A HISTORY OF THE NASH EQUILIBRIUM THEOREM IN THE FIXED POINT THEORY

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Abstract. In 1950, John Nash [N1,2] established his celebrated equilibrium theorem by applying the Brouwer or the Kakutani fixed point theorem. Since then there have appeared several fixed point theorems from which generalizations of the Nash theorem can be derived. In this paper, we introduce several stages of such developments.

1. Introduction

In 1928, John von Neumann found his celebrated minimax theorem [V1] and, in 1937, his intersection lemma [V2], which was intended to establish his minimax theorem and his theorem on optimal balanced growth paths. In 1941, Kakutani [K] obtained a fixed point theorem for multimaps, from which von Neumann's minimax theorem and intersection lemma were easily deduced.

In 1950, John Nash [N1,2] established his celebrated equilibrium theorem by applying the Brouwer or the Kakutani fixed point theorem. In 1952, Fan [F1] and Glicksberg [G] extended Kakutani's theorem to locally convex Hausdorff topological vector spaces, and Fan generalized the von Neumann intersection lemma by applying his own fixed point theorem.

In 1961, Ky Fan [F2] obtained his own KKM lemma and, in 1964 [F3], applied it to another intersection theorem for a finite family of sets having convex sections. This was applied in 1966 [F4] to a proof of the Nash equilibrium theorem. This is the origin of the application of the KKM theory to the Nash theorem.

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Since then there have appeared many generalizations of the Nash theorem and studies on related topics. In fact, there are diverse alternative formulations of the Nash equilibrium: *as a fixed point of the best response correspondence, as a fixed point of a function, as a solution of a nonlinear complementarity problem, as a solution of a stationary point problem, as a minimum of a function on a polytope, as an element of semi-algebraic set*; see, for example, [MM].

In our previous works [P27,28], we noticed that our studies on the Nash equilibrium were based on the following three methods:

(1) Fixed point method — Applications of the Kakutani theorem and its various generalizations (for example, for acyclic valued multimaps, admissible maps, or better admissible maps in the sense of Park); see [BK,D,F1,F3,G,H,IP,K,L2,Lu,M,N1,2, Ni,P3,4,10-12,15,26,30,31,IP,PP,TM] and others.

(2) Continuous selection method — Applications of the fact that Fan-Browder type maps have continuous selections under certain assumptions like Hausdorffness and compactness of relevant spaces; see [Br,H1,HL,P6,8,14,T] and others.

(3) The KKM method — As for the Sion minimax theorem [S], direct applications of the KKM theorem [KK] or its equivalents like as the Fan-Browder fixed point theorem [Br] for which we do not need the Hausdorffness; see [BH,CG,C,F2,4,5,GK,Gr,H1-3,HL,Kh,KS,Ko,L1,Li,Lo,P8,9,13,18-20,24,25,28,29,S] and others.

The history on the studies based on (2) and (3) was given recently in [P27,28].

In the present paper, we review the study based on the method (1). In fact, we introduce several stages of such developments of generalizations of the Nash theorem and related results within the frame of fixed point theory. We are mainly concerned with the works of the present author.

2. From von Neumann to Nash

In 1928, J. von Neumann [V1] obtained the following minimax theorem, which is one of the fundamental theorems in the theory of games developed by himself:

Theorem [K]**.** *Let $f(x,y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \le \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \ge \beta$ is convex, then we have*

$$\max_{x\in K}\min_{y\in L} f(x,y) = \min_{y\in L}\max_{x\in K} f(x,y).$$

For the history of earlier proofs of the theorem, see von Neumann [V3] and Dantzig [D].

The theorem is later extended by himself [V2] to the following intersection lemma:

Lemma [K]**.** *Let K and L be two bounded closed convex sets in the Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ in $\mathbf{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. We adopted the above formulations of Theorem and Lemma from Kakutani [K].

According to Debreu (*A commentary on the Kakutani fixed point theorem* in [Ka]),

> "Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved elementary means."

