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CONTINUOUS SELECTION THEOREMS IN GENERALIZED CONVEX SPACES: REVISITED

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Abstract. Since our previous work [25] appeared, some new results on continuous selection problem on G-convex spaces were obtained by a number of other authors. Some of them claimed to obtain already known results. In this paper, we show that selection theorems in [25] with a few generalized forms of them contain certain selection theorems in more than a dozen papers mainly appeared after [25].

1. Introduction

The fixed point theory of multimaps in topological vector spaces has numerous applications in many fields in mathematical sciences. This theory began with the celebrated Kakutani fixed point theorem in 1941 on upper semicontinuous multimaps with closed convex values. Recall that Ernest Michael's groundbreaking theory of continuous selections on multimaps began in 1956 and was mainly concerned with the class of lower semicontinuous multimaps with closed convex values. For applications of Michael's selection theorems to fixed point theory, see [30].

It is well-known that any multimap from a paracompact (or Hausdorff compact) space to a convex space has a continuous selection whenever it has nonempty convex values and open fibers. This fact was first used by Browder [4,5] in order to establish the Fan-Browder fixed point theorem. Later, it was explicitly formulated by Ben-El-Mechaiekh, Deguire, and Granas [2,3] and Yannelis and Prabhakar [39], and has been applied by many authors.

Moreover, Horvath [15-17] extended the continuous selection theorem by replacing convex spaces by C-spaces (or H-spaces) due to himself. Later,

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the present author extended C-spaces to generalized convex spaces (or G-convex spaces) and studied the KKM theory and fixed point theory on these spaces; see [23-29, 33-37].

In our earlier work [25] in 1999, we obtained some continuous selection theorems related to G-convex spaces. In fact, first, we showed that any multimap from a Hausdorff compact space to a G-convex space has a continuous selection whenever it has nonempty generalized convex values and open fibers. Second, if the domain is paracompact, we obtained a local selection result only. This was applied to selection theorems for convex spaces. In fact, in [25], some new results as well as generalizations of many known results were applied to fixed point theorems and existence of equilibria.

Actually, theorems in [25] unify, extend, or improve a large number of known results until that time. After [25] appeared, some results on continuous selection problem were further obtained by a number of other authors. Some of them claimed to obtain already known results. Our aim in this short paper is to clarify such rather confusing situation. In fact, we show that selection theorems in [25] with a few generalized forms of them contain certain selection theorems in more than a dozen papers mainly appeared after [25].

Section 2 deals with preliminaries on G-convex spaces and examples. In Section 3, we recall some selection theorems in [25] with a few generalized forms of them. Section 4 concerns with a list of more than a dozen papers mainly appeared after [25], which contain particular selection theorems to our results.

2. Generalized convex spaces

A *multimap* or a *map* $T : X \multimap Y$ is a function $T : X \to 2^Y$ from a set X into the power set of a set Y. Let $T^- : Y \multimap X$ be defined by $x \in T^-(y)$ if and only if $y \in T(x)$. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of D.

This section concerns with the concepts and typical examples of generalized convex spaces:

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

For details on G-convex spaces, see [23-29, 33-37], where basic theory was extensively developed and lots of examples of G-convex spaces were given. Here we give only a few as follows:

Examples.

(1) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$; see [19].

(2) A G-convex space $(X, D; \Gamma)$ is called an *H-space* if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D \rangle$. When $X = D$, an H-space reduces to a *C-space* due to Horvath [15,16].

(3) A *space having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a *ϕ_A-space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [31].

A ϕ_A-space is called a GFC-space by some authors and an FC-space by other authors when $X = D$.

In this paper, all paracompact spaces are Hausdorff. Note that every Hausdorff compact space is paracompact, and a paracompact space is normal.

For each locally finite open cover $\mathcal{U}$ of a normal space there is a partition of unity which is subordinate to $\mathcal{U}$.

