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REMARKS ON MARGINALLY CLOSED-VALUED KKM MAPS AND RELATED MATTERS

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Abstract. In this review, we introduce equivalent formulations of the minimax theorem for marginally semicontinuous functions and the KKM theorem for marginally closed-valued multimaps due to Greco and Moschen [6]. We note that many results in that paper are equivalent to the Brouwer fixed point theorem. Moreover, we give more equivalent formulations as in our previous works [10,11] and remarks on some related matters.

1. Introduction

In 1998, Greco and Moschen [6] obtained a "finite-dimensional minimax inequality" which includes Brouwer fixed point theorem, Fan-Browder fixed point theorem, KKM theorem, Fan minimax inequality, Liu minimax theorem, Sion minimax theorem, and other minimax theorems due to Greco and his collaborators.

In our previous work [10] in 1999 on the history of the Brouwer fixed point theorem, we were concerned with equivalent formulations and generalizations of the theorem. There we could state around three dozens of such equivalent statements.

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. A partial KKM space is an abstract convex space satisfying the partial KKM principle. In our recent work [11], we clearly derived a sequence of a dozen statements which characterize the KKM spaces and the partial KKM spaces. As their applications, we added more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, Nash equilibrium theorem, and Fan type minimax inequalities for any partial KKM spaces. Consequently, [11] unifies and enlarges previously

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known several proper examples of such statements for particular types of partial KKM spaces.

Note that our previous works [10, 11] did not reflect equivalent formulations of the minimax inequality of [6] and that many of them are also equivalent to the Brouwer fixed point theorem.

In the present review, we introduce equivalent formulations of the minimax theorem for marginally semicontinuous functions and the KKM theorem for marginally closed-valued multimaps due to Greco and Moschen [6]. We note that many results in that paper are equivalent to the Brouwer fixed point theorem. Moreover, we give more equivalent formulations as in our previous works [10, 11] and add remarks on some related matters.

2. Minimax theorems for marginally semicontinuous functions

A *multimap* or simply a *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of *nonempty* subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$ and, for any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

A map $F : X \multimap Y$ for topological spaces X and Y is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

Recall that an extended real function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower semicontinuous* (l.s.c.) if $\{x \in X \mid f(x) > r\}$ is open for each $r \in \overline{\mathbb{R}}$.

For a convex set X, a function $f : X \to \overline{\mathbb{R}}$ is said to be *quasi-concave* if $\{x \in X \mid f(x) \geq r\}$ is convex for each $r \in \overline{\mathbb{R}}$.

Similarly, the upper semicontinuity (u.s.c.) and the quasi-convexity can be defined.

A function $f : X \times Y \to \overline{\mathbb{R}}$ for topological spaces X and Y is said to be *marginally u.s.c. on* X (resp., *marginally l.s.c. on* Y) [6], if for every open subset V of Y (resp., U of X)

$$x \mapsto \inf_{y \in V} f(x, y) \text{ is u.s.c. on } X \text{ (resp., } y \mapsto \sup_{x \in U} f(x, y) \text{ is l.s.c. on } Y).$$

Obviously, every u.s.c (resp., l.s.c.) function on X is marginally u.s.c (resp., marginally l.s.c.) on X. The example given in [2, p. 249] shows that the converse is not true.

In the sequel X, Y will always denote non-empty convex subsets of any topological vector spaces (t.v.s.) otherwise explicitly stated.

The following is the main theorem in [6] based on the finite dimensional Michael Selection Theorem and the Brouwer Fixed Point Theorem:

Theorem A (Minimax Theorem for Marginally Semicontinuous Functions)**.** *Let X, Y be finite-dimensional convex sets such that either X or Y is compact. Let $f,\ g : X \times Y \to \overline{\mathbb{R}}$ be functions with $f \le g$. If f is both marginally l.s.c. on Y and quasi-concave on X and if g is both marginally u.s.c. on X and quasi-convex on Y, then*

$$\inf_Y \sup_X f \le \sup_X \inf_Y g.$$

Greco and Moschen [6] deduced the following corollaries from Theorem A:

Corollary 1 (Minimax Theorem [2])**.** *The case $f = g$ in Theorem A.*

Corollary 2 (Fan Minimax Inequality [5])**.** *Let X be a compact convex set (not necessarily finite-dimensional). Let $f : X \times X \to \overline{\mathbb{R}}$ be a function which is quasi-concave in the first variable and l.s.c. in the second one. Then*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \le \sup_{x \in X} f(x, x).$$

Corollary 3 (Liu Minimax Inequality [8])**.** *Let $X,\ Y$ be convex sets (not necessarily finite-dimensional) such that either X or Y is compact. Let $f,\ g : X \times Y \to \overline{\mathbb{R}}$ be functions with $f \le g$. If f is both quasi-concave on X and l.s.c. on Y and if g is both u.s.c. on X and quasi-convex on Y, then*

$$\inf_Y \sup_X f(x, y) \le \sup_X \inf_Y g(x, y).$$

Moreover, from Theorem A, Greco and Moschen [6] derived four "equivalent" reformulation of Theorem A, in terms of intersection property, of coincidence property, of fixed point property, and of KKM property, resp.

