Nonlinear Analysis Forum **17**, pp. 43–51, 2012
Available electronically at http://www.na-forum.org

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Reprinted from the
Nonlinear Analysis Forum
Vol. 17, August 2012

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REMARKS ON THE PAPER "VARIATIONAL RELATION PROBLEM WITHOUT THE KKM PROPERTY WITH APPLICATIONS" BY Y. J. PU AND Z. YANG

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Abstract. Recently, by means of variational relation problems, Y. J. Pu and Z. Yang [3] obtained new existence theorems of solutions for generalized KKM theorems, variational inclusion problems, the generalized (vector) Ky Fan minimax inequality, generalized Ky Fan section theorems, n-person noncooperative generalized games and n-person noncooperative multi-objective generalized games.

In the present paper, based on recent works on the KKM theory, we show that the Hausdorffness and the fixed point property imposed on relevant spaces are redundant in all of the results in [3].

1. Introduction

In our recent study on the KKM theory originated from the celebrated Knaster–Kuratowski–Mazurkiewicz theorem in 1929, we introduced abstract convex spaces and a particular type of them, ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$; see [1] and the references therein. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on abstract convex spaces. Therefore, our abstract convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces in [2].

In a recent paper [3], Y. J. Pu and Z. Yang used certain methods to establish variational relation problems without the KKM property. Moreover, by means of variational relation problems, they obtained new existence theorems of solutions for generalized KKM theorems, variational inclusion problems, a generalized (vector) Ky Fan minimax inequality, generalized

2010 Mathematics Subject Classification: 47H04, 47H10, 47J20, 47N10, 49J53, 52A99, 54C60, 54H25, 58E35, 90C47, 91A13, 91B50.

Key words and phrases: Generalized KKM theorem, (vector) Ky Fan minimax inequality, Ky Fan section theorem, variational relation problems, Nash equilibrium, weakly Pareto–Nash equilibrium.

Ky Fan section theorems, n-person noncooperative generalized games and n-person noncooperative multi-objective generalized games.

Their basic techniques are the partition of unity argument, continuous selection method, and fixed point method. Hence they had to inevitably assume the Hausdorffness and the fixed point property of relevant spaces. However, in the present paper, based on recent works on the KKM theory, we show that such assumptions are redundant in all of the results in [3].

Section 2 is devoted to preliminaries on the KKM theory of abstract convex spaces. In Section 3, we obtain generalized versions of the main results in [3]. Section 4 deals with corrected versions of various applications of main results in [3] to variational inclusion problems, a generalized (vector) Ky Fan minimax inequality, generalized Ky Fan section theorems, n-person noncooperative generalized games and n-person noncooperative multi-objective generalized games.

2. Preliminaries

A multimap $F : X \multimap Y$ is a function $F : X \to 2^Y$ from a set X into the power set of Y and $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. Multimaps are also called simply maps.

Let

$$\Delta_n := \{(\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0\}$$

be the standard n-simplex. (Note that Δ_n denoted an $(n-1)$-simplex throughout in [3].)

The following is the origin of the KKM theory; see [4,5].

The KKM Theorem. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map* (*that is,* co $A \subset G(A)$ *for each* $A \subset D$) *with closed* [*resp., open*] *values. Then* $\bigcap_{z \in D} G(z) \neq \emptyset$.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [1] and the references therein:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Exmaple. Typical examples of abstract convex spaces are convexity spaces, convex spaces, H-spaces, G-convex spaces, ϕ_A-spaces, KKM spaces, partial KKM spaces and numerous examples of them; see [1] and references therein.

We need the following:

Definition. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ due to Park consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see the references in [4].

Definition. A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Exmaple. Known examples of ϕ_A-spaces are given in [1,2,4].

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

Exmaple. (1) Let X be a nonempty subset of a topological vector space E. A multimap $F : X \multimap E$ is a KKM map if $\mathrm{co}\,\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ for each finite subset $\{x_1, \ldots, x_n\} \subset X$.

(2) [4] For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \ \text{ for each } \ A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is a KKM map on some G-convex space $(X, D; \Gamma)$, where Δ_J is the face of the simplex $\Delta_{|A|+1}$ corresponding to $J \subset A$.