A *continuous selection* $f : X \to Y$ of a map $T : X \multimap Y$ is a continuous function such that $f(x) \in T(x)$ for all $x \in X$.

Throughout this paper, let X be a normal space and $(Y, D; \Gamma)$ a G-convex space.

A multimap $T : X \multimap Y$ is called a *Φ-map* provided that there exists a companion multimap $S : X \multimap D$ satisfying

(a) for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$; and

(b) $X = \bigcup\{\operatorname{Int} S^-(y) \mid y \in D\}$.

Recall that some authors adopted certain terminology related to condition (b) as follows: If S has nonempty values, then S has the local intersection property if and only if S^- is transfer open-valued and if and only if condition

$$\bigcup\{S^-(y) \mid y \in D\} = \bigcup\{\operatorname{Int} S^-(y) \mid y \in D\} \quad [S^- \text{ is transfer open-valued}]$$

holds. Someone also stated that S is *transfer open inversed valued.* Comments on such terminology, see [32].

3. Selection theorems

In this section, we recall some continuous selection results given in [25] and add some new results related to them.

Lemma 3.1. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, and $S : X \multimap D$ a multimap such that*

$$X = \bigcup\{\mathrm{Int}\, S^{-}(y) \mid y \in A\}.$$

for some $A \in \langle S(X)\rangle$. Then there exist a continuous map $s : X \to \Gamma_A$ such that $s(x) \in \Gamma(A \cap S(x))$ for all $x \in X$.

Proof. Let $A := \{y_0, y_1, \ldots, y_n\} \in \langle D\rangle$ such that $X = \bigcup_{i=0}^{n} \mathrm{Int}\, S^{-}(y_i)$. Note that $y_i \in S(X)$ for all i. Since $(Y, D; \Gamma)$ is G-convex, there exists a continuous map $\phi_A : \Delta_n \to Y$ such that $\phi_A(\Delta_n) \subset \Gamma_A$ and $\phi_A(\Delta_J) \subset \Gamma_J$ for each $J \in \langle A\rangle$. Let $\{\alpha_i\}_{i=0}^{n}$ be a partition of unity subordinated to the cover $\{\mathrm{Int}\, S^{-}(y_i)\}_{i=0}^{n}$ of the normal space X; that is,

(1) for each i, $\alpha_i : X \to [0, 1]$ is continuous;

(2) $\mathrm{Supp}\, \alpha_i \subset \mathrm{Int}\, S^{-}(y_i)$ for each i; and

(3) for each $x \in X$, $\sum_{i=0}^{n} \alpha_i(x) = 1$.

Define a continuous map $p : X \to \Delta_n$ by

$$p(x) = \sum_{i=0}^{n} \alpha_i(x) e_i = \sum_{y_i \in A_x} \alpha_i(x) e_i \quad \text{for } x \in X,$$

where $\{e_i\}_{i=0}^{n}$ are vertices of Δ_n,

$$y_i \in A_x \subset A \iff \alpha_i(x) \neq 0 \implies x \in \mathrm{Int}\, S^{-}(y_i) \implies y_i \in S(x)$$

and hence $A_x \in \langle A \cap S(x)\rangle$ and $p(x) \in \Delta_{A_x}$. Therefore,

$$(\phi_A \circ p)(x) \in \phi_A(\Delta_{A_x}) \subset \phi_A(\Delta_{A \cap S(x)}) \subset \Gamma(A \cap S(x))$$

and $s := \phi_A \circ p : X \to Y$ is a continuous map such that $s(x) \in \Gamma(A \cap S(x))$.

Remark. Lemma 3.1 was first given in [28, Theorem J] and [29, Lemma 3.1]. When $X = K$ is Hausdorff compact, then Lemma 3.1 reduces to [25, Lemma 1], which originates from Browder's proof [4,5] of the so-called Fan-Browder fixed point theorem.