In order to give simple proofs of von Neumann's Lemma and the minimax theorem, Kakutani in 1941 obtained the following generalization of the Brouwer theorem to multimaps:

Theorem [K]**.** *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary [K]**.** *Theorem is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary readily implies von Neumann's Lemma, and it is known later that those two results are directly equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

According to Debreu (*op. cit.*) again:

> "However the formulation given by Kakutani is far more convenient to use, and his proof is distinctly more appealing.

> One of the earliest, and most important, applications of the theorem of Kakutani was made by Nash [1950] in his proof of the existence of an equilibrium for a finite game. It was followed by several hundred applications in the theory of games and in economic theory. In the latter Kakutani's theorem has been more than three decades the main tool for proving the existence of an economic equilibrium (a recent survey by Debreu [1982] quotes some three hundred fifty instances). Other areas of applications were Mathematical Programming, Control Theory and the theory of Differential Equations."

The first remarkable one of generalizations of von Neumann's minimax theorem was the Nash theorem [N1,2] on equilibrium points of non-cooperative games. The following Nash theorem is formulated by Fan [F4, Theorem 4] :

Theorem. [F4] *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j \neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasiconcave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \Pi_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \operatorname*{Max}_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

3. Generalizations of Debreu's work

In 1998 [P4], an acyclic version of the social equilibrium existence theorem of Debreu [De] is obtained.

A *polyhedron* is a set in $\mathbf{R}^n$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R}^n$. The product of two polyhedra is a polyhedron [De].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery [EM] or, more generally, to Begle [B]:

Lemma 3.1. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map (that is, u.s.c. with acyclic values). Then T has a fixed point $\hat{x} \in Z$; that is, $\hat{x} \in T(\hat{x})$.*

Let $\{X_i\}_{i \in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j \in I} X_j \quad \text{and} \quad X^i = \prod_{j \in I \setminus \{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: Its ith coordinate is x_i and, for $j \neq i$, its jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

The following recent result is a collectively fixed point theorem equivalent to Lemma.

Theorem 3.2. [P30] *Let $\{X_i\}_{i\in I}$ be an any family of acyclic polyhedra, and $T_i : X \multimap X_i$ an acyclic map for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

From Theorem 3.2, we have the following extension of the social equilibrium existence theorem of Debreu [De].

Theorem 3.3. [P30] *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $A_i : X^i \multimap X_i$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \leq f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) = \max_{y\in A_i(x^i)} g_i[x^i, y]$ is an l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) := \{x_i \in A_i(x^i) \mid f_i[x^i, x_i] \geq \varphi_i(x^i)\}$$

is acyclic.

Then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\hat{a}^i) \quad \text{and} \quad f_i(\hat{a}) = \max_{a_i\in A(\hat{a}^i)} g_i[\hat{a}^i, a_i] \quad \text{for all} \quad i \in I.$$

This is applied in [P4] to deduce acyclic versions of theorems on saddle points and minimax theorems. The following acyclic version of the Nash equilibrium theorem is given in [P4] for a finite I and in [P30] for arbitrary I:

Corollary 3.4. *Let $\{X_i\}_{i\in I}$ be a family of acyclic polyhedra, $X = \Pi_{i=1}^n X_i$, and for each i, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

Then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i\in X_i} f_i[\hat{a}^i, y_i] \quad \text{for all} \quad i \in I.$$

4. From the Idzik fixed point theorem

Let E be a real Hausdorff topological vector space (in short, a *t.v.s.*). A set $B \subset E$ is said to be *convexly totally bounded* (c.t.b.) whenever for every neighborhood V of $0 \in E$, there exist a finite subset $\{x_i \mid i \in I\} \subset E$ and a finite family of convex sets $\{C_i \mid i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i \mid i \in I\}$. See Idzik [I].

The following particular form of Idzik's theorem [I, Theorem 4.3]:

Theorem 4.1. [I] *Let X be a nonempty convex subset of a t.v.s. E and $T : X \multimap X$ a closed map with convex values. If $\overline{T(X)}$ is a compact c.t.b. subset of X, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Theorem 4.1 generalizes earlier results due to Zima, Rzepecki, Himmelberg, and Hadžić. For references, see [I].

As an application of the Idzik theorem, in this section, we consider a noncompact infinite optimization problem for a non-locally convex t.v.s.