(3) [3, 5] Let X be a nonempty subset of a (Hausdorff) topological vector space E. A multimap $F : X \multimap E$ is called a generalized KKM mapping iff, for any finite $A := \{x_0, x_1, \ldots, x_n\} \subset X$, there is a continuous function $\phi_n : \Delta_n \to E$ such that, for any $\lambda \in \Delta_n$, there exists $i \in J(\lambda)$ such that $\phi_n(\lambda) \in F(x_i)$, where

$$J(\lambda) := \{i \in \{0, 1, \ldots, n\} \mid \lambda_i > 0\}.$$

Remark. In (3), note that $\phi_n(\Delta_J) \subset F(J) = \bigcup_{x_i \in J} F(x_i)$ for all $J \subset A$. Hence $(E, X; \{\phi_A\}_{A \in \langle X \rangle})$ is a ϕ_A-space and $F : X \multimap E$ is a KKM map in our sense.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

3. Main theorem

The following KKM theorem in [2] shows that any ϕ_A-space is a KKM space and its proof is given here for the completeness:

Theorem 3.1. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have*

$$\phi_N(\Delta_n) \cap \bigcap_{z \in N} G(z) \neq \emptyset.)$$

Further, if $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$, then we have

$$\bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1} G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM theorem, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z \in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear.

Definition ([6, 7])**.** Let A and B be nonempty subsets of topological spaces E_1 and E_2, respectively, and $R(a, b)$ be a relation linking $a \in A$ and $b \in B$. For each fixed $b \in B$, we say that $R(\cdot, b)$ is closed in the first variable, if, for every net $\{a_\alpha\}$ converges to some a and $R(a_\alpha, b)$ holds for any α, then the relation $R(a, b)$ holds.

The following existence of solutions for variational relation problem extends [3, Theorem 2.1]:

Theorem 3.2. *Let X be a nonempty and compact subset of a topological vector space E. Let $R(x, y)$ be a relation linking elements $x \in X$, $y \in X$ with the following conditions:*

(i) *for any fixed $y \in X$, $R(\cdot, y)$ is closed;*

(ii) *for any finite subset* $\{x_1, \dots, x_n\}$ *of* X*, there exists a continuous mapping* $\phi_n : \Delta_n \to X$ *such that, for any* $\lambda = (\lambda_0, \dots, \lambda_n) \in \Delta_n$*, there exists* $i \in J(\lambda)$ *such that* $R(\phi_n(\lambda), x_i)$ *holds, where* $\Delta_n = \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}_n \mid \sum_{i=0}^n \lambda_i = 1,\ \lambda_i \geq 0\}$, $J(\lambda) = \{i \in \{0, 1, \dots, n\} \mid \lambda_i > 0\}$.

Then there exists $x^* \in X$ *such that* $R(x^*, y)$ *holds for any* $y \in X$.

Proof. Define $G : X \multimap X$ by $G(y) := \{x \in X \mid R(x, y)\}$ for $y \in X$. Since X is compact, by (i), each $G(y)$ is closed and hence compact. By (ii), for any $A \in \langle X \rangle$ with the cardinality $|A| = n+1$, there exists $\phi_A : \Delta_n \to X$. Hence $(X; \phi_A\}_{A \in \langle X \rangle})$ is a ϕ_A-space. Moreover, for each $\lambda \in \Delta_n$, there exists $i \in J(\lambda) = \{i \mid \lambda_i > 0\}$. This implies $\phi_A(\Delta_J) \subset G(J) = \bigcup_{x_i \in J} G(x_i)$ for all $J \subset A$. Hence $G : X \multimap X$ is a KKM map in our sense. Therefore, by Theorem 3.1, there exists an $x^* \in \bigcap_{y \in x} G(y) \neq \emptyset$. This is the conclusion.

Note that [3, Theorem 2.1] is a particular form of Theorem 3.2 under the redundant assumptions that E is Hausdorff and X has the fixed point property.