The following extends a part of [25, Theorem 1]:

Theorem 3.2. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, $T : X \multimap Y$ a Φ-map with a companion map $S : X \multimap D$ such that*

$$X = \bigcup\{\mathrm{Int}\, S^{-}(y) \mid y \in A\}.$$

for some $A \in \langle S(X)\rangle$.

Then we have the following:

(i) *T has a continuous selection $f : X \to Y$ such that $f(X) \subset \Gamma_A$. More precisely, if $|A| = n + 1$, there exist two continuous maps $p : X \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$.*

(ii) *If* $g : Y \to X$ *is a continuous map, then there exists a* $y_0 \in Y$ *such that* $y_0 \in T(g(y_0))$.

(iii) *If* $R : X \multimap Y$ *is a map such that* R^- *has a continuous selection, then* R *and* T *have a coincidence point* $x_0 \in X$*; that is,* $R(x_0) \cap T(x_0) \neq \emptyset$.

Proof. (i) By Lemma 3.1, there exist a continuous map $f = s : X \to Y$ such that $f = \phi_A \circ p$ and

$$f(x) \in \phi_A(\Delta_{A\cap S(x)}) \subset \phi_A(\Delta_n) \subset \Gamma_A \quad \text{for all} \quad x \in X.$$

By (a), $\Gamma(A \cap S(x)) \subset T(x)$ and hence f is the required continuous selection of T.

(ii) Since $g : Y \to X$ is continuous, as in the proof of Lemma 3.1, the composition

$$\Delta_n \xrightarrow{\phi_A} Y \xrightarrow{g} X \xrightarrow{p} \Delta_n$$

has a fixed point $v \in \Delta_n$ by the Brouwer fixed point theorem. Let $y_0 = \phi_A(v)$. Since $\phi_A \circ p$ is a selection of T by (i), we have

$$(\phi_A \circ p)(g(y_0)) \in T(g(y_0)).$$

Note that

$$(\phi_A \circ p)(g(y_0)) = \phi_A(p \circ g \circ \phi_A)(v) = \phi_A(v) = y_0,$$

and hence $y_0 \in T(g(y_0))$.

(iii) Let $g : Y \to X$ be a continuous selection of the multimap R^-. Then for any $y \in Y$, we have $g(y) \in R^-(y)$ or $y \in R(g(y))$. Since there exists a $y_0 \in Y$ such that $y_0 \in T(g(y_0))$ by (ii), we have

$$y_0 \in R(g(y_0)) \cap T(g(y_0)).$$

Remark. Note that if X is a compact subset of a Hausdorff space, Theorem 3.2 reduces to [25, Theorem 1 (i)-(iii)].

It should be noted that in case Y is a convex subset of a topological vector space (t.v.s.), Theorem 3.2 is very useful. We now show that Theorem 3.2 also works when Y is a certain non-convex subset.

Recall that a nonempty subset Y of a t.v.s. E is called *almost convex* [14] if for any neighborhood V of the origin 0 of E and for any finite set $\{y_1, \ldots, y_n\}$ of Y, there exists a finite set $\{z_1, \ldots, z_n\}$ of Y such that $z_i - y_i \in V$ for all $i = 1, \ldots, n$ and $\text{co}\{z_1, \ldots, z_n\} \subset Y$.

Clearly, each convex set is almost convex, but the converse is not true in general.

Lemma 3.3. *Any almost convex subset* Y *of a t.v.s.* E *is a* G*-convex space.*

Proof. Fix a neighborhood V of 0 in E and let $D = Y$. For each $A = \{y_1, \ldots, y_n\} \in \langle Y \rangle$, choose a set $B = \{z_1, \ldots, z_n\}$ as above and define $\Gamma_A = \text{co}\, B$. Then $(Y, D; \Gamma) = (Y, \Gamma)$ becomes a G-convex space.

Note that Lemma 3.3 provides us an example of a G-convex space and a new application of Theorem 3.2.