From Theorem 4.1, we deduced the following:

Theorem 4.2. [PP] *Let I be an index set, and for each $i \in I$, X_i be a convex subset of a t.v.s. E_i, D_i be a nonempty compact subsets of X_i such that $D = \prod_{i\in I} D_i$ is a c.t.v. subset of $E = \prod_{i\in I} E_i$. For each $i \in I$, let $f_i : X = \prod_{i\in I} X_i \to \mathbf{R}$ be a u.s.c. function, and $S_i : X^i \multimap D_i$ a closed map such that*

(1) *the function M_i defined on X^i by*

$$M_i(x^i) := \sup_{y\in S_i(x^i)} f_i[x^i, y] \quad \text{for} \quad x^i \in X^i$$

is l.s.c.; and

(2) *for each $x^i \in X^i$, the set*

$$T_i(x^i) := \{y \in S_i(x^i) \mid f_i[x^i, y] = M_i(x^i)\}$$

is convex.

Then there exists an $\overline{x} \in D$ such that for each $i \in I$,

$$\overline{x}_i \in S_i(\overline{x}^i) \quad \text{and} \quad f_i[\overline{x}^i, \overline{x}_i] = M_i(\overline{x}^i).$$

Remarks 1. If each E_i is locally convex and each f_i and S_i are continuous, then Theorem 4.2 includes results due to Idzik, Im and Kim, and Kaczynski and Zeidan.

2. Instead of the compactness of S_i, as in Theorem 4.2, we may obtain a result for Φ-condensing maps S_i.

From Theorem 4.2, we obtain the following infinite version of the Nash equilibrium theorem:

Theorem 4.3. [PP,IP] *Let I be an index set, and for each $i \in I$, X_i be a nonempty compact convex subset of a t.v.s. E_i such that $X = \prod_{i\in I} X_i$ is a c.t.b. subset of $E = \prod_{i\in I} E_i$. For each $i \in I$, let $f_i : X \to \mathbf{R}$ be a continuous function such that for each given point $x^i \in X^i$, $x_i \mapsto f[x^i, x_i]$ is a quasiconcave function on X_i. Then there exists an $\overline{x} \in X$ such that*

$$f_i(\overline{x}) = f_i[\overline{x}^i, \overline{x}_i] = \max_{y_i \in X_i} f_i[\overline{x}^i, y_i] \quad \text{for each} \quad i \in I.$$

Remarks 1. Note that Ma already established Theorem 5 without assuming that X is c.t.b. A generalization of Ma's theorem was given by Idzik.

2. Nash's original theorem is the case E_i are Euclidean spaces and I is finite.

Moreover, in 1998 [IP], we considered two applications of a fixed point theorem due to Idzik [I]. First, we extended the Leray-Schauder theorem to t.v.s. which are not necessarily locally convex. As an application we derived some well-known fixed point theorems. Second, we deduced a variation of the social equilibrium existence theorem of Debreu. This was applied to results on saddle points, minimax theorems, and the Nash equilibria. These were generalizations of results of von Neumann, Kakutani, Nash, and von Neumann and Morgenstern; for the literature, see Debreu [De].

5. Fixed points of compositions of acyclic maps

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a topological vector space, convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely in each stage.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is called an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X,Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X,Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following theorems are only few examples of our previous works; for more general results, see [P22,23].

Theorem 5.1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $T \in \mathbb{V}_c(X,X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

A nonempty subset X of a topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood

V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex Hausdorff topological vector space is admissible. Other examples of admissible topological vector spaces are ℓ^p, $L^p(0,1)$, H^p for $0 < p < 1$, and many others; see [P5,7,16,21-23] and references therein.

Theorem 5.2. *Let E be a Hausdorff topological vector space and X an admissible convex subset of E. Then any compact map $T \in \mathbb{V}_c(X,X)$ has a fixed point.*

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Examples of Klee approximable sets can be seen in [P17].