Remark. Balaj [1, Theorem 10] obtained a variant of Theorem A in the class of G-convex spaces in the sense of Park.

3. Intersection property

The following is [6, Proposition 5]:

Proposition 1 (Intersection Property for Multimaps with Marginally Closed Values)**.** *Let X, Y be finite-dimensional convex sets such that either X or Y is compact. Let $\Delta : X \multimap Y$ be a multimap such that*

(1) *$\bigcap_{x \in U} \Delta x$ is closed for every open subset U of X (that is, Δ has marginally closed-values);*

(2) *$\Delta \tilde{x} \subset \Delta x_0 \cup \Delta x_1$ for every $x_0,\ x_1 \in X$ and $\tilde{x} \in [x_0, x_1]$.*

Then $\bigcap_{x \in X} \Delta x \neq \emptyset$ iff there is an l.s.c. map $\Omega : X \multimap Y$ with non-empty convex values such that $\Omega \subset \Delta$.

The following definitions are given in [12] with some examples:

Definition. A *γ-convex space* $(E, D; \gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\gamma : D \times D \multimap E$ with nonempty values $\gamma(a, b)$ for any $a, b \in D$.

For any $D' \subset D$, the *γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\gamma D' := \bigcup\{\gamma(a, b) \mid a, b \in D'\} \subset E.$$

A subset X of E is called a *γ-convex subset* of $(E, D; \gamma)$ relative to D' if for any $a, b \in D'$, we have $\gamma(a, b) \subset X$, that is, $\mathrm{co}_\gamma D' \subset X$.

In case $E \supset D$, let $(E \supset D; \gamma) := (E, D; \gamma)$ and let $(E; \gamma) := (E, E; \gamma)$.

Definition. Let $(E, D; \gamma)$ be a γ-convex space. If a map $G : D \multimap E$ satisfies

$$\gamma(a, b) \subset G(a) \cup G(b) \qquad \text{for any} \quad a, b \in D,$$

then G is called a 2-*KKM map*.

Definition. For a γ-convex space $(E, D; \gamma)$ and a given family $\mathcal{F}(E)$ of nonempty subsets of E, the 2-*KKM principle* with respect to $\mathcal{F}(E)$ is the statement that, for any 2-KKM map $G : D \to \mathcal{F}(E)$, the family $\{G(y)\}_{y \in D}$ has the nonempty intersection.

From Proposition 1, we have the following:

Exmaple. Under the situation of Proposition 1, define a multimap $\gamma : X \times X \multimap Y$ by $\gamma(x_0, x_1) := \Delta[x_0, x_1]$ for $x_0,\ x_1 \in X$. Then

(a) $(Y, X; \gamma)$ is a γ-convex space;

(b) Δ is a 2-KKM map; and

(c) $\bigcap_{x \in X} \Delta x \neq \emptyset$ iff there is an l.s.c. map $\Omega : X \multimap Y$ as in the conclusion of Proposition 1.

4. Coincidence property and fixed point property

The following are Propositions 6 and 7 in [6], resp.:

Proposition 2 (Coincidence Property for Lower Semicontinuous Multimaps)**.** *Let X, Y be finite-dimensional convex sets such that either X or Y is compact. If $\Omega : X \multimap Y$ and $\Gamma : Y \multimap X$ are l.s.c. multimaps with non-empty convex values (not necessarily closed !), then $\Omega \cap \Gamma^- \neq \emptyset$.*

Proposition 3 (Fixed Point Property for Lower Semicontinuous Multimaps)**.** *Let X be a finite-dimensional compact convex set. If $\Omega : X \multimap X$ is an l.s.c. multimap with non-empty convex values (not necessarily closed!), then there exists $x \in X$ such that $x \in \Omega x$.*

Proposition 2 yields the following due to Fan [4]:

Corollary 4 (Fan Coincidence Theorem [8])**.** *If either X or Y is compact, then $\Omega \cap \Gamma^- \neq \emptyset$ whenever $\Omega : X \multimap Y$ and $\Gamma : Y \multimap X$ have open fibers and non-empty convex values.*

Greco and Moschen [6] noted that Proposition 3 includes the Brouwer fixed point theorem and the following:

Corollary 5 (Fan–Browder Fixed Point Theorem). *There is $x \in X$ with $x \in \Omega x$, if X is compact and $\Omega : X \multimap X$ has open fibers and non-empty convex values.*

5. Marginally closed-valued KKM maps

The following is [6, Proposition 8]:

Proposition 4 (KKM Property for Multimaps with Marginally Closed Values). *Let X be a finite-dimensional compact convex set and $\Delta : X \multimap X$ be a KKM multimap with marginally closed values. Then $\bigcap_{x\in X} \Delta x \neq \emptyset$.*

Here, $\Delta : X \multimap X$ is called a *KKM map* if $\operatorname{co} A \subset \bigcup_{x\in A} \Delta x$ for every finite subset $A \subset X$.

