INTERNATIONAL PUBLICATIONS (USA)

Communications on Applied Nonlinear Analysis
Volume 19(2012), Number 1, 59–69

Remarks on Simplicial Spaces and L^*-spaces of Kulpa and Szymanski

Sehie Park
The National Academy of Sciences
Republic of Korea, Seoul 137–044, and
Seoul National University
Department of Mathematical Sciences
Seoul 151–747, Korea
shpark@math.snu.ac.kr
parkcha38@daum.net

Communicated by the Editors
(Received December 15, 2011; Accepted January 5, 2012)

Abstract

In a recent paper, Kulpa and Szymanski [15] introduced a series of theorems called Infimum Principles in simplicial spaces. As applications, they derive fixed point theorems due to Schauder, Tychonoff, Kakutani, and Fan-Browder; minimax theorems; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Ky Fan minimax inequality. Their study is based on and utilizes the techniques of simplicial structure and the Fan-Browder map. In this paper, we recall that for any abstract convex spaces satisfying abstract KKM principle, we can deduce such classical theorems without using any Infimum Principles. Moreover, we note that the newly defined L^*-spaces in [15] are particular types of abstract convex spaces satisfying the abstract KKM principle, and add some remarks on them.

AMS Subject Classification: 47H10, 49J53, 52A07, 54H25, 90A56, 90D06, 91A13.

Key words: Abstract convex space, KKM principle, Matching property, Fixed point theorems (of Schauder, Tychonoff, Kakutani, and Fan-Browder), Minimax theorem, Nash equilibrium theorem, Gale-Nikaido-Debreu theorem, Fan minimax inequality.

1 Introduction

The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In our recent works [24-26], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are closely related to KKM spaces or spaces satisfying the partial KKM principle. Moreover, a number of such results are equivalent to each other and can be applied to various problems in many fields.

In a recent paper, Kulpa and Szymanski [15] introduced a series of theorems called *Infimum Principles* and applied them to some classical results. Note that most of results in [15] originate from the *Theorem of Indexed Families* (Theorem 3) and that this is an easy consequence of a Fan type matching theorem ([17], Theorem 1). Their study is based on and utilizes the techniques of simplicial structures and CO families (equivalently, multimaps with nonempty convex values and open fibers). Here a simplicial space is a topological space having a certain collection of singular simplexes. As applications, they derived fixed point theorems due to Schauder, Tychonoff, Kakutani, and Fan-Browder; minimax theorem; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Fan minimax inequality. Finally, the authors of [15] suggested a way of extending their results to a wider class of topological spaces called L^*-*spaces.*

In the present paper, we recall that for any abstract convex spaces satisfying abstract KKM principle, we can deduce such classical theorems without using any Infimum Principles. In fact, such theorems are consequences of some equivalents of the KKM theorem on a simplicial space and hence are typical particular results of the KKM theory for abstract convex spaces. Moreover, we show that simplicial spaces and L^*-spaces are of particular type of KKM spaces due to ourselves and that some of main results in [15] are consequences of corresponding ones in KKM spaces.

In Section 2, we introduce the concepts and some examples of abstract convex spaces and KKM spaces. Section 3 concerns with some equivalents of the KKM principle and, as an application, a generalization of basic theorems in [15]. In Section 4, we introduce the main contents of [15] for simplicial spaces and give some remarks on them. Finally, Section 5 deals with L^*-spaces. We note that they are of particular type of abstract convex spaces satisfying the partial KKM principle and add some remarks.

2 Abstract convex spaces and KKM spaces

Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [24-26]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. The following are known examples of abstract convex spaces:

(1) The original KKM theorem [13] is for the triple $(\Delta_n, V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \to 2^{\Delta_n}$ the convex hull operation.

(2) A triple $(X, D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [5] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

(3) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [16] for $X = D$. However he obtained several KKM type theorems w.r.t. $(X, D; \Gamma)$.

(4) A triple $(X, D; \Gamma)$ is called an *H-space* if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a c-space by Horvath [8,9].

(5) Hyperconvex metric spaces due to Aronszajn and Panitchpakdi are particular cases of c-spaces; see [9].

(6) Hyperbolic spaces due to Reich and Shafrir [31] are particular cases of c-spaces. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic.

(7) Any topological semilattice $(X, \le)$ with path-connected interval, introduced by Horvath and Llinares [11]. See also [18].