The following was given as [25, Theorem 8]:

Theorem 3.4. *Let X be a paracompact space, $(Y, D; \Gamma)$ an H-space, and $T : X \multimap Y$ a Φ-map. Then T has a continuous selection.*

Remark. In [25], we noted that it was not known yet whether Theorem 3.4 holds for any G-convex space $(Y, D; \Gamma)$. The following extension of Theorem 3.2(i) is the best of our solution to this matter:

Theorem 3.5. *Let X be a normal space, $(Y, D; \Gamma)$ a G-convex space, $T : X \multimap Y$ a Φ-map with a companion map $S : X \multimap D$ such that*

(1) *the cover $\{\operatorname{Int} S^-(y) \mid y \in D\}$ of X has a locally finite refinement; and*

(2) *for each $A, B \in \langle D \rangle$ with $B \subset A$ implies $\phi_B = \phi_A|_{\Delta_n}$ where $|B| = n+1$.*

Then T has a continuous selection $f : X \to Y$.

Proof. Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ be the locally finite refinement in (1). Since X is normal, there is a continuous partition of unity $\{\lambda_\alpha\}_{\alpha \in I}$ subordinated to $\mathcal{U}$ such that

$$\forall \alpha \in I \quad \exists y_\alpha \in D \quad \text{such that } U_\alpha \subset \operatorname{Int} S^-(y_\alpha).$$

We may suppose that I is well-ordered. Let us define a map $g : X \to \bigcup_{r=0}^{\infty} \Delta_r$ by

$$g(x) := \sum_{k=0}^{n} \lambda_{\alpha_k}(x) e_k \quad \text{for each } x \in X$$

where $N := \{y_{\alpha_0}, \ldots, y_{\alpha_n}\} = \{y_\alpha \in D \mid \lambda_\alpha(x) > 0,\ \alpha \in I\} \in \langle S(x) \rangle$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$.

Let V_x be a neighborhood of x. Then it intersects a finite number of U_α, say $U_{\beta_0}, \ldots, U_{\beta_m} \in \mathcal{U}$. Then, for any $x' \in V_x$, we have

$$g(x') = \sum_{k=0}^{m} \lambda_{\beta_k}(x') e_k$$

and $g(x)$ can also has the form

$$g(x) = \sum_{k=0}^{m} \lambda_{\beta_k}(x) e_k.$$

Therefore, for any $x \in X$ and $\varepsilon > 0$, there exist an m and a V_x such that $g(x), g(x') \in \Delta_m$ for $x' \in V_x$ and

$$dist_{\Delta_m}(g(x), g(x')) < \varepsilon.$$

Hence g is continuous.

Since $g(X) \subset \Delta_n \subset \Delta_m$, we can define $f := \phi_N \circ g$. Then $f = \phi_M \circ g$ for each $M \in \langle D \rangle$ with $N \subset M$ by (2). Hence f is well-defined and continuous.

It remains to show that f is a selection of T. In fact, for any $x \in X$ with $\lambda_{\alpha_k}(x) > 0$ for all k,

$$x \in U_{\alpha_k} \subset \operatorname{Int} S^-(y_{\alpha_k}) \subset S^-(y_{\alpha_k}) \text{ implies } y_{\alpha_k} \in S(x)$$

and hence $N \in \langle S(x) \rangle$. Since S is a companion of the Φ-map T, we have $\Gamma_N \subset T(x)$. Since $f(x) \in \phi_N(\Delta_n) \subset \Gamma_N$, we have $f(x) \in T(x)$. This completes our proof.

Remarks. 1. When X is paracompact, condition (1) is automatically satisfied.

2. When (Y, D) is a convex space, then condition (2) is automatically satisfied.

3. When X is paracompact and (Y, D) is a convex space, then Theorem 3.5 reduces to [25, Theorem 6].

4. Comments on other works

In this section, we collect some consequences of the selection theorems in Section 3 mainly due to other authors. These consequences are not cited in [25] and most of them have appeared after [25].

