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## ABSTRACT CONVEX SPACES, KKM SPACES AND $\phi_A$ -SPACES

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ABSTRACT. In our previous works, we showed that every  $\phi_A$ -space  $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$  can be made into a G-convex space in several ways. In this work, we show that a  $\phi_A$ -space can be made into a G-convex space  $(X, D; \Gamma)$  iff it has a KKM map  $G : D \multimap X$ , and that it is a KKM space. Moreover, we show that recent examples of GFC-spaces due to Khanh et al. and Ding are not adequate to claim that GFC-spaces or FC-spaces properly include G-convex spaces.

### 1. Introduction

Since we introduced the KKM theory as an independent branch of Nonlinear Analysis in 1992 [13], there have appeared more than twelve hundred publications related to the theory. Many of them are concerned with simple modifications or imitations of known results by adopting certain artificial concepts. Moreover, recently the KKM theory tends to the study of abstract convex spaces properly including generalized convex spaces (simply, G-convex spaces) due to the author.

The concept of G-convex spaces has also a number of modifications or imitations. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [1], FC-spaces due to Ding [3], GFC-spaces due to Khanh et al. [9], and some others. Since then, there have appeared many papers whose contents are slightly modified versions of known results in the G-convex space theory. In order to destroy such inadequate examples and to upgrade the KKM theory, the present author has published a sequence of papers [14-33] criticizing or improving such trends.

In some of our previous works, we showed that most of the above mentioned spaces are unified to the concept of  $\phi_A$ -spaces  $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$  which can be made into G-convex spaces in several ways [14-19, 22].

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In this paper, we show that a  $\phi_A$ -space can be made into a  $G$ -convex space  $(X, D; \Gamma)$  iff it has a KKM map  $G : D \multimap X$ , and that it is a KKM space. Moreover, we show that recent examples of GFC-spaces due to Khanh et al. [10] and FC-spaces of Ding [3,5] are not adequate to verify their claims that GFC-spaces or FC-spaces properly include  $G$ -convex spaces.

In Section 2, the basic concepts on abstract convex spaces in the sense of Park are introduced. We recall some subclasses of abstract convex spaces; namely, convexity spaces, convex spaces,  $H$ -spaces, and generalized ( $G$ -) convex spaces. Section 3 deals with  $\phi_A$ -spaces and their examples. We show that a  $\phi_A$ -space can be made into a  $G$ -convex space  $(X, D; \Gamma)$  iff it has a KKM map  $G : D \multimap X$ , and that it is a KKM space. We also discuss on recent examples of GFC-spaces due to Khanh et al. [10]. In Section 4, we deal with Ding's recent examples of FC-spaces which are not  $L$ -spaces in the sense of Ben-El-Mechaiekh et al. Several comments on the examples are added.

## 2. Abstract convex spaces

We follow our recent works [20, 21, 23, 28-32]:

**Definition.** An *abstract convex space*  $(E, D; \Gamma)$  consists of a topological space  $E$ , a nonempty set  $D$ , and a multimap  $\Gamma : \langle D \rangle \multimap E$  with nonempty values  $\Gamma_A := \Gamma(A)$  for  $A \in \langle D \rangle$ , where  $\langle D \rangle$  is the set of all nonempty finite subsets of  $D$ .

For any  $D' \subset D$ , the  $\Gamma$ -convex hull of  $D'$  is denoted and defined by

$$\text{co}_\Gamma D' := \bigcup \{ \Gamma_A \mid A \in \langle D' \rangle \} \subset E.$$

A subset  $X$  of  $E$  is called a  $\Gamma$ -convex subset of  $(E, D; \Gamma)$  relative to  $D'$  if for any  $N \in \langle D' \rangle$ , we have  $\Gamma_N \subset X$ ; that is,  $\text{co}_\Gamma D' \subset X$ .

When  $D \subset E$ , a subset  $X$  of  $E$  is said to be  $\Gamma$ -convex if  $\text{co}_\Gamma(X \cap D) \subset X$ ; in other words,  $X$  is  $\Gamma$ -convex relative to  $D' := X \cap D$ . In case  $E = D$ , let  $(E; \Gamma) := (E, E; \Gamma)$ .