(8) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For $X = D$, G-convex spaces reduce to L-spaces due to Ben-El-Mechaiekh et al. [1]. Recall that all examples (1)-(7) are G-convex spaces. For details, see references of [20-22].

(9) A ϕ_A-*space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [27].

(10) Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [12]. Then $(H, X; \Gamma)$ is an abstract convex space.

(11) According to Horvath [10], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

(12) A $\mathbb{B}$-space due to Briec and Horvath [3] is an abstract convex space.

Note that each of the above has a large number of concrete examples.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \to 2^E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [24-26], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Example. We give known examples of KKM spaces:

(1) Every G-convex space is a KKM space [22].

(2) A connected linearly ordered space $(X, \le)$ can be made into a KKM space [23].

(3) The extended long line L^* is a KKM space $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space; see [23,25].

(4) For a closed convex subset X of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each $A \in \langle X \rangle$, the triple $(H, X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [12]. Later we found that $(H \supset X; \Gamma)$ is a KKM space [28].

(5) For Horvath's convex space $(X; \Gamma)$ with the weak Van de Vel property is a KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$; see [10].

(6) A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [3].

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

It is not known yet whether there is a space satisfying the partial KKM principle that is not a KKM space.

3 Some equivalents of the KKM principle

For an abstract convex space $(E, D; \Gamma)$, let us consider the following statements:

(A) The KKM principle. *For any closed-valued [resp., open-valued] KKM map* $G : D \to 2^E$*, the family* $\{G(z)\}_{z \in D}$ *has the finite intersection property.*

(B) The Fan matching property. *Let* $S : D \to 2^E$ *be a map satisfying*
(B.1) $S(z)$ *is open [resp., closed] for each* $z \in D$*; and*
(B.2) $E = \bigcup_{z \in M} S(z)$ *for some* $M \in \langle D \rangle$.
Then there exists an $N \in \langle M \rangle$ *such that*

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

(C) The Fan-Browder fixed point property. *Let* $S : E \to 2^D$, $T : E \to 2^E$ *be maps satisfying*
(C.1) $S^-(z) := \{x \in E \mid z \in S(x)\}$ *is open [resp., closed] for each* $z \in D$*;*
(C.2) *for each* $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*
(C.3) $E = \bigcup_{z \in M} S^-(z)$ *for some* $M \in \langle D \rangle$.
Then T *has a fixed point* $x_0 \in X$*; that is,* $x_0 \in T(x_0)$.

Remark. Note that (A) is the definition of the KKM space. For the origins of (A)-(C), see [26] and references therein.

Theorem 1. (Characterizations of the KKM spaces) *For an abstract convex space* $(E, D; \Gamma)$*, the statements* (A), (B), *and* (C) *are equivalent.*

Proof. (A) $\Longrightarrow$ (B). Let $G : D \to 2^E$ be a map given by $G(z) := E \setminus S(z)$ for $z \in D$. Then G has closed [resp., open] values. Suppose, on the contrary to the conclusion, that for any $N \in \langle M \rangle$, we have $\Gamma_N \cap \bigcap_{z \in N} S(z) = \emptyset$; that is, $\Gamma_N \subset E \setminus \bigcap_{z \in N} S(z) = \bigcup_{z \in N}(E \setminus S(z)) = G(N)$. Then $G|_M : M \to 2^E$ is a KKM map. Since $(E, M; \Gamma)$ satisfies the KKM principle (A), there exists a $\widehat{y} \in \bigcap_{z \in N} G(z) = \bigcap_{z \in N}(E \setminus S(z))$. Hence $\widehat{y} \notin S(z)$ for all $z \in N$. Since N is arbitrary, this violates condition (B.2).

(B) $\Longrightarrow$ (C). Considering S^- in (B) instead of S, by (C.1) and (C.2), there exist $N \in \langle M \rangle$ and $x_0 \in \Gamma_N \cap \bigcap_{z \in N} S^-(z) \neq \emptyset$. Since $x_0 \in S^-(z) \Leftrightarrow z \in S(x_0)$ for all $z \in N$, we have $N \subset S(x_0)$. Therefore, by (C.2), we have $\Gamma_N \subset T(x_0)$. Since $x_0 \in \Gamma_N$, we have $x_0 \in T(x_0)$.