We define a class $\mathfrak{B}$ of maps from a subset X of a t.v.s. E into a topological space Y as follows [P12,16,17]:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

We call $\mathfrak{B}$ the 'better' admissible class. Recently it is known that any u.s.c. map with compact values having *trivial shape* (that is, contractible in each neighborhood) belongs to $\mathfrak{B}(X,Y)$. Note that the class $\mathfrak{B}^p$ in [P16,17] should be replaced by $\mathfrak{B}$.

The following results appeared in our previous work [P17]:

Theorem 5.3. [P17, Corollary 2.3] *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

6. For admissible sets

In 2000 [P11] and 2002 [P15], we applied Theorem 5.2 to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.

The following examples are generalized forms of quasi-equilibrium theorem or social equilibrium existence theorems which directly imply generalizations of the Nash-Ma type equilibrium existence theorem.

Theorem 6.1. [P11] *Let* $\{X_i\}_{i=1}^n$ *be a family of convex sets, each in a t.v.s.* E_i, K_i *a nonempty compact subset of* X_i, $S_i : X \multimap K_i$ *a closed map, and* $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ *u.s.c. functions for each* i.

Suppose that for each i,

(i) $g_i(x) \leq f_i(x)$ *for each* $x \in X$;

(ii) *the function* M_i *defined on* X *by*

$$M_i(x) := \max_{y \in S_i(x)} g_i[x^i, y] \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(iii) *for each* $x \in X$, *the set*

$$\{y \in S_i(x) \mid f_i[x^i, y] \geq M_i(x)\}$$

is acyclic.

If X *is admissible in* $E = \Pi_{j=1}^n E_j$, *then there exists an* $\hat{x} \in K$ *such that for each* i,

$$\hat{x}_i \in S_i(\hat{x}) \quad \textit{and} \quad f_i[\hat{x}^i, \hat{x}_i] \geq g_i[\hat{x}^i, y] \quad \textit{for all} \quad y \in S_i(\hat{x}).$$

Theorem 6.2. [P15] *Let* X_0 *be a topological space and* $\{X_i\}_{i=1}^n$ *be a family of convex sets, each in a t.v.s.* E_i. *For each* $i = 0, 1, \ldots, n$, *let* $S_i : X^i \multimap X_i$ *be a closed map with compact values, and* $f_i, g_i : X = \prod_{i=0}^n X_i \to \mathbf{R}$ *u.s.c. real-valued functions.*

Suppose that for each i,

(i) $g_i(x) \leq f_i(x)$ *for each* $x \in X$;

(ii) *the function* $M_i : X^i \to R$ *defined by*

$$M_i(x^i) := \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad \textit{for} \quad x^i \in X^i$$

is l.s.c.; and

(iii) *for each* $x^i \in X^i$, *the set*

$$\{x_i \in S_i(x^i) \mid f_i[x^i, x_i] \geq M_i(x^i)\}$$

is acyclic.

If X^0 *is admissible in* $E^0 = \Pi_{j=1}^n E_j$ *and if all the maps* S_i *are compact except possibly* S_n *and* S_n *is u.s.c., then there exists an equilibrium point* $\hat{x} \in X$; *that is,*

$$\hat{x}_i \in S_i(\hat{x}^i) \quad \textit{and} \quad f_i(\hat{x}) \geq \max_{y_i \in S_i(x^i)} g_i[\hat{x}^i, y] \quad \textit{for all} \quad i \in Z_{n+1}.$$

7. For Klee approximable sets

In 2008 [P21], we deduce some collectively fixed point theorems for families of maps and, then, various von Neumann type intersection theorems.

Theorem 7.1. [P21] *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be a subset of E_i, K_i a nonempty compact subset of X_i, and $F_i : X \multimap K_i$ a closed map with acyclic values (resp., values of trivial shape). If $K := \prod_{i=1}^n K_i$ is Klee approximable into X, then there exists an $\overline{x} = (\overline{x_i})_{i=1}^n \in X$ such that $\overline{x_i} \in F_i(\overline{x})$ for each i.*

From Theorem 7.1, we obtain the following von Neumann type intersection theorem:

Theorem 7.2. [P21] *Let $\{X_i\}_{i=1}^n$ be a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X such that $A_i(x^i)$ is an acyclic subset of K_i for each $x^i \in X^i$, where $1 \le i \le n$. If X is an almost convex admissible subset of E, then $\bigcap_{j=1}^n A_j \neq \emptyset$.*