**Definition.** Let  $(E, D; \Gamma)$  be an abstract convex space and  $Z$  a topological space. For a multimap  $F : E \multimap Z$  with nonempty values, if a multimap  $G : D \multimap Z$  satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle,$$

then  $G$  is called a *KKM map* with respect to  $F$ . A *KKM map*  $G : D \multimap E$  is a KKM map with respect to the identity map  $1_E$ .

A multimap  $F : E \multimap Z$  is called a  $\mathfrak{K}\mathfrak{C}$ -map [resp., a  $\mathfrak{K}\mathfrak{D}$ -map] if, for any closed-valued [resp., open-valued] KKM map  $G : D \multimap Z$  with respect to  $F$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. In this case, we denote  $F \in \mathfrak{K}\mathfrak{C}(E, D, Z)$  [resp.,  $F \in \mathfrak{K}\mathfrak{D}(E, D, Z)$ ].

**Exmample.** The following are typical examples of abstract convex spaces. Others can be seen in [29, 32] and the references therein.

(1) A *convexity space*  $(E, \mathcal{C})$  in the classical sense consists of a topological space  $E$  and a family  $\mathcal{C}$  of subsets of  $E$  such that  $E$  itself is an element of  $\mathcal{C}$  and  $\mathcal{C}$  is closed under arbitrary intersection.

(2) A *convex space*  $(X, D) = (X, D; \Gamma)$  is a triple where  $X$  is a subset of a vector space,  $D \subset X$  such that  $\text{co } D \subset X$ , and each  $\Gamma_A$  is the convex hull of  $A \in \langle D \rangle$  equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for  $X = D$ .

(3) An abstract convex space  $(X, D; \Gamma)$  is called an *H-space* by Park if  $\Gamma = \{\Gamma_A\}$  is a family of contractible (or, more generally,  $\omega$ -connected) subsets of  $X$  indexed by  $A \in \langle D \rangle$  such that  $\Gamma_A \subset \Gamma_B$  whenever  $A \subset B \in \langle D \rangle$ . If  $D = X$ ,  $(X; \Gamma)$  is called a *c-space* by Horvath.

(4) A *generalized convex space* or a *G-convex space*  $(X, D; \Gamma)$  is an abstract convex space such that for each  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ , there exists a continuous function  $\phi_A : \Delta_n \rightarrow \Gamma(A)$  such that  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) \subset \Gamma(J)$ .

Here,  $\Delta_n$  is the standard  $n$ -simplex with vertices  $\{e_i\}_{i=0}^n$ , and  $\Delta_J$  the face of  $\Delta_n$  corresponding to  $J \in \langle A \rangle$ ; that is, if  $A = \{a_0, a_1, \dots, a_n\}$  and  $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$ , then  $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$ . We may write  $\Gamma_A := \Gamma(A)$  for each  $A \in \langle D \rangle$ . In case  $X \supset D$ , the *G-convex space* is denoted by  $(X \supset D; \Gamma)$ .

**Definition.** The *partial KKM principle* for an abstract convex space  $(E, D; \Gamma)$  is the statement  $1_E \in \mathfrak{K}\mathfrak{C}(E, D, E)$ ; that is, for any closed-valued KKM map  $G : D \multimap E$ , the family  $\{G(y)\}_{y \in D}$  has the finite intersection property. The *KKM principle* is the statement  $1_E \in \mathfrak{K}\mathfrak{C}(E, D, E) \cap \mathfrak{K}\mathfrak{O}(E, D, E)$ ; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

**Exmample.** We give known examples of KKM spaces as in [28, 29], where the references can be seen:

- (1) Every G-convex space is a KKM space.
- (2) A connected linearly ordered space  $(X, \leq)$  can be made into a KKM space.
- (3) The extended long line  $L^*$  is a KKM space  $(L^*, D; \Gamma)$  with the ordinal space  $D := [0, \Omega]$ . But  $L^*$  is not a G-convex space.
- (4) For a closed convex subset  $X$  of a complete  $\mathbb{R}$ -tree  $H$ , and  $\Gamma_A := \text{conv}_H(A)$  for each  $A \in \langle X \rangle$ , the triple  $(H \supset X; \Gamma)$  is a KKM space.
- (5) For Horvath's convex space  $(X; \Gamma)$  with the weak Van de Vel property is a KKM space, where  $\Gamma_A := [[A]]$  for each  $A \in \langle X \rangle$ .