From [3, Theorem 2.1], its authors deduced a new existence theorem [3, Theorem 2.2] of solutions for the generalized variational relation problem without the KKM property and [3, Corollary 2.1], under redundant requirements of Hausdorffness and the fixed point property.

As a special case of Theorem 3.2, we obtain the following extended version of [3, Corollary 2.1]:

Corollary 3.1. *Let* X *be a nonempty and compact subset of a topological vector space,* $S_1 : X \multimap X$*,* $S_2 : X \multimap X$ *be two set-valued maps with nonempty values. Let* $R(x, y)$ *be a relation linking elements* $x \in X$ *and* $y \in X$*. Assume that*

(i) $A := \{x \in X \mid x \in S_1(x)\}$ *is closed;*

(ii) $S_2(x) \subset S_1(x)$ *for any* $x \in X$*, and* $S_2^-(y)$ *is open in* X *for any* $y \in X$*;*

(iii) *for any fixed* $y \in X$*,* $R(\cdot, y)$ *is closed;*

(iv) *for any finite subset* $\{x_0, \dots, x_n\}$ *of* X*, there exists a continuous map* $\phi_n : \Delta_n \to X$ *such that, for any* $\lambda = (\lambda_0, \dots, \lambda_n) \in \Delta_n$*, there exists* $i \in J(\lambda)$ *such that* $R(\phi_n(\lambda), x_i)$ *holds; if* $x_i \in S_2(\phi_n(\lambda))$ *for any* $i \in J(\lambda)$*, then* $\phi_n(\lambda) \in S_2(\phi_n(\lambda))$*, where* $J(\lambda) = \{i \in \{0, 1, \dots, n\} \mid \lambda_i > 0\}$.

Then there exists $x^* \in X$ *such that* $x^* \in S_1(x^*)$ *and* $R(x^*, y)$ *holds for any* $y \in S_2(x^*)$.

The following is a disguised form of the KKM theorem for ϕ_A-spaces:

Corollary 3.2. *Let* X *be a nonempty and compact subset of a topological vector space* E*, a multivalued mapping* $F : X \multimap X$ *be a closed-value generalized KKM-mapping in the sense of [3]. Then* $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof. Apply Corollary 3.1 when $R(x, y)$ holds iff $x \in F(y)$.

Note that Corollary 3.2 extends [3, Theorem 2.3], where the Hausdorffness of E and the fixed point property of X are redundant.

From Corollary 3.1, a generalized Ky Fan section theorem is obtained:

Theorem 3.3. *Let X be a nonempty and compact subset of a topological vector space E. Let $B \subset X \times X$ satisfy*

(i) *for any $y \in X$, $\{x \in X \mid (x, y) \in B\}$ is open in X;*

(ii) *for any finite subset $\{x_0, x_1, \ldots, x_n\}$ of X, there exists a continuous mapping $\phi_n : \Delta_n \to X$ such that, for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$, there exists $i \in J(\lambda)$ such that $(\phi_n(\lambda), x_i) \notin B$.*

Then there exists $x^* \in X$ such that $(x^*, y) \notin B$ for any $y \in X$.

Proof. Define variational relation $R(x, y)$ by

$$R(x, y) \text{ holds if and only if } (x, y) \notin B.$$

By condition (i), $R(\cdot, y)$ is closed for any $y \in X$. By condition (ii), for any finite subset $\{x_0, x_1, \ldots, x_n\}$ of X, there exists a continuous mapping $\phi_n : \Delta_n \to X$ such that, for any $\lambda \in \Delta_n$, there exists $i \in J(\lambda)$ such that $R(\phi_n(\lambda), x_i)$ holds. Hence, by Corollary 3.1, there exists $x^* \in X$ such that $R(x^*, y)$ holds for any $y \in X$, that is, $(x^*, y) \notin B$ for any $y \in X$.

Note that [3, Theorem 2.4] is a particular form of Theorem 3.3 under the extra assumption that E is Hausdorff and X has the fixed point property.