Similarly, we can obtain a more general result than Theorem 7.2 as follows:

Theorem 7.2.′ [P21] *Let I be any index set, $\{X_i\}_{i\in I}$ a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X for each $i \in I$. Suppose that for each $x^i \in X^i$, $A_i(x^i)$ is a convex subset of K_i except a finite number of i's for which $A_i(x^i)$ is an acyclic subset of K_i. If X is an almost convex admissible subset of E, then $\bigcap_{j\in I} A_j \neq \emptyset$.*

Remark. If $I = \{1, 2\}$, E_i are Euclidean, $X_i = K_i$, and $A_i(x^i)$ are nonempty and convex, then Theorem 7.2 or 4.2′ reduces to the intersection lemma of von Neumann [V2].

We have another intersection theorem:

Theorem 7.3. [P21] *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For each $i = 0, 1, 2, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n and $F_i \in \mathbb{V}_c(X^i, X_i)$. If K^0 is Klee approximable into X^0, then $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$.*

Remarks. 1. In case when each X_i is convex for $i \ge 1$ and X^0 is admissible in E^0, Theorem 7.3 reduces to [P15, Theorem 4].

2. Particular forms of Theorem 7.3 were given by von Neumann, Fan, Lassonde, Chang, and Park; see [P15]. The following is one of them:

Corollary 7.4. *Let X be a topological space, Y a subset in a t.v.s. E, and $F \in \mathbb{V}_c(X, Y)$ and $G \in \mathbb{V}_c(Y, X)$. If F is compact and $F(X)$ is Klee approximable into Y, then* $\mathrm{Gr}(F) \cap \mathrm{Gr}(G) \neq \emptyset$.

From Corollary 7.4, we have the following:

Corollary 7.5. *Let X be a topological space and Y a compact subset of a t.v.s. E. Let A and B be two closed subsets of $X \times Y$ such that*

(1) *for each $x \in X$, $A(x) := \{y \in Y \mid (x, y) \in A\}$ is acyclic; and*

(2) *for each $y \in Y$, $B(y) := \{x \in X \mid (x, y) \in B\}$ is acyclic.*

If $A(X) := \bigcup\{A(x) \mid x \in X\}$ is Klee approximable into Y, then $A \cap B \neq \emptyset$.

Remarks. 1. If Y is an admissible, compact, and almost convex subset of E, then $A(X)$ is Klee approximable into Y. Especially, for the particular case when X is compact and Y is convex, Corollary 7.5 was obtained in [P11].

2. For other particular forms of Corollary 7.5, see [P11].

From Theorem 7.3, we deduce the following generalized form of the quasi-equilibrium theorem or the social equilibrium existence theorem in the sense of Debreu [De]:

Theorem 7.6. [P21] *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, $S_i : X^i \multimap K_i$ be a closed map with compact values, and $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ u.s.c. real functions.*

Suppose that for each $i = 0, 1, \cdots, n$,

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(ii) *the real function $M_i : X^i \to \mathbf{R}$ defined by*

$$M_i(x^i) := \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for \quad x^i \in X^i$$

is l.s.c.; and

(iii) *for each $x^i \in X^i$, the set*

$$\{y_i \in S_i(x^i) \mid f_i[x^i, y_i] \geq M_i(x^i)\}$$

is acyclic.

If K^0 is Klee approximable into X^0 and if S_n is u.s.c., then there exists an equilibrium point $\hat{x} \in X$; that is,,

$$\hat{x}_i \in S_i(\hat{x}^i) \quad and \quad f_i[\hat{x}^i, \hat{x}_i] \geq \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for\ each \quad i \in Z_{n+1}.$$

Therefore, from Theorem 7.6, we have the following particular form:

Theorem 7.7. [P21] *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \cdots, n$, let K_i be a nonempty subset of X_i which is compact* *except possibly* K_n, $S_i : X^i \multimap K_i$ *be a continuous multimap with compact values, and* $f_i : X = X^i \times X_i \to \mathbf{R}$ *a continuous real function.*

Suppose that for each $i = 0, 1, \cdots, n$, *the following holds:*

(0) *for each* $x^i \in X^i$ *and each* $\alpha \in \mathbf{R}$, *the set*

$$\{x_i \in S_i(x^i) \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If K^0 *is Klee approximable into* X^0, *there exists an equilibrium point* $\hat{x} \in X$.