(6) A  $\mathbb{B}$ -space due to Bricc and Horvath is a KKM space.

**Exmaple.** Recently, Kulpa and Szymanski [11] obtained examples of partial KKM spaces which are not KKM spaces.

Now we have the following diagram for triples  $(E, D; \Gamma)$ :

$$\begin{aligned} \text{Simplex} &\implies \text{Convex subset of a t.v.s.} \implies \text{Convex space} \implies \text{H-space} \\ &\implies \text{G-convex space} \implies \phi_A\text{-space} \implies \text{KKM space} \\ &\implies \text{Partial KKM space} \implies \text{Abstract convex space} \end{aligned}$$

This diagram with “G-convex space  $\iff \phi_A$ -space” has appeared in a number of our previous works. In the next section, we will explain why we have to replace it by “G-convex space  $\implies \phi_A$ -space”.

### 3. $\phi_A$ -spaces

In 2007, we introduced a variant of G-convex spaces as follows [14-18, 22]:

**Definition.** A space having a family  $\{\phi_A\}_{A \in \langle D \rangle}$  or simply a  $\phi_A$ -space

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space  $X$ , a nonempty set  $D$ , and a family of continuous functions  $\phi_A : \Delta_n \rightarrow X$  (that is, singular  $n$ -simplexes) for  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ .

We define a KKM map  $G : D \multimap X$  on a  $\phi_A$ -space  $(X, D; \{\phi_A\})$  if, for each  $N \in \langle D \rangle$  and  $J \subset N$  with  $|J| = k + 1$ , we have

$$\phi_N(\Delta_J) \subset G(J)$$

where  $\Delta_J$  is the face of  $\Delta_{|N|-1}$  corresponding to  $J$ ; see Park [14] and Khanh et al. [9].

**Exmaple.** We give examples of  $\phi_A$ -spaces as follows:

- (1) An L-space [1], which is a G-convex space  $(E; \Gamma)$ .
- (2) An MC-space [20], which is known to be a G-convex space.
- (3) A topological space  $Y$  with property (H) [8], that is, if, for each  $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ , there exists a continuous mapping  $\varphi_N : \Delta_n \rightarrow Y$ .
- (4) Ding’s FC-spaces; see the next section.
- (5) Particular forms of G-convex spaces in [2, 7].
- (6) Any G-convex space is a  $\phi_A$ -space. Conversely, a  $\phi_A$ -space  $(X, D; \{\phi_A\})$  can be made into a G-convex space  $(X, D; \Gamma)$  in several ways [14-19, 22].
- (7) Later  $\phi_A$ -spaces are called GFC-spaces by Khanh et al. [9] in 2009. Note that they studied on such spaces in several papers and their study can be regarded as a part of G-convex space theory; see [10].

The following is known [14-19, 22]:

**Proposition 3.1.** *A  $\phi_A$ -space  $(X, D; \{\phi_A\})$  can be made into a G-convex space  $(X, D; \Gamma)$  with  $\Gamma_A = \Gamma(A) \supset \phi_A(\Delta_n)$  for each  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ .*

Moreover, we have the following:

**Proposition 3.2.** *A  $\phi_A$ -space  $(X, D; \{\phi_A\})$  can be made into a G-convex space  $(X, D; \Gamma)$  iff it has a KKM map  $G : D \multimap X$ .*

**Proof.** Suppose that  $(X, D; \{\phi_A\})$  has a KKM map  $G : D \multimap X$ . Then we have

$$\phi_A(\Delta_J) \subset G(J) \text{ for each } A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Define a map  $\Gamma : \langle D \rangle \multimap X$  by  $\Gamma(J) := G(J) = \bigcup \{G(a) \mid a \in J\}$  for each  $A \in \langle D \rangle$  and each  $J \in \langle A \rangle$ . Then we have

$$\phi_A(\Delta_J) \subset \Gamma(J) \text{ for each } A \in \langle D \rangle \text{ and } J \in \langle A \rangle.$$

Therefore  $(X, D; \Gamma)$  is a G-convex space.