(C) $\Longrightarrow$ (A). Let $G : D \to 2^E$ be a KKM map with closed [resp., open] values. Suppose the family $\{G(z)\}_{z \in D}$ does not have the finite intersection property; that is, there exists an $M \in \langle D \rangle$ such that $\bigcap_{z \in M} G(z) = \emptyset$. Define a map $S : E \to 2^D$ by $S^-(z) := E \setminus G(z)$ for $z \in D$ and a map $T : E \to 2^E$ by $T(x) := \mathrm{co}_\Gamma S(x)$ for each $x \in E$. Note that $E = \bigcup_{z \in M}(E \setminus G(z)) = \bigcup_{z \in M} S^-(z)$. Then all of the requirements (C.1)–(C.3) are satisfied. Hence there exists an

$x_0 \in E$ such that $x_0 \in T(x_0)$. Then $x_0 \in T(x_0) = \mathrm{co}_\Gamma S(x_0)$ and hence, there exists an $N \in \langle S(x_0) \rangle$ such that $x_0 \in \Gamma_N \subset \mathrm{co}_\Gamma S(x_0)$. Therefore, for each $z \in N$, we have $x_0 \in S^-(z)$ or $x_0 \notin G(z)$; that is, $\Gamma_N \not\subset G(N)$. Hence G is not a KKM map, a contradiction. Therefore $(E, D; \Gamma)$ satisfies the KKM principle (A).

Remark. Here we recall briefly applications of particular forms of (A)-(C) on particular types of KKM spaces in the chronological order.

(1) At the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Ky Fan in 1961-84. A number of intersection theorems and applications to equilibrium problems followed; see [21].

(2) The origin of (C) is due to Browder in 1968 [4] with applications to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. Later this is also applied to the existence of maximal elements in mathematical economics.

(3) In 1981, Gwinner [7] displayed relations and connections between some of the most fundamental results of modern nonlinear analysis in the form of a circular tour. His tour starts in a traditional way, but also ends with the classical KKM theorem; thus eight results in [7] are in some wide sense equivalent to the KKM theorem. Nowadays there are nearly one hundred such equivalent results; see [21] and references therein. Especially, in [7], an infinite dimensional analogue of the Walras' excess demand theorem was given, which is equivalent to the Fan-Glicksberg fixed point theorem.

(4) Granas [6] studied generalizations and applications of early results in the KKM theory on convex subsets of topological vector spaces and unified classical results systematically. In fact, Granas expanded Fan's works systematically and established new topological methods in convex analysis and nonlinear analysis.

(5) Then, the KKM theory has been extended to convex spaces by Lassonde in 1983 [16], and to c-spaces (or H-spaces) by Horvath in 1984-93 [8,9] and others. For other literature, see [21].

(6) Since 1993, the theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the present author and others. In our previous work [22], for G-convex spaces, there exist more than 15 equivalent formulations of the KKM principle such as Alexandroff-Pasynkoff theorem, Fan type matching theorem, Tarafdar type intersection theorem, geometric or section properties, Fan-Browder type fixed point theorems, maximal element theorems, analytic alternatives, Fan type minimax inequalities, variational inequalities, and others. This is also true for KKM spaces.

(7) Park and Jeong [30] showed that the Brouwer fixed point theorem is equivalent to a number of results closely related to the Euclidean spaces or n-simplexes or n-balls. Among them are the Sperner lemma, the KKM theorem, some intersection theorems, various fixed point theorems, an intermediate value theorem, various non-retract theorems, the non-contractibility of spheres, and others.

(8) There are several more statements equivalent to the KKM principle for abstract convex spaces; see [26,28]. In [24-26,28], we applied (A)-(C) and other equivalents of KKM spaces to obtain fixed point, minimax, and equilibrium theorems including the Nash equilibrium theorem. Especially, in [25], we showed that some of well-known results in the KKM theory on G-convex spaces also hold on the KKM spaces. Examples of such results are theorems

of Sperner and Alexandroff-Pasynkoff, Horvath type fixed point theorem, Fan-Browder type coincidence theorems, Fan type minimax inequalities, variational inequalities, von Neumann type minimax theorem, Nash type equilibrium theorem, and Himmelberg type fixed point theorem. Moreover, in our other works [19,29], we obtained generalizations of the Gale-Nikaido-Debreu theorem.