Remarks. 1. If each X_i is convex and if X is admissible in E, then Theorem 7.7 reduces to [P21, Theorem 6].

2. For other particular forms of Theorems 7.6 and 7.7, see [P11,15].

The following generalizes the Nash theorem:

Corollary 7.8. [P21] *Let* X_0 *be a compact topological space, and* $\{X_i\}_{i=1}^n$ *a family of convex sets, each in a t.v.s.* E_i, *such that each* X_i *is compact except* X_n. *For* $i = 0, 1, \cdots, n$, *let* $f_i : X = X^i \times X_i \to \mathbf{R}$ *be a continuous real function such that*

(1) *for each* $x^i \in X^i$ *and each* $\alpha \in \mathbf{R}$, *the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If X^0 *is admissible, then there exists an equilibrium point* $\hat{x} \in X$; *that is,*

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad for\ all \quad i \in \mathbf{Z}_{n+1}.$$

Remarks. 1. This slightly extends [P11, Theorem 7].

2. If all X_i are compact convex subsets of Euclidean spaces and if $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave for each $x^i \in X^i$, then Corollary 7.8 reduces to Nash [N2, Theorem].

From Corollaries 7.5 or 7.8, we have the following von Neumann type minimax theorem:

Theorem 7.9. [P21] *Let* X *be a compact space and* Y *an admissible compact convex subset of a t.v.s., and* $f : X \times Y \to \mathbf{R}$ *a continuous real function. Suppose that for each* $x_0 \in X$ *and* $y_0 \in Y$, *the sets*

$$\{x \in X \mid f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y \mid f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are acyclic. Then

(1) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, y) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

(2) *We have the minimax inequality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Remarks. 1. Theorem 7.9 includes [P11, Theorem 4] and [P15, Corollary 6.2], where some particular forms were noted.

2. For Euclidean spaces or locally convex t.v.s., if acyclicity is replaced by convexity, then Theorem 7.9 reduces to the von Neumann minimax theorem [V2] or Fan [F1, Theorem 3], resp.

The following generalization of the von Neumann minimax theorem is a simple consequence of Corollary 7.5 or Theorem 7.9:

Theorem 7.10. *Let X, Y, and f be the same as in Theorem 7.9. Suppose that*

(1) *for every $x \in X$ and $\alpha \in \mathbf{R}$, $\{y \in Y \mid f(x, y) \leq \alpha\}$ is acyclic; and*

(2) *for every $y \in Y$ and $\beta \in \mathbf{R}$, $\{x \in X \mid f(x, y) \geq \beta\}$ is acyclic.*

Then we have

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

8. Existence of pure-strategy Nash equilibrium

In this section, we introduce the contents of a recent work [P31]. The concept of generalized convex spaces is well known:

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$.

We follow [Lu]. Let $I := \{1, \dots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple $\Lambda := \{X_1, \dots, X_n; u_1, \dots, u_n\}$, where the nonempty set X_i is the ith player's pure strategy space and $u_i : X := \prod_{i=1}^n X_i \to \mathbb{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Let $X_{-i} := \prod_{j \in I \setminus \{i\}} X_j$ and denote by x and x_{-i} an element of X and X_{-i}, resp. A strategy n-tuple $(x_1^*, \dots, x_n^*)$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*) \ \text{ for all } y_i \in X_i \ \text{ and } \ i \in I.$$

As usual, we define an aggregate payoff function $U : X \times X \to \mathbb{R}$ as follows:

$$U(x, y) := \sum_{i=1}^n [u_i(y_i, x_{-i}) - u_i(x)] \ \text{ for any } \ x = (x_i, x_{-i}), y = (y_i, y_{-i}) \in X.$$

The following is given in [Lu, Proposition 1]:

Lemma 8.1. *Let Λ be a non-cooperative game, K a nonempty subset of X, and $x^* = \{x_1^*, \dots, x_n^*\} \in K$. Then the following are equivalent:*
(a) *x^* is a Nash equilibrium;*
(b) *$\forall i \in I,\ \forall y_i \in X_i,\ \ u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*)$;*
(c) *$\forall y \in X,\ U(x^*, y) \leq 0$.*

Note that (c) implies $U(x^*, y) \leq 0$ for all $y \in D \subset X$.