Conversely, suppose that  $(X, D; \{\phi_A\})$  is a G-convex space  $(X, D; \Gamma)$ . By putting  $G(z) := \Gamma(\{z\})$  for  $z \in D$ , we have a KKM map  $G : D \multimap X$  as above. In fact, for each  $A$  with  $|A| = n + 1$ , we have a continuous function  $\phi_A : \Delta_n \rightarrow G(A) =: \Gamma(A)$  such that  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) \subset G(J) =: \Gamma(J)$ . Hence  $\phi_A(\Delta_J) \subset \Gamma(J) \subset G(J)$  for each  $A \in \langle D \rangle$  and  $J \in \langle A \rangle$ . Therefore  $G : D \multimap X$  is a KKM map on  $(X, D; \{\phi_A\})$ .

In [10], its authors gave an example to show that their GFC-spaces properly generalize our G-convex spaces and stated that: “A G-convex space  $(X, A, \Gamma)$  is called trivial iff, for all  $N \in \langle A \rangle$ ,  $\Gamma(N) = X$ . Of course, any GFC-space can be made into a trivial G-convex space, but a trivial G-convex space has no use.”

Similarly, let us call that a KKM map  $G : D \multimap X$  on a  $\phi_A$ -space  $(X, D; \{\phi_A\})$  is said to be ‘trivial’ if  $G(a) = X$  for all  $a \in D$ .

From the proof of Proposition 3.2, we immediately have the following:

**Proposition 3.3.** *A  $\phi_A$ -space  $(X, D; \{\phi_A\})$  can be made into a trivial G-convex space  $(X, D; \Gamma)$  iff it has a trivial KKM map  $G : D \multimap X$ .*

As expressed in [10], of course, any GFC-space without nontrivial KKM map has no use in the KKM theory. Recall that the authors of [10] gave an example of GFC-space which is a trivial G-convex space and, hence, has a trivial KKM map. Consequently, in order to show that GFC-spaces properly generalize G-convex spaces, the authors of [10] should give an example of a GFC-space without trivial KKM maps which is not G-convex. This is impossible by Proposition 3.3.

In order to apply our theory of abstract convex spaces to  $\phi_A$ -spaces, we propose the following:

**Proposition 3.4.** *A  $\phi_A$ -space  $(X, D; \{\phi_A\})$  becomes an abstract convex space  $(X, D; \Gamma)$  with  $\Gamma_A = \Gamma(A) = \phi_A(\Delta_n)$  for  $A \in \langle D \rangle$  with the cardinality  $|A| = n + 1$ .*

In this case, we denote the  $\phi_A$ -space  $(X, D; \{\phi_A\})$  by  $(X, D; \Gamma)$ . This space is not necessarily G-convex; see the next section.

Recall that, for an abstract convex space  $(X, D; \Gamma)$ , a KKM map  $G : D \multimap X$  is the one satisfying

$$\Gamma_N \subset G(N) \text{ for each } N \in \langle D \rangle.$$

We have the following:

**Proposition 3.5.** *Any  $\phi_A$ -space  $(X, D; \{\phi_A\})$  is a KKM space.*

**Proof 1.** As in Proposition 3.1, the space can be made into a G-convex space  $(X, D; \Gamma)$  such that  $\phi_N(\Delta_J) \subset \Gamma_J$  for all  $J \subset N \in \langle D \rangle$  as above. It is well-known that any G-convex space is a KKM space.

**Proof 2.** As in Proposition 3.4, let  $G : D \multimap X$  be a closed-valued [resp., open-valued] KKM map, and let  $N \in \langle D \rangle$ . Define a new abstract convex space  $(X, N; \Gamma')$  by

$$\Gamma'_J = \Gamma'(J) := \phi_N(\Delta_J) \text{ for all } J \subset N.$$

Then  $\Gamma' : \langle N \rangle \multimap X$  is well-defined. For any  $A \in \langle N \rangle$ , define  $\phi_A := \phi_N|_{\Delta_A} : \Delta_A \rightarrow \Gamma'(A)$ . Hence  $J \in \langle A \rangle$  implies  $\phi_A(\Delta_J) = \phi_N(\Delta_J) = \Gamma'(J)$ . Therefore  $(X, N; \Gamma')$  is a G-convex space and hence a KKM space. Moreover,  $G|_N : N \multimap X$  is a KKM map. In fact, for all  $J \subset N$ , we have

$$\Gamma'(J) = \phi_N(\Delta_J) \subset G(J)$$

since  $G : D \multimap X$  is a KKM map. Therefore  $\bigcap_{y \in N} G(y) \neq \emptyset$  and hence  $\{G(y)\}_{y \in D}$  has the finite intersection property. Consequently, the  $\phi_A$ -space  $(X, D; \Gamma)$  is a KKM space.

