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REMARKS ON CERTAIN COERCIVITY IN GENERAL KKM THEOREMS

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Abstract. In a previous work of the author [5], we obtained three general KKM type theorems for abstract convex spaces. In a recent review [10], we showed that two of them can be stated for intersectionally closed-valued KKM maps in the sense of Luc et al. [1]. Each of such KKM type theorems contains a large number of previously known particular forms. In the present review, we recall some of them due to the author in order to give comments or minor corrections of the proofs of them.

1. Introduction

In a previous work [5], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in this paper, resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors. In a resent paper [10], we showed that two of them can be stated for intersectionally closed-valued KKM maps in the sense of Luc et al. [1], and recalled some historically important previous particular versions of our KKM type theorems in [5] in order to give a short history on each of them. Moreover, further comments on related works were given.

Since 2006, while the author was constructing the abstract convex space theory, we used to adopt slightly incorrect forms of the following coercivity or compactness condition (ii) for KKM type theorems:

Let $(E, D; \Gamma)$ be an abstract convex space and $G : D \multimap E$ a KKM map such that there exists a nonempty compact subset K of E such that

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(ii) *for each nonempty finite subset* $N \subset D$*, there exists a compact* Γ-*convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

After the appearance of [5] and [10], we noticed that a number of our previous works adopted incorrect forms of condition (ii) which have to be corrected in view of the new method developed in the KKM theory. In the present review, we recall some KKM type theorems with such incorrect coercivity due to the author and give comments or minor corrections of the proofs of them.

2. General KKM theorems for abstract convex spaces

We follow our recent works [5-10]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup \{G(y) \mid y \in A\} \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

We have abstract convex subspaces as the following simple observation shows:

Proposition 2.1 ([?]). *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D' \rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ for \ A \in \langle D' \rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace relative to D'.

Proposition 2.2 ([?]). *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a topological space. If $F \in \mathfrak{KC}(E, D, Z)$, then $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$.*

Proof. Suppose that a closed-valued map $G' : D' \multimap \overline{F(X)}$ satisfies

$$F|_X(\Gamma'_A) \subset G'(A) \ \text{ for all } A \in \langle D' \rangle.$$

Define a map $G : D \to 2^Z$ by

$$G(y) := \begin{cases} G'(y) & \text{for } y \in D' \\ \overline{F(X)} & \text{otherwise} \end{cases}$$

Then

$$F(\Gamma_A) = F|_X(\Gamma'_A) \subset G'(A) = G(A) \ \text{ for } A \in \langle D' \rangle; \text{ and}$$

$$F(\Gamma_A) \subset \overline{F(X)} = G(A) \ \text{ for } A \in \langle D \rangle \setminus \langle D' \rangle.$$

Since $F \in \mathfrak{KC}(E, D, Z)$ and G has closed values, the family $\{G(y)\}_{y \in D}$ has the finite intersection property, and hence so does its subfamily $\{G'(y)\}_{y \in D'}$. Therefore, $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$.

Remark 2.3. In Proposition 2.2, we immediately have the following:

1. If $1_E \in \mathfrak{KC}(E, D, E)$ and $1_{\overline{X}} : \overline{X} \to \overline{X}$ are the identity maps, then $1_{\overline{X}} = 1_E|_{\overline{X}} \in \mathfrak{KC}(\overline{X}, D', \overline{X})$.

2. Further if X is closed in E, then $1_X = 1_E|_X \in \mathfrak{KC}(X, D', X)$ and $(X, D'; \Gamma')$ satisfies the partial KKM principle.

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems [5,10]:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) G has closed [resp., open] values; and

(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).

Then $\{G(z)\}_{z \in D}$ has the finite intersection property.

Further, if
(3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some* $M \in \langle D\rangle$,
then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Proof. The first part is a simple consequence of definition. For the second part, let $K := \bigcap_{z\in M} \overline{G(z)}$. Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property.

Consider the following related four conditions:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [1]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [1], its authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c).