The following is motivated from [15]:

Theorem 2. *Let $(E, D; \Gamma)$ be a KKM space, X a topological space, $g : E \to X$ a continuous map, and $S : X \to 2^D$, $T : X \to 2^E$ multimaps satisfying*
(D.1) *$S^-(z)$ is open [resp., closed] for each $z \in D$;*
(D.2) *for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*
(D.3) *$g(E) = \bigcup_{z \in M} S^-(z)$ for some $M \in \langle D \rangle$.*
Then there exists a point $a \in E$ such that $a \in T(g(a))$.

Proof. Consider the compositions $Sg : E \to 2^D$, $Tg : E \to 2^E$. Then we have the following:
(1) for each $z \in D$, $(Sg)^-(z) = g^- S^-(z)$ is open [resp., closed];
(2) for each $x \in E$, $\mathrm{co}_\Gamma (Sg)(x) \subset Tg(x)$; and
(3) by (D.3), $E = \bigcup_{z \in M} g^- S^-(z)$ for some $M \in \langle D \rangle$.
Therefore, by (C) with (Sg, Tg) instead of (S, T), we have a fixed point of Tg.

4 Simplicial spaces of Kulpa and Szymanski

Let X be a a topological space. The following due to Kulpa [14] based on the Brouwer fixed point theorem is the main tool of [15]:

Theorem 3. (Theorem on Indexed Families) [15] *Let $[p_0, p_1, \dots, p_n]$ be an n-dimensional simplex and let $\sigma : [p_0, p_1, \dots, p_n] \to X$ be a continuous function. For any covering $U_0, U_1, \dots, U_n$ of the subspace $\sigma([p_0, p_1, \dots, p_n])$ by non-empty open subsets of X, there exists a non-empty subset of indices $\{i_0, i_1, \dots, i_k\} \subset \{0, 1, \dots, n\}$ such that $\sigma([p_{i_0}, p_{i_1}, \dots, p_{i_k}]) \cap U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k} \neq \emptyset$.*

Theorem 3 is a particular case of a Fan type matching theorem given as Theorem 1 of [17] and the Fan matching property (B) for $(E, D; \Gamma)$ where $E = X$, $D = \{p_0, p_1, \dots, p_n\}$, $\Gamma_N = \sigma([N])$ for each $N \subset D$, and $S(p_i) = U_i$. Recall that this $(E, D; \Gamma)$ is a G-convex space and hence satisfies the KKM principle (A).

Definition. [15] A collection $\mathcal{S}$ of singular simplexes in a space X is called a simplicial structure on X if: (a) For any finite subset $\{a_0, a_1, \dots, a_n\}$ of (not necessarily distinct) points of X, there exists $\sigma \in \mathcal{S}$ such that $\sigma : [e_0, e_1, \dots, e_n] \to X$ and $\sigma(e_i) = a_i$ for each $i = 0, 1, \dots, n$; (b) If $\sigma \in \mathcal{S}$, then the restriction of σ to any face of the domain of σ belongs to $\mathcal{S}$.

A simplicial space $(X, \mathcal{S})$ is a topological space X together with a simplicial structure $\mathcal{S}$.

The simplicial convexity due to Bielawski [2] is extended to *simplicial spaces* in [15], which are particular to *ϕ_A-spaces* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ for the case $X = D$. Every ϕ_A-space can be made into a G-convex space [27], for which we obtained lots of results comparable to corresponding ones in [15]. These are further extended to KKM spaces in [24-26,28].

Definition. [15] Let $\mathcal{F} = \{F(x) \mid x \in X\}$ be a family of subsets of a simplicial space Y indexed by elements of a topological space X. We say that $\mathcal{F}$ is a convex open (simply, CO) family if $F(x)$ is a non-empty convex subset of Y for each $x \in X$ and $F^{-}(y)$ is an open subset of X for each $y \in Y$.

Recall that the corresponding multimap $F : X \to 2^Y$ is usually called a Φ-map or a Fan-Browder map.

Theorem 4. [15] *Let Y be a simplicial space and let $g : Y \to X$ be a continuous map such that $\overline{g(Y)}$ is a compact subset of X. Let $\mathcal{F} = \{F(x) \mid x \in X\}$ be a CO family in Y. Then there exists $a \in Y$ such that $a \in F(g(a))$.*

Note that this is a particular case of Theorem 2 for $Y = E = D$ and $F = S = T$.