(I) Wu and Shen 1996 [38]

In this paper, the following was given as a generalization of a result of Yannelis and Prabhakar [39]:

Theorem 1 ([38]). *Let X be a nonempty paracompact subset of a topological space E and Y be a nonempty subset of a Hausdorff topological vector space F. Suppose that $S, T : X \to 2^Y$ are two multivalued mappings with the following conditions:*

(i) *for each $x \in X$, $S(x)$ is nonempty and $\operatorname{co} S(x) \subset T(x)$.*

(ii) *S has local intersection property.*

Then T has a continuous selection.

Note that this was already known ealier by Browder and follows from our Theorem 3.4.

(II) Zhang 1997 [41]

Zhang stated that the above theorem might be the most general form among existing selection theorems in topological linear spaces. He also stated that the supposition in the above theorem is strong and proved approximate selection theorems under suitable conditions.

(III) Ben-El-Mechaiekh et al. 1998 [1]

The following was given:

Proposition 3.8 ([1]). *Let $\mathcal{C}$ be a B'-convexity on a topological space Y. Let X be a paracompact space, and let $\phi : X \to Y$ be a correspondence with nonempty, $\mathcal{C}$-convex values and open lower section $\phi^{-1}(y) = \{x \in X \mid y \in \phi(x)\}$. Then ϕ has a continuous selection.*

Note that $(Y,\mathcal{C})$ is an example of G-convex space satisfying condition (2) of Theorem 3.5; see [1]. Hence the above proposition follows from Theorem 3.5. Moreover, every H-space $(Y;\Gamma)$ has a B'-convexity [1, Proposition 3.6], Theorem 3.4 follows from Theorem 3.5.

(IV) Lin and Park 1998 [20]

The following simple consequence of Theorem 3.2(i) was obtained:

Theorem 1 ([20])**.** *Let X be a Hausdorff compact space, (Y,Γ) a G-convex space, and $F : X \multimap Y$. Suppose that*
(i) *for each $x \in X$, $F(x)$ is G-convex; and*
(ii) *$X = \bigcup_{y\in Y} \operatorname{Int} F^-(y)$ (that is, F^- has transfer open values).*
Then F has a continuous selection.

In [20], its authors studied some equilibrium problems, quasi-equilibrium problems, and generalized quasi-equilibrium problems in G-convex spaces using a new method of fixed point approach based on Theorem 1.

(V) Mao 1998 [21]

In [21], its author presented the following continuous selection theorem in H-space which includes the earlier selection theorems of Tarafdar and Wu as special cases:

Theorem 2.3 ([21])**.** *Let X be a nonempty paracompact subset of a Hausdorff space E. Let $(F,\{\Gamma_A\})$ be an H-space and let Y be a nonempty subset of F. Suppose $S,T : X \to 2^Y$ are two multivalued mappings such that*
(i) *S has local intersection property and $S(x)$ is nonempty for each $x \in X$;*
(ii) *for each $x \in X$, $T(x)$ is H-convex with respect to $S(x)$.*
Then T has a continuous selection.

Note that this is a simple consequence of Theorem 3.4.

(VI) Kim and Tan 2001 [18]

The following is also a simple consequence of Theorem 3.4:

Lemma 2 ([18])**.** *Let X be a non-empty paracompact set in a Hausdorff topological space and Y be a non-empty subset of a Hausdorff topological vector space. Let $S,T : X \to 2^Y$ be correspondences such that*
(1) *for each $x \in X$, $\emptyset \neq \operatorname{co} S(x) \subset T(x)$,*
(2) *S is transfer open inverse valued on X.*
Then T has a continuous selection.

Note that this was already well-known that time and that Y could be a mere convex space; see [25, Theorem 6]. Moreover, Lemma 1 of [18] shows that, if $S : X \multimap Y$ is transfer open valued, then $\bigcup_{x\in X} S(x)$ is open in Y. This simply follows from our definition of that concept.

In [18], from Lemma 2, its authors derived a collectively fixed point theorem, which was applied to two general equilibrium existence theorems and three quasi-variational inequalities.