From the partial KKM principle we have a whole intersection property of the Fan type. The following is a slightly different form of [10, Theorem B] with a slightly corrected proof:

Theorem B. *Let* $(E, D; \Gamma)$ *be an abstract convex space satisfying the partial KKM principle* [*that is,* $1_E \in \mathfrak{KC}(E, D, E)$] *and* $G : D \multimap E$ *a map such that*
(1) $\overline{G}$ *is a KKM map* [*that is,* $\Gamma_A \subset \overline{G}(A)$ *for all* $A \in \langle D\rangle$]*; and*
(2) *there exists a nonempty compact subset* K *of* E *such that either*
(i) $\bigcap_{z\in M} \overline{G(z)} \subset K$ *for some* $M \in \langle D\rangle$*; or*
(ii) *for each* $N \in \langle D\rangle$*, there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{z\in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.
Furthermore,
(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$*;*
(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Suppose that $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} = \emptyset$; that is, $K \subset \bigcup\{E \setminus \overline{G(z)} \mid z \in N\}$ for some $N \in \langle D\rangle$.

Case (i): By Theorem A, we have an $x \in \bigcap\{\overline{G(z)} \mid z \in D\} \subset \bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ by (i). But, we have $x \in K \subset \bigcup\{X \setminus \overline{G(z)} \mid z \in D\}$, a contradiction.

Case (ii): Let L_N be the compact Γ-convex subset of E in (ii). Define $G' : D' \multimap L_N$ by $G'(z) := G(z) \cap L_N$ for $z \in D'$. Then $A \in \langle D'\rangle$ implies

$\Gamma'_A := \Gamma_A \subset \overline{G}(A) \cap L_N \subset \overline{G'}(A) \subset \overline{L_N}$ by (1); and hence $\overline{G'} : D' \multimap \overline{L_N}$ is a KKM map on $(\overline{L_N}, D'; \Gamma')$ with closed values. Note that $1_E \in \mathfrak{KC}(E, D, E)$ implies $1_{\overline{L_N}} \in \mathfrak{KC}(\overline{L_N}, D', \overline{L_N})$ by Remark 2.3. Therefore, by Theorem A, $\{\overline{G'(z)} \mid z \in D'\}$ has the finite intersection property and $\bigcap\{\overline{G'(z)} \mid z \in D'\} \neq \emptyset$. For any $y \in \bigcap\{\overline{G'(z)} \mid z \in D'\} \subset \overline{L_N}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{E \setminus \overline{G(z)} \mid z \in N\}$, we have $y \notin \overline{G(z)}$ for some $z \in N \subset D'$. This is a contradiction.

Therefore, we must have $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

(α) Since G is transfer closed-valued, we have

$$K \cap \bigcap_{z \in D} G(z) = K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued, we have

$$\overline{\bigcap_{z \in D} G(z)} = \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

This implies the conclusion.

Theorem B can be extended to $F \in \mathfrak{KC}(E, D, Z)$ instead of $1_E \in \mathfrak{KC}(E, D, E))$ as the following [10, Theorem C] shows. Its proof is repeated here for completeness:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Proof. Case (i): Since $F(\Gamma_N) \subset \overline{G}(N)$ for each $N \in \langle D \rangle$ by (1), we have

$$F(\Gamma_N) \subset F(E) \cap \overline{G}(N) \subset \overline{F(E)} \cap \overline{G}(N) =: G'(N),$$

where $G'(y) := \overline{F(E)} \cap \overline{G(y)}$ is closed for each $y \in D$. Then, by Proposition 2.2 on $(E, D', \overline{F(E)})$, $\{G'(y) \mid y \in D\}$ has the finite intersection property. Since the requirement (i) implies

$$\overline{F(E)} \cap K \supset \overline{F(E)} \cap \bigcap_{y \in M} \overline{G(y)} = \bigcap_{y \in M} G'(y),$$

$\bigcap_{y \in M} G'(y)$ is compact. Therefore $\bigcap \{G'(y) \mid y \in D\} \neq \emptyset$ by Theorem A and hence

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Case (ii): Suppose that

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} = \emptyset.$$

Since $\overline{F(E)} \cap K$ is compact, $\overline{F(E)} \cap K \subset \bigcup \{Z \setminus \overline{G(y)} \mid y \in N\}$ for some $N \in \langle D \rangle$. Let L_N be the Γ-convex subset of E in (ii). Define $G' : D' \multimap \overline{F(L_N)}$ by $G'(y) := \overline{G(y)} \cap \overline{F(L_N)}$ for $y \in D'$. For each $A \in \langle D' \rangle$, define $\Gamma'_A := \Gamma_A \cap L_N$. Then $(L_N, D'; \Gamma')$ is an abstract convex space. Moreover,

$$(F|_{L_N})(\Gamma'_A) \subset F(\Gamma_A) \cap F(L_N) \subset \overline{G}(A) \cap \overline{F(L_N)} = G'(A)$$