In Section 3 in [15], the existence of certain CO family is given (Theorem 5). Section 4 [15] deals with families of real functions with certain compatibility and their properties. In Section 5.1 [15], the authors derive two double-sided *Infimum Principles* (Finite Version and General Version) from Theorem 4. These are applied to the Tychonoff and Schauder fixed point theorems and a general minimax principle on simplicial spaces. Section 5.2 [15] deals with one-sided Infimum Principle derived from Theorem 4. This principle is applied to generalizations of the Ky Fan minimax inequality and the Nash equilibrium theorem. In Section 5.3 [15], the authors obtained another Infimum Principles related set-valued maps. Among a slew of possible consequences of the last Infimum Principles, the authors state only those of utmost significance, that is, the Kakutani fixed point theorem and the Gale-Nikaido-Debreu theorem.

Here the most simple form of Infimum Principles asserts that for continuous real functions $f, g : X \times Y \to \mathbb{R}$, where X and Y are compact convex subsets of topological vector spaces, if $f(x, \cdot)$ and $g(\cdot, y)$ are quasi-convex, then there exists a point (a, b) such that $f(a, b) = \inf_{x \in X} f(x, b)$ and $g(a, b) = \inf_{y \in Y} g(a, y)$ [15].

Remark on Infimum Principles. Although simplicial spaces and Theorems 3 and 4 are very particular ones in the KKM theory, the Infimum Principles in [15] are very deep results and applicable to generalize important results like fixed point theorems due to Schauder, Tychonoff, Kakutani, and Fan-Browder; minimax theorems; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Ky Fan minimax inequality. Such method supplies surely a new scope in the KKM theory and is not comparable to previous methods mentioned in Section 3 [15].

5 L^*-spaces of Kulpa and Szymanski

Finally, in Section 6 [15], the authors suggested a way of extending their results to a wider class of topological spaces that contains, in particular, the class of L-spaces due to Ben-El-Mechaiekh et al. [1] and defined an L^*-*structure* on a topological space X by means of a map $L : \langle X \rangle \to 2^X$ that satisfies the following condition:

(**) If $A \in \langle X \rangle$ and $\{U_x \mid x \in A\}$ is an open cover of X, then there exists $B \subset A$ such that $L(B) \cap \{U_x \mid x \in B\} \neq \emptyset$.

Accordingly, the authors call (X, L) an L^*-*space*, and a non-empty subset Y of X to be L^*-*convex* if for each non-empty finite subset A of Y, $L(A) \subset Y$.

It follows from Theorem 3 on Indexed Families in [15] that any L-space is an L^*-space. The authors wrote that it is very conceivable that the converse statement is not true, though they have to admit they do not have any counterexample at this moment.

They also restated their Theorem 4 [15] in terms of L^*-spaces as follows:

Theorem 12. [15] *Let Y be an L^*-space and let $g : Y \to X$ be a continuous map into the space X such that $\overline{g(Y)}$ is a compact subset of X. Let $\mathcal{F} = \{F(x) : x \in X\}$ be a family of non-empty convex subsets of the space Y such that $F^-(y)$ is an open subset of X for each $y \in Y$. Then there exists a point $a \in Y$ such that $a \in F(g(a))$.*

Finally, the authors stated that "Theorem 12 enables transferring some results of the paper [15] from *simplicial spaces* to L^*-spaces. In particular, it gives a possibility of eliminating singular simplexes from Nash's equilibrium theorem, and the Nash theorem becomes a set-theoretic schema. However this line of research is not going to be elaborated here.

Remarks on L^*-spaces.

(1) An L^*-*space* (X, L) is a particular case of a KKM-space $(E, D; \Gamma)$ for $X = E = D$ and $L = \Gamma$ since the condition (**) implies the Fan matching property (B) or the definition (A) of the KKM space.

(2) Theorem 12 of [15] is a particular case of Theorem 2 for $Y = E = D$ and $F = S = T : X \to 2^Y$. Therefore, according to Kulpa and Szymanski, Theorem 2 enables transferring some results of the paper [15] from simplicial spaces to KKM spaces. Actually, we obtained such results in [24-26,28] and others.

(3) An L-space due to Ben-El-Mechaiekh et al. [1] is a G-convex space which in turn a KKM space; see [27]. We showed that an example of a KKM space which is not a G-convex space is the extended long line $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [23,25].

References

[1] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J.-V. Llinares, Abstract convexity and fixed points, *J. Math. Anal. Appl.* **222** (1998), 138–150.

