hausdorff topological vector space with nonempty, convex, closed, and pointed cone P with $\operatorname{Int} P \neq \emptyset$. A mapping $f : X \times X \to Z$ satisfies*

(i) *for any fixed* $y \in X$, $x \mapsto f(x, y)$ *is* P*-continuous;*
(ii) *for any fixed* $x \in X$, $y \mapsto f(x, y)$ *is* $\mathcal{C}$-P*-quasiconcave on* X*;*
(iii) *for any* $x \in X$, $f(x, x) \notin \mathrm{Int}\, P$.
Then there exists $x^* \in X$ *such that* $f(x^*, y) \notin \mathrm{Int}\, P$ *for any* $y \in X$.

Proof. Just follow the proof of [21, Theorem 3.2] and apply Theorem 3.1 instead of Corollary 3.1 [21, Theorem 2.1].

Note that Theorem 4.2 was given as [21, Theorem 3.2] under an extra assumption that X has the fixed point property.

4.3 Generalized Ky Fan Section Theorem

The following is one of our versions of Fan's geometric lemma:

Theorem 4.3 [17, Corollary 5.2] *Let* $(X, D; \Gamma)$ *be a compact partial KKM space and* $A \subset X \times X$, $C \subset D \times X$. *Suppose that*
(1) $(x, x) \in A$ *for every* $x \in X$*;*
(2) *for each* $z \in D$, $\{y \in X \mid (z, y) \in C\}$ *is intersectionally closed;*
(3) *for any fixed* $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid (z, y) \notin C\} \subset \{x \in X \mid (x, y) \notin A\}$.
Then there exists a point $y_0 \in X$ *such that* $D \times \{y_0\} \subset C$.

Note that [21, Theorem 3.3] is a particular form of Theorem 4.3 for a ϕ_A-space $(X; \{\phi_A\}_{A \in \langle X \rangle})$ with the redundant fixed point property of X.

4.4 Other applications

In Subsections 3.4 and 3.5 in [21], its authors are concerned with

3.4 Generalized Variational Relation Problems; and

3.5 Multiobjective Game and N-person Noncooperative Game.

In these subsections, they obtain Theorems 3.4–3.6 with a redundant assumption that X has the fixed point property. These results can also be generalized by our method in this paper.

5 Conclusion

In this paper, based on recent works on the KKM theory, we show that the KKM type theorem and all results in [21] are consequences of an already known result for ϕ_A-spaces . As applications, some existence theorems of solutions for (vector) Ky Fan minimax inequality, Ky Fan section theorem, variational relation problems, n-person noncooperative game, and n-person noncooperative multi-objective game in [21] can be all generalized and improved by eliminating redundant restrictions.

References

[1] M. Baye, G. Tian, and J. Zhou, Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs, *Rev. Econom. Stud.* **60** (1993), 935–948.

[2] S.-Y. Chang, Inequalities and Nash equilibria, *Nonlinear Anal.* **73** (2010), 2933–2940.

[3] J.-C. Hou, Characterization of the existence of a pure-strategy Nash equilibrium, *Appl. Math. Lett.* **22** (2009), 689–692.

[4] W. K. Kim and K. H. Lee, The existence of Nash equilibrium in N-person games with $\mathcal{C}$-concavity, *Comput. Math. Appl.* **44** (2002), 1219–1228.

[5] D. T. Luc, E. Sarabi, and A. Soubeyran, Existence of solutions in variational relation problems without convexity, *J. Math. Anal. Appl.* **364** (2010), 544–555.

[6] S. Park, Ninety years of the Brouwer fixed point theorem, *Vietnam J. Math.* **27** (1999), 187–222.

[7] S. Park, Elements of the KKM theory for generalized convex spaces, *Korean J. Comp. Appl. Math.* **7** (2000), 1–28.

[8] S. Park, Remarks on KKM maps and fixed point theorems in generalized convex spaces, *CUBO, Math. J.* **10** (2008), 1–13.

[9] S. Park, Elements of the KKM theory on abstract convex spaces, *J. Korean Math. Soc.* **45** (2008), 1–27.

[10] S. Park, Equilibrium existence theorems in KKM spaces, *Nonlinear Anal.* **69** (2008), 4352–4364.

[11] S. Park, New foundations of the KKM theory, *J. Nonlinear Convex Anal.* **9** (2008), 331–350.

[12] S. Park, Comments on the KKM theory on ϕ_A-spaces, *PanAmerican Math. J.* **18** (2008), 61–71.

[13] S. Park, Fixed point theory of multimaps in abstract convex uniform spaces, *Nonlinear Anal.* **71** (2009), 2468–2480.

[14] S. Park, The KKM principle in abstract convex spaces: Equivalent formulations and applications, *Nonlinear Anal.* **73** (2010), 1028–1042.

[15] S. Park, Applications of the KKM theory to fixed point theory, *J. Nat. Acad. Sci., ROK, Nat. Sci. Ser.* **50** (2011), 21–69.

[16] S. Park, A genesis of general KKM theorems for abstract convex spaces, *J. Nonlinear Anal. Optim.* **2** (2011), 133–146.

[17] S. Park, New generalizations of basic theorems in the KKM theory, *Nonlinear Anal.* **74** (2011), 3000–3010.

[18] S. Park, Applications of some basic theorems in the KKM theory [in: The series of papers on S. Park's Contribution to the Development of Fixed Point Theory and KKM Theory], *Fixed Point Theory Appl.* vol. 2011:98 doi:10.1186/1687-1812-2011-98.

[19] S. Park, Remarks on certain coercivity in general KKM theorems, *Nonlinear Anal. Forum* **16**(2011), 1–10.

[20] S. Park, On S.-Y. Chang's inequalities and Nash equilibria, *J. Nonlinear Convex Anal.* **12** (2011), 455–471.

[21] Z. Yang and Y. J. Pu, Generalized Knaster-Kuratowski-Mazurkiewicz theorem without convex hull, *J. Optim. Theory Appl.* DOI 10.1007/s10957-012-9994-8.