4. Related applications

4.1. Variational inclusion problem

In Subsection 3.1 of [3], three types of variational inclusion problems for Hausdorff topological vector spaces and a compact subset X are considered. Their existences of solutions are proved by [3,Theorem 2.2]. Therefore, all of the results in that Subsection [3, Theorems 3.1-3.3 and Corollaries 3.1-3.3] can be improved by removing the redundant assumptions that topological vector spaces are Hausdorff and its compact subset X has the fixed point property.

4.2. Generalized Ky Fan Minimax Inequality with $\mathcal{C}$-quasiconcavity

Here we give a new usage of ϕ_A-spaces. In [8], Hou introduced the following $\mathcal{C}$-quasiconcavity:

Definition. Let X be a topological space, and $D, Y \subset X$. A real function $f : X \times Y \to \mathbb{R}$ is called $\mathcal{C}$-*quasiconcave* on D if, for any $N = \{x^0, x^1, \ldots, x^n\} \in \langle D \rangle$, there exists a continuous map $\phi_N : \Delta_n \to Y$ such that

$$\min\{f(x^i, \phi_N(\lambda)) \mid i \in J(\lambda)\} \leq f(\phi_N(\lambda), \phi_N(\lambda))$$

for all $\lambda := (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, where $J := \{i \mid \lambda_i \neq 0\}$.

Note that $(Y, D; \{\phi_N\}_{N \in \langle D \rangle})$ is a ϕ_A-space.

As in [3, Subsection 3.2], we will give an existence theorem of solution for the generalized Ky Fan minimax inequality. As a special case, the Ky Fan minimax inequality with $\mathcal{C}$-quasiconcavity is given.

Theorem 4.1. *Let X be a nonempty and compact subset of a topological vector space. Let $S_1 : X \multimap X$, $S_2 : X \multimap X$ be two maps with nonempty values, and $f : X \times X \to \mathbb{R}$ be a real-valued function. Assume that* (i)-(iii) *of Corollary 3.1 hold and*

(1) *for any fixed $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous;*

(2) *for any $x \in X$, $f(x, x) \leq 0$;*

(3) *for any finite subset $\{x_0, \ldots, x_n\}$ of X, there exists a continuous map $\phi_n : \Delta_n \to X$ such that, for any $\lambda \in \Delta_n$, $f(\phi_n(\lambda), \phi_n(\lambda)) \geq \min_{i \in J(\lambda)} f(\phi_n(\lambda), x_i)$; if $x_i \in S_2(\phi_n(\lambda))$ for any $i \in J(\lambda)$, then $\phi_n(\lambda) \in S_2(\phi_n(\lambda))$.*

Then there exists $x^* \in S_1(x^*)$ such that $f(x^*, y) \leq 0$ for any $y \in S_2(x^*)$.

Proof. Just follow that of [3, Theorem 3.4] and apply Corollary 3.1.

When the space is Hausdorff and X has the fixed point property, Theorem 4.1 reduces to [3, Theorem 3.4]. In the same way, we have the following improved version of [3, Corollary 3.4]:

Corollary 4.1. *Let X be a nonempty and compact subset of a topological vector space, and X have the fixed point property. A function $f : X \times X \to \mathbb{R}$ satisfies*

(i) *for any fixed $y \in X$, $x \mapsto f(x, y)$ is lower semicontinuous;*

(ii) *for any fixed $x \in X$, $y \mapsto f(x, y)$ is $\mathcal{C}$-quasiconcave on X;*

(iii) *for any $x \in X, f(x, x) \leq 0$.*

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for any $y \in X$.

4.3. Generalized vector Ky Fan minimax inequality with $\mathcal{C}$-P-quasiconcavity

As in [3, Subsection 3.3], we generalize the Ky Fan minimax inequality with $\mathcal{C}$-quasiconcavity, and obtain a vector Ky Fan minimax inequality with $\mathcal{C}$-P-quasiconcavity.