In [29], we clearly derived a sequence of a dozen statements which characterize the KKM spaces or are equivalent formulations of the partial KKM principle. As their applications, we added more than a dozen statements including generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, [29] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces. Therefore, the whole contents of [29] is applicable to any  $\phi_A$ -space  $(X, D; \{\phi_A\})$ .

#### 4. Ding's examples of FC-spaces

In 2005, Ding [3] introduced the following notion of “a finitely continuous” topological space (in short, FC-space):

**Definition** ([3]).  $(Y, \{\varphi_N\})$  is said to be an FC-space if  $Y$  is a topological space and for each  $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$  where some elements in  $N$  may be same, there exists a continuous mapping  $\varphi_N : \Delta_n \rightarrow Y$ . A subset  $D$  of  $(Y, \{\varphi_N\})$  is said to be an FC-subspace of  $Y$  if for each  $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$  and for each  $\{y_{i_0}, \dots, y_{i_k}\} \subset N \cap D$ ,  $\varphi_N(\Delta_k) \subset D$  where  $\Delta_k = \text{co}\{e_{i_j} : j = 0, \dots, k\}$ .

Then Ding [3] wrote that “it is clear that the class of G-convex spaces is a true subclass of FC-spaces,” without any justification. He repeated the same claim in more than a dozen of his subsequent papers; see [22].

Actually, in 2007-2009 [14-19, 22], we showed that FC-spaces due to Ding are particular types of L-spaces due to Ben-El-Mechaiekh et al., and hence particular types of G-convex spaces. Some counter-examples are given and related matters are also discussed in [22].

Recently, in 2011, Ding [5] continued to insist his claim in [4] as follows: “By comparing the definitions of H-spaces, G-convex spaces, L-convex spaces and FC-spaces, it is easy to see that each H-spaces must be a G-convex space, each G-convex space must be a L-convex space, and each L-convex space must be an FC-space. But some examples of FC-spaces which is not an L-convex space have given by Ding [4] and Ding et al. [6] as follows.” After giving two examples, he added as follows:

“Hence some critiques on L-spaces and FC-spaces given by Park [22] are not fair. I believe that the readers will give the fairest judgement.”

This statement is incorrect by the following arguments.

**Exmaple 2.1** ([3, 5]). Let  $X_1$  and  $X_2$  be two nonempty convex subsets of a topological vector space  $X$  with  $\text{cl } X_1 \cap \text{cl } X_2 = \emptyset$  and  $g : X_2 \rightarrow X_1$  be a single-valued mapping. Then  $E = X_1 \cup X_2$  is not convex. For each  $N = \{x_0, \dots, x_n\} \in \langle E \rangle$ , define a mapping  $\varphi_N : \Delta_n \rightarrow 2^X$  by

$$\varphi_N(\alpha) = \begin{cases} \sum_{i=0}^n \alpha_i x_i & \text{if } N \subset X_1 \text{ or } N \subset X_2; \\ \sum_{i=0}^j \alpha_i x_i + \sum_{i=j+1}^n \alpha_i g(x_i) & \text{if } N = N_1 \cup N_2, \end{cases}$$

for all  $\alpha = (\alpha_0, \dots, \alpha_n) \in \Delta_n$  where  $N_1 = \{x_0, \dots, x_j\} \subset X_1$ ,  $N_2 = \{x_{j+1}, \dots, x_n\} \subset X_2$ . It is easy to see that  $\varphi_N$  is continuous and hence  $(E, \varphi_N)$  is an FC-space.  $\dots$