Now we have our main result:

Theorem 8.2. *Let $I = \{1, \dots, n\}$ be a set of players, K a nonempty compact subset of a Hausdorff product G-convex space $(X, D; \Gamma) = \prod_{i=1}^n (X_i, D_i; \Gamma_i)$ and Λ a non-cooperative game. Suppose that*

(i) *the function $U : X \times X \to \mathbf{R}$ satisfies that*

$$\{(x, y) \in X \times X \mid U(x, y) > 0\}$$

is open;

(ii) *for each $x \in K$, $\{y \in X \mid U(x, y) > 0\}$ is Γ-convex [that is, $M \in \langle\{y \in D \mid U(x, y) > 0\}\rangle$ implies $\Gamma_M \subset \{y \in X \mid U(x, y) > 0\}$];*

(iii) *for each $y \in X$, the set $\{x \in K \mid U(x, y) \leq 0\}$ is acyclic.*

Then there exists a point $x^* \in K$ such that x^* is an equilibrium point for the non-cooperative game.

Note that condition (i) can be replaced by the following:

(i)$'$ the function $U(x, y)$ is lower semicontinuous on $X \times X$.

In this case, when $X = D$ is a topological vector space, Theorem 8.2 reduces to [Lu, Theorem 1].

9. Historical remarks

John von Neumann's 1928 minimax theorem [N1] and 1937 intersection lemma [N2] have numerous generalizations and applications. Kakutani's 1941 fixed point theorem [K] was to give simple proofs of the above-mentioned results. In 1950, John Nash [N1,2] obtained his equilibrium theorem based on the Brouwer or Kakutani fixed point theorem.

In the 1950's, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [BK] and to locally convex Hausdorff topological vector spaces by Fan [F1] and Glicksberg [G]. These extensions were mainly used to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma [M] and others. For the literature, see [P6] and references therein.

An upper semicontinuous (u.s.c.) multimaps with nonempty compact convex values is called a *Kakutani map.* The Fan-Glicksberg theorem was extended by Himmelberg [H] in 1972 for compact Kakutani maps instead of assuming compactness of domains. In 1988, Idzik [I] extended the Himmelberg theorem to convexly totally bounded sets instead of convex subsets in locally convex t.v.s. This result is applied in [P3,PP,IP] to various problems. In 1990, Lassonde [L2] extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in Hausdorff topological vector spaces. Moreover, Lassonde applied his theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem comparable to Debreu's social equilibrium existence theorem [De].

On the other hand, in 1946, the Kakutani fixed point theorem was extended for acyclic maps by Eilenberg and Montgomery [EM]. Moreover, the Kakutani theorem was known to be included in the extensions, due to Eilenberg and Montgomery [EM] or Begle [B], of Lefschetz's fixed point theorem to u.s.c. multimaps of a compact *lc*-space into the family of its nonempty compact acyclic subsets. This result was applied by Park [P4] to give acyclic versions of the social equilibrium existence theorem due to Debreu [De], saddle point theorems, minimax theorems, and the Nash equilibrium theorem.

Moreover, Park [P1,2,4,15-17,21,22] obtained a sequence of fixed point theorems for various classes of multimaps (including compact compositions of acyclic maps) defined on very general subsets (including Klee approximable subsets) of topological vector spaces. Especially, our cyclic coincidence theorem for acyclic maps were applied to generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, the von Neumann minimax theorems, and many other results; see [P26].

Finally, recall that there are several thousand published works on the KKM theory and fixed point theory and we can cover only a part of them. For the more historical background for the related fixed point theory and for the more involved or related results in this paper, see the references of [P7,22,23,26,28,29] and the literature therein.

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