for each $A \in \langle D' \rangle$ by (1); and hence $G' : D' \multimap \overline{F(L_N)}$ is a KKM map w.r.t. $F|_{L_N}$ on the abstract convex space $(L_N, D'; \Gamma')$ with closed values in $\overline{F(L_N)}$. Since $F \in \mathfrak{KC}(E, D, Z)$, by Proposition 2.2, we have $F|_{L_N} \in \mathfrak{KC}(L_N, D', \overline{F(L_N)})$ and hence, $\{G'(y) \mid y \in D'\} = \{\overline{G(y)} \cap \overline{F(L_N)} \mid y \in D'\}$ has the finite intersection property. Since we assumed that $\overline{F(L_N)}$ is compact, each $G'(y)$ is compact. Hence $\bigcap \{G'(y) \mid y \in D'\} \neq \emptyset$ as in Theorem A and there exists a

$$z \in \bigcap_{y \in D'} G'(y) = \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K$$

by (ii). Since $z \in K$ and $z \in \overline{F(L_N)}$, we have $z \in \bigcup \{Z \setminus \overline{G(y)} \mid y \in N\}$ by our assumption. So $z \notin \overline{G(y)}$ for some $y \in N \subset D'$, and hence $z \notin \bigcap \{\overline{G(y)} \mid y \in D'\}$. This contradicts $z \in \bigcap \{G'(y) \mid y \in D'\}$. Therefore, we must have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(α) Since G is transfer closed-valued, we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} G(y) = \overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued, we have

$$\overline{\bigcap_{y \in D} G(y)} = \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

This implies the conclusion.

Remark 2.4. 1. For $E = Z$ and $F = 1_E$, Theorem C reduces to Theorem B.

2. We give another relatively simple proof of Theorem C just following the method in [2] as follows:

Another Proof of Theorem C. As in [2, Proposition 3], by Condition (1), for any $N \in \langle D \rangle$, we have $F(\Gamma_N) \subset F(E) \cap \overline{G}(N) \subset \bigcup_{y \in N} \{\overline{F(E)} \cap \overline{G(y)}\}$. Since $F \in \mathfrak{KC}(E, D, Z)$, the family $\{\overline{F(E)} \cap \overline{G(y)}\}_{y \in D}$ has the finite intersection property.

(i) Let $K' := \overline{F(E)} \cap \bigcap_{y \in M} \overline{G(y)} \cap K \neq \emptyset$. Then the family $\{\overline{G(y)} \cap K'\}_{y \in D}$ is a family of closed sets in the compact set K and hence has the whole intersection property, that is, $\bigcap_{y \in D} \overline{G(y)} \cap \overline{F(E)} \cap K \neq \emptyset$.

(ii) Recall that $\{\overline{F(L_N)} \cap \overline{G(y)}\}_{y \in D'}$ has the finite intersection property as in the first proof. Since $\overline{F(L_N)}$ is compact, it has the nonempty whole intersection that is in K by the assumption (ii). Let $K' := \overline{F(X)} \cap \bigcap_{y \in N} \overline{G(y)} \cap K \supset \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \cap K \neq \emptyset$. Then as in the above, we have the same conclusion.

This proves the first part of Theorem C. For (α) and (β), follow the first proof.

3. Comments on the KKM theorems in our previous works

Consider the following coercivity condition in Theorem C:

(2) *there exists a nonempty compact subset K of Z such that*

C(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)}.$$

For the particular case $F = 1_E$, condition C(ii) reduces to the following condition in Theorem B:

B(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

In a number of our previous works, instead of B(ii), we adopted the following:

B(ii)′ *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

In each of the following special cases, condition B(ii)$'$ becomes B(ii):

(1) Replace L_N by its closure $\overline{L_N}$.

(2) $E = Z$ is Hausdorff (that is, every compact subset is closed).

(3) K is closed.

(4) L_N is closed.

In this section, we review our previous papers related to Theorem B with B(ii)$'$ instead of B(ii), and give comments or corrections on them if necessary:

(I) 2006 [2]

In [2], an *abstract convex space* $(E, D; \Gamma)$ and related concepts were defined for a nonempty set E, not necessarily topologized.