[2] R. Bielawski, Simplicial convexity and its applications, *J. Math. Anal. Appl.* **127** (1987), 155–171.

[3] W. Briec and C. Horvath, Nash points, Ky Fan inequality and equilibria of abstract economies in Max-Plus and $\mathbb{B}$-convexity, *J. Math. Anal. Appl.* **341**(1) (2008), 188–199.

[4] F. E. Browder, The fixed point theory of multi-valued mappings in topological vector spaces, *Math. Ann.* **177** (1968), 283–301.

[5] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305–310.

[6] A. Granas, Quelques méthodes topologique èn analyse convexe, *Méthdes topologiques en analyse convexe, Sém. Math. Supér.* **110**, 11–77, Press. Univ. Montréal., 1990.

[7] J. Gwinner, On fixed points and variational inequalities — A circular tour, *Nonlinear Anal.* **5** (1981), 565–583.

[8] C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.* **156** (1991), 341–357.

[9] C. D. Horvath, Extension and selection theorems in topological spaces with a generalized convexity structure, *Ann. Fac. Sci. Toulouse* **2** (1993), 253–269.

[10] C. D. Horvath, Topological convexities, selections and fixed points, *Topology Appl.* **155** (2008), 830–850.

[11] C. D. Horvath and J. V. Llinares Ciscar, Maximal elements and fixed points for binary relations on topological ordered spaces, *J. Math. Econom.* **25** (1996), 291–306.

[12] W. A. Kirk and B. Panyanak, Best approximations in $\mathbb{R}$-trees, *Numer. Funct. Anal. Optimiz.* **28**(5-6) (2007), 681–690.

[13] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe, *Fund. Math.* **14** (1929), 132–137.

[14] W. Kulpa, Convexity and the Brouwer fixed point theorem, *Topology Proc.* **22** (1997), 211–235.

[15] W. Kulpa and A. Szymanski, Applications of general infimum principles to fixed-point theory and game theory, *Set-valued Anal.* **16** (2008), 375–398.

[16] M. Lassonde, On the use of KKM multifunctions in fixed point theory and related topics, *J. Math. Anal. Appl.* **97** (1983), 151–201.

[17] S. Park, Generalizations of Ky Fan's matching theorems and their applications, *J. Math. Anal. Appl.* **141** (1989), 164–176.

[18] S. Park, Five episodes related to generalized convex spaces, *Nonlinear Funct. Anal. Appl.* **2** (1997), 49–61.

[19] S. Park, A generalized minimax inequality related to admissible multimaps and its applications, *J. Korean Math. Soc.* **34** (1997), 719–730.

[20] S. Park, Continuous selection theorems in generalized convex spaces, *Numer. Funct. Anal. Optimiz.* **20** (1999), 567–583.

[21] S. Park, Ninety years of the Brouwer fixed point theorem, *Vietnam J. Math.* **27** (1999), 187–222.

[22] S. Park, Elements of the KKM theory for generalized convex spaces, *Korean J. Comp. Appl. Math.* **7** (2000), 1–28.

[23] S. Park, Examples of *KC*-maps and *KO*-maps on abstract convex spaces, *Soochow J. Math.* **33**(3) (2007), 477–486.

[24] S. Park, Elements of the KKM theory on abstract convex spaces, *J. Korean Math. Soc.* **45**(1) (2008), 1–27.

[25] S. Park, Equilibrium existence theorems in KKM spaces, *Nonlinear Anal.* **69** (2008), 4352–4364.

[26] S. Park, New foundations of the KKM theory, *J. Nonlinear Convex Anal.* **9**(3) (2008), 331–350.

[27] S. Park, Generalized convex spaces, *L*-spaces, and *FC*-spaces, *J. Global Optim.* **45**(2) (2009), 203–210.

[28] S. Park, The KKM principle in abstract convex spaces: Equivalent formulations and applications, *Nonlinear Anal.* **73** (2010), 1028–1042.

[29] S. Park, Further generalizations of the Gale-Nikaido-Debreu theorem, *J. Appl. Math. Comp.* **32**(1) (2010), 171–176.

[30] S. Park and K. S. Jeong, Fixed point and non-retract theorems – Classical circular tours, *Taiwan. J. Math.* **5** (2001), 97–108.

[31] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.* **15** (1990), 537-558.