Definition ([3, Definition 3.2])**.** Let X be a topological space, Z be a topological vector space with nonempty, convex, closed, and pointed cone P with $\operatorname{Int} P \neq \emptyset$, and $D, Y \subset X$. A function $f : X \times Y \to Z$ is called $\mathcal{C}$-P-quasiconcave on D iff, for any finite subset $\{x_0, x_1, \ldots, x_n\}$ of D, there exists a continuous map $\phi_n : \Delta_n \to Y$ such that, for any $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$, there exists $i \in J(\lambda)$ such that

$$f(\phi_n(\lambda), \phi_n(\lambda)) \in f(\phi_n(\lambda), x_i) + P,$$

where $J(\lambda) := \{i \in \{0, 1, \ldots, n\} \mid \lambda_i > 0\}$.

In the above definition, note that $(Y, D; \{\phi_n\})$ is a ϕ_A-space.

Definition ([3, Definition 3.3])**.** A vector-valued function $f : X \to Z$ is called P-continuous at $x_0 \in X$ iff, for any open neighborhood V of the zero element in Z, there exists an open neighborhood U of x_0 in X such that, for any $x \in U$, $f(x) \in f(x_0) + V + P$, and f is called P-continuous on X if it is P-continuous at every point of X.

From these definitions and [3, Corollary 2.1], its authors obtain [3, Theorem 3.5], where the Hausdorffness of the topological vector space and the fixed point property of X are redundant. From [3, Theorem 3.5], we obtain the following:

Theorem 4.2. *Let X be a nonempty and compact subset of a topological vector space E, and Z be a topological vector space with nonempty, convex, closed, and pointed cone P with $\operatorname{Int} P \neq \emptyset$. A mapping $f : X \times X \to Z$ satisfies*
(i) *for any fixed $y \in X$, $x \mapsto f(x, y)$ is P-continuous;*
(ii) *for any fixed $x \in X$, $y \mapsto f(x, y)$ is $\mathcal{C}$-P-quasiconcave on X;*
(iii) *for any $x \in X$, $f(x, x) \notin \operatorname{Int} P$.*
Then there exists $x^* \in X$ such that $f(x^*, y) \notin \operatorname{Int} P$ for any $y \in X$.

Note that Theorem 4.2 was given as [3, Corollary 3.5] under extra assumptions that X has the fixed point property and the Hausdorffness of E and Z.

4.4. Multi-objective generalized game and n-person noncooperative generalized game

In Subsection 3.4 of [3], its authors considered the n-person multi-objective generalized game and showed the existence of a weakly Pareto-Nash equilibrium point of the game. The existences given as [3, Theorems 3.6 and 3.7] are easy consequences of [3, Corollary 3.5], which is improved as Theorem 4.2. Therefore [3, Theorems 3.6 and 3.7] hold without assuming the Hausdorffness of the strategy sets X_i and the fixed point property of their product X.

5. Conclusion

In a recent paper [3], its authors used methods to establish variational relation problems without the KKM property. Furthermore, by means of variational relation problems, they obtained new existence theorems of solutions for generalized KKM theorems, variational inclusion problems, the generalized (vector) Ky Fan minimax inequality, generalized Ky Fan section theorems, n-person noncooperative generalized games and n-person noncooperative multi-objective generalized games.

In an another paper [5] of the same authors, a KKM type theorem was proved and applied to some existence theorems of solutions for (vector) Ky Fan minimax inequality, Ky Fan section theorem, variational relation problems, n-person noncooperative game, and n-person noncooperative multiobjective game. In a previous paper, we showed that the KKM type theorem and all results in [5] are consequences of an already known result for ϕ_A-space and can be improved.

Basic techniques of [3, 5] assume the Hausdorffness and the fixed point property of relevant spaces. However, in the present paper, based on recent works on the KKM theory, we show that such assumptions are redundant in all of the results in [3].

Finally, recall that, in our previous survey [9], we compared the fixed point method and the KKM method in nonlinear analysis. Especially, we considered two methods in the proofs of the following important theorems in the chronological order: the von Neumann minimax theorem, the von Neumann intersection lemma, the Nash equilibrium theorem, the social equilibrium existence theorem of Debreu, the Gale-Nikaido-Debreu theorem, the Fan-Browder fixed point theorem, generalized Fan minimax inequality, and the Himmelberg fixed point theorem. Our implicit conclusion there is that the KKM method seems to be preferable.

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