We have abstract convex subspaces:

Proposition 1 ([2]). *For an abstract convex space $(E \supset D; \Gamma)$, let X be a Γ-convex subset of E, and D' a nonempty subset of $X \cap D$. Let $\Gamma' : \langle D' \rangle \multimap X$ be a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ \ for \ A \in \langle D' \rangle.$$

Then $(X \supset D'; \Gamma')$ itself is an abstract convex space called a subspace.

Proposition 2 ([2]). *Let $(E \supset D; \Gamma)$ be an abstract convex space, $(X \supset D'; \Gamma')$ a subspace, and Z a set. If $F \in \mathfrak{K}(E, Z)$, then $F|_X \in \mathfrak{K}(X, Z)$.*

These are origins of Propositions 2.1 and 2.2.

(II) 2006 [3]

The following generalize [2, Propositions 1 and 2] with the same terminology, resp.:

Lemma 1 ([?]). *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D' \rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ \ for \ A \in \langle D' \rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace.

Lemma 2 ([?]). *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a set. If $F \in \mathfrak{K}(E, Z)$, then $F|_X \in \mathfrak{K}(X, Z)$.*

Note that Lemma 2 holds for $\mathfrak{KC}$ and $\mathfrak{KO}$ instead of $\mathfrak{K}$ whenever Z is a topological space.

(III) 2008 [4]

In [4], we claimed the following:

Theorem 8.2 ([4]). *Let $(X, D; \Gamma)$ be an abstract convex topological space satisfying the partial KKM principle, K be a nonempty compact subset of X, and $G : D \multimap X$ a map such that*

(8.2.1) $\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ [*that is, G is transfer closed-valued*];
(8.2.2) $\overline{G}$ *is a KKM map; and*
(8.2.3) *either*
(i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D\rangle$; *or*
(ii) *for each* $N \in \langle D\rangle$, *there exists a compact* Γ-*convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

This is the first result adopting condition B(ii)$'$. In the proof, we made incorrect statement that "since $(X, D; \Gamma)$ is a KKM space, so is $(L_N, D'; \Gamma')$." Now Theorem B is a correct generalized form of the above statement.

(IV) 2009 [5]

In [5], the following are obtained:

(1) Propositions 2.1 and 2.2 in this paper were given.

(2) Theorem A was given as [5, Proposition 2.7].

(3) Theorem B with the coercivity condition B(ii)$'$ for a transfer closed-valued map $G : D \multimap Z$ [4, Theorem 8.2] was introduced as [5, Theorem 2.9].

(4) Theorem C was proved as [5, Theorem 2.10] by applying Proposition 2.2.

(V) 2010 [6]

In [6], an equivalent form of [4, Theorem 8.2] was introduced and not affected to other parts of [6].

(VI) 2010 [7]

In [7], we need the following:

(1) The coercivity condition B(ii)$'$ in [7, Theorem 3] should be replaced by B(ii).

(2) All X in Theorem 4 (p.1034) should be replaced by E.

(VII) 2011 [8]

In [8], the coercivity condition B(ii)$'$ or its equivalents appeared in Theorem 2.1, Corollary 2.1, Theorems 6.1, 6.2 and 6.3. They should be replaced by the condition B(ii) or corresponding equivalents.

This is caused by the incorrect statement that "since $(E, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$; see [3, Lemma 2]" in the proof of [8, Theorem 2.1].

(VIII) 2011 [9]

The paper [9] is based on the following with an incorrect proof:

Proposition 3.2 ([9]). *Let $(E, D; \Gamma)$ be an abstract convex space and $(X, D'; \Gamma')$ a subspace. If $(E, D; \Gamma)$ satisfies the partial KKM principle, then so does $(X, D'; \Gamma')$.*

Note that a correct form of this is given in Remark 2.3.1, that is, we have to assume X is closed.

In [9], based on the preceding proposition, incorrect forms [9, Theorems 4.1 and 4.2] of Theorem B with condition B(ii)$'$ instead of B(ii) were claimed.

(IX) 2011 [10]

After introducing Propositions 2.1 and 2.2 from [5], we obtained the following:

(1) Theorem A same as in this paper was proved.

(2) Theorem B with condition B(ii)$'$ instead of (B.ii) was claimed. The main part of its proof was based on results in [4,5,7]. Note that Theorem B with B(ii)$'$ does not follow from Theorem C with $F = 1_E$ and hence, seems to be incorrect. Note that, in Section 4 of [10], the set L_N should be replaced by $\overline{L_N}$.

(3) Theorem C was proved by the first proof in this paper.

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