(VII) Ding and J. Y. Park 2002 [11]

Motivated by [25, Theorem 1(i)], the following was obtained:

Theorem 2.1 ([11]). *Let X be a normal space, (Y, F) be an L-convex space, and $G : X \to 2^Y$ be a set-valued mapping such that*

(i) *G has nonempty L-convex values and satisfies the local intersection property, and*

(ii) *there exists a compact subset K of X and a finite subset M of Y such that $X \backslash K \subset \bigcup_{y \in M} \operatorname{Int} G^{-}(y)$.*

Then there exists a continuous selection $f : X \to Y$ of G such that $f = \phi \circ p$ where $\phi : \Delta_n \to Y$ and $p : X \to \Delta_n$ are both continuous and n is some positive integer.

Here L-convex spaces are particular forms of G-convex spaces and note that X is covered by finite number of $\operatorname{Int} G^{-}(y)'$s.

Based on Ding's misconception that L-spaces (not L-convex spaces) in [1] extend our G-convex spaces, he and his followers produced a very large number of papers on L-convex spaces or FC-spaces just imitating the G-convex space theory; see [31]. The same phenomenon is occurring for the so-called GFC-spaces.

(VIII) Ding and J. Y. Park 2002 [12]

The following particular form of Theorem 3.2(i) appeared:

Lemma 2.1 ([12]). *Let X be a normal space, (Y, Γ) be a G-convex space and $G : X \to 2^Y$ be a set-valued mapping such that*

(i) *G has a nonempty G-convex values,*

(ii) *$X = \bigcup_{y \in Y} \operatorname{Int} G^{-1}(y)$,*

(iii) *there exists a nonempty compact subset K of X and a finite subset M of Y such that $X \backslash K \subset \bigcup_{y \in M} \operatorname{Int} G^{-1}(y)$.*

Then there exists a continuous selection $f : X \to Y$ of G such that $f = \phi \circ \psi$ where $\phi : \Delta_n \to Y$ and $\psi : X \to \Delta_n$ are both continuous and n is some positive integer.

Note that $X = \bigcup \{\operatorname{Int} S^{-}(y) \mid y \in A\}$ for some $A \in \langle S(X) \rangle$.

(IX) Yu and Lin 2003 [39]

The following theorem was given as above:

Theorem 1 ([39]). *Let X be a paracompact topological space, $(Y, D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $S : X \multimap D$, $T : X \multimap Y$ be two maps satisfying the following conditions:*

(i) *for each $x \in X$, $M \in \langle S(x) \rangle$ implies $\Gamma_M \subset T(x)$;*

(ii) *$X = \bigcup \{\operatorname{Int} S^{-}(y) \mid y \in D\}$; and*

(iii) *$X \backslash K \subset \bigcup \{\operatorname{Int} S^{-}(y) \mid y \in M\}$ for some $M \in \langle D \rangle$.*

Then there exist $A \in \langle D \rangle$, with $|A| = n+1$ for some $n \in N$ and continuous mappings $g : \Delta_n \to \Gamma_A$, $\phi : X \to \Delta_n$ such that $f = g\phi$ is a continuous selection of T.

In [39], all spaces are assumed to be Hausdorff. Note that the paracompactness in Theorem 1 [39] can be replaced by the normality. In case X has an infinite open cover, a number of selection results were given in [25].

(X) Fakhar and Zafarani 2005 [13]

The following is a consequence of Theorem 3.5:

Proposition 2.1 ([13]). *Let $(X, D; \Gamma)$ be a G-convex space and Y be a normal space. Suppose that $S : Y \to 2^D$ and $T : Y \to 2^X$ are two multivalued mappings such that:*
(1) *for each $y \in Y$ and for every $L \in \langle S(y)\rangle$ one has $\Gamma(L) \subseteq T(y)$,*
(2) *there exists a nonempty paracompact subset K of Y and finite subset M of D such that $Y \setminus K \subseteq \bigcup\{\operatorname{Int} S^-(x) : x \in M\}$,*
(3) *$K = \bigcup\{\operatorname{Int} S^-(x) : x \in S(K)\}$,*
(4) *for each $A \in \langle D\rangle$ we have $\phi_A(\Delta_l) = \phi_L(\Delta_l)$ for any $L \subseteq A$ where $l + 1 = |L|$.*
Then T has a continuous selection.

