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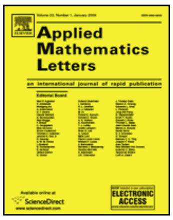

The Fan minimax inequality implies the Nash equilibrium theorem

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ABSTRACT

We show that in an abstract convex space $(E, D; \Gamma)$, the partial KKM principle implies the Ky Fan minimax inequality, from which we deduce a generalization of the Nash equilibrium theorem.

1. Introduction

It is well-known that the Brouwer fixed point theorem, the Sperner combinatorial lemma, the Knaster–Kuratowski–Mazurkiewicz (for short, KKM), the Kakutani fixed point theorem, the Nash equilibrium theorem, Ky Fan's theorem on sets with convex sections, the Fan minimax inequality, the Fan–Browder fixed point theorem and many others, mainly in the KKM theory, are mutually equivalent; see [1,2].

Recall that the celebrated Nash equilibrium theorem was first proved using the Brouwer or the Kakutani fixed point theorem; see [3,4]. Later, on the basis of his own KKM lemma, Fan proved the Nash theorem by applying his result on sets with convex sections within the framework of the KKM theory; see [5,6]. This section theorem is equivalent to the Fan–Browder fixed point theorem, from which we can deduce the Nash theorem, the von Neumann–Sion minimax theorem and a number of important results; see [2,7,8] and the references therein.

Nowadays the Nash theorem is known to be one of the most important applications of the Fan minimax inequality [9]. The inequality and its various generalizations are very useful tools in various fields in mathematical sciences, for example, nonlinear analysis, especially in fixed point theory, variational inequalities, various equilibrium theory, mathematical programming, partial differential equations, game theory, impulsive control, and mathematical economics; see [10,11] and the references therein.

Since the inequality appeared in 1972, it has been followed by a large number of generalizations and applications in the KKM theory for convex subsets of topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, generalized convex spaces in the sense of Park, and spaces of other types. These are all unified within the category of abstract convex spaces; see [2] and the references therein.

In fact, in our recent works [2,12–14], we studied elements or foundations of the KKM theory on abstract convex spaces. The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. We noticed that many important results in the KKM theory are closely related to spaces satisfying the partial KKM principle. Moreover, many such results are equivalent to each other.

In this work, we introduce a new form of an abstract KKM theorem related to the multimaps having intersectionally closed values in the sense of Luc et al. [15]. We show that this KKM theorem implies various forms of the Fan minimax

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inequality in our recent works [16–18]. Our aim in this work is to show that such a minimax inequality implies a new generalization of the Nash equilibrium theorem in abstract convex spaces.

In Section 2, we introduce the basic facts on abstract convex spaces from our previous works [12–14]. Section 3 deals with a new general KKM theorem on KKM maps having intersectionally closed values due to Luc et al. [15] and its applications to several Fan type analytic alternatives or minimax inequalities. In Section 4, we deduce a generalization of the Nash theorem in abstract convex spaces from an analytical alternative or a minimax inequality.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Recall the following from [2,12–14]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the Γ*-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_N \mid N \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ*-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D'\rangle})$ is called a Γ*-convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be Γ*-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In the case $E = D$, suppose that $(E; \Gamma) := (E, E; \Gamma)$.

For examples of abstract convex spaces, see [2,7,8,12–14] and the references therein.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [2,12–14], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the partial KKM principle.

Examples of KKM spaces are given in [2,7,8,12–14] and the references therein. Here we give only two examples due to the author as follows.

Example. (1) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, for Δ_n with vertices $\{e_i\}_{i=0}^n$, Δ_J is the face corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

(2) A ϕ_A*-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous maps $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [19]. Some authors' GFC-spaces or FC-spaces are ϕ_A-spaces or particular forms of them, respectively.

(3) Note that G-convex spaces contain convex subsets of topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces, ϕ_A-spaces, and spaces of other types. Note also that every G-convex space satisfies the KKM principle.

3. From the KKM principle to the minimax inequality

In this section, we follow mainly our recent work [14–16].

Consider the following four related conditions from [13]:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.
(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued*).
(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

In [13], the authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is an appropriate time to deal with condition (b) instead of (c) in the KKM theory.

For a multimap $G : D \multimap E$, consider the following four related conditions:

(a) $\bigcup_{z\in D} G(z) = E$ implies $\bigcup_{z\in D} \operatorname{Int} G(z) = E$.
(b) $\operatorname{Int} \bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *unionly open-valued* [13]).
(c) $\bigcup_{z\in D} G(z) = \bigcup_{z\in D} \operatorname{Int} G(z)$ (G is *transfer open-valued*).
(d) G is open-valued.