The following is a variant of the original Knaster-Kuratowski-Mazurkiewitz (simply, KKM) theorem; see [3]:

Corollary 6 (A KKM type Theorem). *In Proposition 4, Δ has closed values.*

The following is known to be equivalent to Corollary 6; see Kim [7] and Shih–Tan [14]:

Corollary 7 (An open-valued KKM type Theorem). *In Proposition 4, if Δ has open values, $\{\Delta x\}_{x\in X}$ has the finite intersection property.*

6. Intersectionally closed-valued KKM maps

Let $\Gamma : X \multimap Y$ be a multimap between topological spaces X and Y. We say that the values of Γ is *topologically closed, marginally closed, or quasi-topologically closed*, resp., whenever

$$\bigcap_{x\in U} \Gamma x = \bigcap_{x\in U} \overline{\Gamma x}, \quad \bigcap_{x\in U} \Gamma x \text{ is closed, or } \overline{\bigcap_{x\in U} \Gamma x} = \bigcap_{x\in U} \overline{\Gamma x}, \text{ resp.,}$$

for any open set $U \subset X$; see [6].

Consider the following related four conditions for a map $G : D \multimap X$ in [9], where D is a non-empty set:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [9]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

If D is an open subset of a topological space, then topologically closedness or quasi-topologically closedness implies (c) or (b), resp.

In our recent paper [13], we obtained a new KKM type theorem for intersectionally closed-valued KKM maps and some useful new basic consequences. Typical examples of them are abstract forms of Fan's matching theorem, Fan's geometric lemma, the Fan–Browder fixed point theorem, maximal element theorems, Fan's minimax inequality, variational inequalities, and others.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle.

Definition. For an abstract convex space $(E, D; \Gamma)$, a subset X of E is said to be *intersectionally closed* (resp., *transfer closed*) if there is an intersectionally (resp., transfer) closed-valued map $G : D \multimap E$ such that $X = G(z)$ for some $z \in D$.

The following are sample results in [13] similar to Proposition 4 and Corollary 2, resp.:

Proposition 4′ (KKM Property for Multimaps with Intersectionally Closed Values)**.** *Let X be a compact convex set in any t.v.s. and $\Delta : X \multimap X$ be a KKM map with intersectionally closed values. Then $\bigcap_{x \in X} \Delta x \neq \emptyset$.*

Corollary 2′ (Minimax inequality)**.** *Let X be a compact convex subset of a t.v.s., $f, g : X \times X \to \overline{\mathbb{R}}$ be functions and $\gamma \in \mathbb{R}$ such that*

(1) *for any $x, y \in X$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq \gamma$;*

(2) *for each $x \in X$, $\{y \in X \mid f(x, y) \leq \gamma\}$ is intersectionally closed in X; and*

(3) *for each $y \in X$, $\{x \in X \mid g(x, y) > \gamma\}$ is convex in X.*

Then (i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \leq \gamma \quad \textit{for all} \;\; x \in X;$$

(ii) *if $\gamma := \sup_{x \in X} g(x, x)$, then*

$$\inf_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

Furthermore,

(iii) *if $y \mapsto \sup_{x \in X} f(x, y)$ is l.s.c. on X, then we have*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} g(x, x).$$

7. Equivalents of the Brouwer theorem and other results

It is well-known that the Brouwer fixed point theorem, the Sperner combinatorial lemma, the KKM theorem, the Fan minimax inequality, the Fan–Browder fixed point theorem, and many other theorems are mutually equivalent to each other; see [10].

In [6], the authors deduced the Brouwer theorem ⇒ Theorem A ⇒ Proposition 1 ⇒ Proposition 2 ⇒ Proposition 3 ⇒ Proposition 4 ⇒ Theorem A ⇒ Fan minimax inequality, and hence these are all equivalent to each other. However, there is no evidence for that Corollaries 1 and 3 are also equivalent to the Brouwer theorem.

Since Corollaries 6 and 7 imply that a finite-dimensional compact convex set X satisfies the KKM principle in the sense of [11], it satisfies 12 mutually equivalent properties; namely, matching property, finite intersection properties, geometric properties, Fan–Browder fixed point property, existence of maximal elements, analytic formulation, minimax inequalities, analytic alternatives as in (0)–(XI) [11, Theorem 1]. Moreover, X satisfies various forms of minimax inequalities and variational inequalities as in (XII)–(XVII) [11, Theorem 5]. Furthermore, for a pair or a finite family of finite-dimensional compact convex sets, we have a sequence of applications; namely, basic minimax theorems, von Neumann–Sion type minimax theorems, collectively fixed point theorem, von Neumann–Fan intersection theorem, Fan type analytic alternative, generalized Nash–Fan type equilibrium theorems as in (XVIII) – (XXVI) [11, Theorem 6].

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