If we define a set-valued mapping  $\Gamma : \langle E \rangle \rightarrow 2^E$  by

$$\Gamma(N) = \varphi_N(\Delta_n), \quad \forall N = \{x_0, \dots, x_n\} \in \langle E \rangle,$$

then we have that for each  $N = \{x_0, \dots, x_n\} \in \langle E \rangle$ ,  $\varphi_N(\Delta_n) \subset \Gamma(N)$ . But if  $N = N_1 \cup N_2$  where  $N_1 = \{x_0, \dots, x_j\} \subset X_1$  and  $N_2 = \{x_{j+1}, \dots, x_n\} \subset X_2$ , then we have  $\Gamma(N_2) = \varphi_{N_2}(\Delta_J) \subset X_2$  and  $\varphi_N(\Delta_J) \subset X_1$ , where  $\Delta_J = \text{co}\{e_k \mid k = j+1, \dots, n\}$ . Hence we have  $\varphi_N(\Delta_J) \not\subset \Gamma(N_2)$ . Hence  $(E, \Gamma)$  is not an L-convex space.

*Comments.* In this example, Ding showed that  $(E, \{\varphi_N\})$  is an FC-space, and  $(E, \Gamma)$  with the particular  $\Gamma(N) = \varphi_N(\Delta_n)$  is not an L-space. Yes, this particular  $\Gamma$  does not work.

However, we can make  $(E, \{\varphi_N\})$  into a G-convex space, and we give an example of Proposition 3.1. In fact, if we define a multimap  $\Gamma : \langle E \rangle \rightarrow 2^E$  by

$$\Gamma(N) = \begin{cases} \varphi_N(\Delta_n) \cup X_1 & \text{if } N \subset X_1 \text{ or } N \subset X_2; \\ \varphi_N(\Delta_n) & \text{if } N = N_1 \cup N_2, \end{cases}$$

where  $N_1 = \{x_0, \dots, x_j\} \subset X_1$ ,  $N_2 = \{x_{j+1}, \dots, x_n\} \subset X_2$ .

Then it is easily checked that  $\varphi_N(\Delta_n) \subset \Gamma(N)$  and  $\varphi_N(\Delta_J) \subset \Gamma(J)$  for any  $J \subset N$ . Therefore  $(E, \Gamma)$  becomes a G-convex space.

**Exmaple 2.2** ([3, 5]). Let  $(X, \|\cdot\|)$  be a strictly convex and reflexive Banach space and  $X_1$  is a nonempty closed convex subset of  $X$  and  $X_2$  be a nonempty convex subset of  $X$  with  $X_1 \cap X_2 = \emptyset$ . Then  $E = X_1 \cup X_2$  is not convex. For each  $N = \{x_0, \dots, x_n\} \in \langle E \rangle$ , define a mapping  $\varphi_N : \Delta_n \rightarrow 2^X$  as in Example 2.1 where  $g : X_2 \rightarrow X_1$  is replaced by the metric project mapping  $P_{X_1} : X_2 \rightarrow X_1$ . Then  $(E, \varphi_N)$  is an FC-space which is not an L-convex space.

*Comments.* Similarly to the preceding comments, Ding's claim is incorrect. We add further comments on Ding's FC-spaces as follows:

(1) Recall that the G-convex space  $(X, D; \Gamma)$  is originated from the original KKM theorem and the celebrated Ky Fan lemma from the beginning, where  $X \neq D$ . The case  $X = D$  is not applicable to them and this is the most serious defect of L-spaces or FC-spaces since they are inadequate for the KKM theory.

(2) Note that G-convex spaces are triples, but FC-spaces are pairs. One wonders how could a pair generalize a triple!

(3) According to the definition of FC-spaces, for each  $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$  where some elements in  $N$  may be same, there should be an infinite number of continuous mappings  $\varphi_N : \Delta_n \rightarrow Y$ .

(4) By defining  $\Gamma(N) = E$  for all  $N \in \langle E \rangle$ , Ding's FC-space becomes trivially an L-space, and hence a G-convex space.

(5) In Ding's examples, he has to show that any  $\Gamma$  does not work, not a particular one. Moreover, his  $\Gamma$  is not well chosen. There exists a suitable  $\Gamma$  such that his claim is false; see Proposition 3.2.

(6) Ding's careless citing of L-convex spaces instead of L-spaces in [1] caused confusions to many of his followers; for example, see [12, 34].

(7) For some other comments on Ding's examples, see [33].

However, Ding's examples can be used to strengthen Proposition 3.4 so that the abstract convex space is not necessarily G-convex.

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