Note that Y has a locally finite subcover of the cover $\{\operatorname{Int} S^-(x) : x \in X\}$ and hence condition (1) of Theorem 3.5 is satisfied.

(XI) Ding 2006 [10]

The following is a particular form of Theorem 3.2(i) for $Y = D$:

Theorem 2.1 ([10]). *Let X be a normal space and (Y, φ_N) be an FC-space. Let $F, G : X \to 2^Y$ be two set-valued mappings satisfying the following conditions:*
(i) For each $x \in X$, $G(x)$ is an FC-subspace of Y relative to $F(x)$,
(ii) There exists an $N = \{y_0, y_1, \dots, y_n\} \in \langle Y\rangle$ such that $X = \bigcup_{i=0}^n \operatorname{Int} F^-(y_i)$.
Then there exists a continuous selection $f : X \to Y$ of G such that $f = \varphi_N \circ \psi$ where $\varphi_N : \Delta_n \to Y$ and $\psi : X \to \Delta_n$ are both continuous.

Recall that FC-spaces are particular G-convex spaces; see [31].

Ding claimed that, as applications, some new collectively fixed point theorems and coincidence theorems for two families of set-valued mappings defined on product space of noncompact topological spaces are obtained under very weak assumptions.

(XII) Chen and Chang 2007 [6]

In [6], its authors proved the following well-known result of Browder and many others:

Lemma 2 ([6]). *Let Y be a compact set, X be a nonempty convex subset of a topological vector space E and $T : Y \to 2^X$ be a Φ-mapping. Then there exists a continuous function $f : Y \to X$ such that for each $y \in Y$, $f(y) \in T(y)$; that is, T has a continuous selection.*

Here Y should be a Hausdorff compact space.

(XIII) Chen, Chang and Liao 2007 [9]

In [9], the authors proved the following well-known result in [25]:

Lemma 2.1 ([9]). *Let Y be a compact set, X a G-convex space. Let and $T : Y \to 2^X$ be a Φ-mapping. Then there exists a continuous function $f : Y \to X$ such that for each $y \in Y$, $f(y) \in T(y)$, that is, T has a continuous selection.*

Here Y should be a Hausdorff compact space.

(XIV) Chen and Chang 2007 [7]

In [7], the authors cited the following result:

Theorem 1 ([22]). *Let X be a paracompact topological space, Y a hyperconvex metric space, and $F : X \to SA(Y)$ a quasi-lower semicontinuous mapping. Then F has a continuous selection.*

From this, the authors proved the following:

Theorem 2 ([7]). *Let X be a paracompact topological space, and Y a hyperconvex metric space. If $T : X \to 2^Y$ is a Φ-mapping, then T has a continuous selection.*

However, this follows from the earlier Theorem 3.4 since Horvath [17] noted that every hyperconvex metric space is an H-space.

(XV) Chen, Chang and Chung 2009 [8]

In [8], the following was proved:

Lemma 2 ([8]). *Let X be a compact set, and Y a nonempty almost-convex subset of a Hausdorff topological vector space E. If $T : X \to 2^Y$ is a generalized Φ-mapping with a companion mapping $F : X \to 2^Y$, then there exists a continuous function $f : X \to Y$ such that for each $x \in X$, $f(x) \in T(x)$, that is, T has a continuous selection.*

Here X should be a Hausdorff compact space and E is not necessarily Hausdorff. This lemma is well-known for a long period and follows from Lemma 3.3 and Theorem 3.2(i).

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