Lemma 1 (*[13]*). *The multimap G is intersectionally closed-valued (resp., transfer closed-valued) if and only if its complement G^c is unionly open-valued (resp., transfer open-valued).*

We have the following form of the KKM type theorems [14–16]:

Theorem 1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $G : D \multimap Z$ a map such that:*

(1) *$\overline{G}$ is a KKM map; and*
(2) *there exists a nonempty compact subset K of E such that either:*
 (i) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, and*

$$\overline{L_N} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore:

(α) if G is transfer closed-valued, then $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$;
(β) if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Theorem 1 can be reformulated to many equivalent statements as in [2,10–12]. We give only the following *analytic alternative*:

Theorem 2. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and suppose that $\alpha, \beta \in \mathbb{R}$, and that $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ are extended real-valued functions. Suppose that:*

(1) *for each $z \in D$, $G(z) := \{y \in E \mid f(z, y) \le \alpha\}$ is intersectionally closed;*
(2) *for each $y \in E$, we have*

$$\operatorname{co}_\Gamma\{z \in D \mid f(z, y) > \alpha\} \subset \{x \in E \mid g(x, y) > \beta\}; \quad \textit{and}$$

(3) *the compactness condition (2) in Theorem 1 holds.*

Then either:

(i) *there exists a $y_0 \in E$ such that $f(z, y_0) \le \alpha$ for all $z \in D$; or*
(ii) *there exists an $\hat{x} \in E$ such that $g(\hat{x}, \hat{x}) > \beta$.*

Lemma 2. *Under the hypothesis of Theorem 2, assume (3) and the negation of (ii). Then the map $G : D \multimap E$ is a KKM map.*

Proof. The negation of (ii) is that $g(x, x) \le \beta$ for all $x \in E$. Suppose, on the contrary, that there exists a finite $N \subset D$ such that $\Gamma_N \not\subset G(N)$. Then there exists a $y \in \Gamma_N$ such that $y \notin G(z)$ or $f(z, y) > \alpha$ for all $z \in N$. Hence $N \subset \{z \in D \mid f(z, y) > \alpha\}$ and, by (2), we have $\Gamma_N \subset \{y \in E \mid g(z, y) > \beta\}$. Since $y \in \Gamma_N$, we have $g(y, y) > \beta$. This contradicts our supposition. □

Proof of Theorem 2. Suppose (ii) does not hold. Then, by Lemma 2, G is a KKM map. Therefore, all the requirements of the KKM Theorem 1 are satisfied and $\{G(z)\}_{z\in D}$ has the nonempty intersection. Hence, there exists a $y_0 \in \bigcap_{z\in D} G(z) \subset E$. So, $f(z, y_0) \le \alpha$ for all $z \in D$. Hence (i) holds. □

Corollary 2.1. *Under the hypothesis of Theorem 2 with $\alpha = \beta = 0$, if $g(x, x) \le 0$ for all $x \in E$, then*
 (i) *there exists a $y_0 \in E$ such that $f(z, y_0) \le 0$ for all $z \in D$.*

We define new concepts as follows; see [14–16]:

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. An extended real-valued function $f : D \times E \to \overline{\mathbb{R}}$ is said to be *generally lower* (resp., *upper*) *semicontinuous* (g.l.s.c.) (resp., (g.u.s.c.)) on E if, for each $z \in D$, $\{y \in E \mid f(z, y) \le r\}$ (resp., $\{y \in E \mid f(z, y) \ge r\}$) is intersectionally closed for each $r \in \overline{\mathbb{R}}$.

This is a generalization of the transfer l.s.c. due to Tian. If the intersectionally closed sets are replaced by mere closed sets, then $f(z, \cdot)$ is said to be l.s.c. (resp., u.s.c.).

Definition. For an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* (resp., *quasiconvex*) if $\{x \in E \mid f(x) > r\}$ (resp., $\{x \in E \mid f(x) < r\}$) is Γ-convex for each $r \in \overline{\mathbb{R}}$.

From Corollary 2.1, we obtain the following:

Corollary 2.2. *Let $(E; \Gamma)$ be a compact abstract convex space and $f, g : E \times E \to \overline{\mathbb{R}}$ two functions such that:*

(1) *$f(x, y) \le g(x, y)$ for every $(x, y) \in E \times E$ and $g(x, x) \le 0$ for all $x \in E$;*
(2) *$y \mapsto f(x, y)$ is g.l.s.c. on E for every $x \in E$; and*
(3) *$x \mapsto g(x, y)$ is quasiconcave on E for every $y \in E$.*

Then there exists a $y_0 \in E$ such that $f(x, y_0) \le 0$ for all $x \in E$.

From Theorem 2, we clearly have the following Fan type minimax inequality:

Theorem 3. *Under the hypothesis of Theorem 2, if $\alpha = \beta = \sup_{x \in X} g(x, x)$, then*

(a) *there exists a $y_0 \in E$ such that*

$$f(z, y_0) \le \sup_{x \in E} g(x, x) \quad \text{for all } z \in D; \text{ and}$$

(b) *we have the following minimax inequality:*

$$\inf_{y \in E} \sup_{z \in D} f(z, y) \le \sup_{x \in E} g(x, x).$$

Note that Theorem 3 is equivalent to Theorem 2; see [2] and the references therein.

4. From the minimax inequality to the Nash equilibrium theorem

In this section, we apply Theorem 2 to a direct proof of a generalization of the Nash theorem in [6].

Let $I = \{1, \dots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple

$$\Lambda := \{X_1, \dots, X_n; u_1, \dots, u_n\},$$

where the nonempty set X_i is the ith player's pure strategy space and $u_i : X = \prod_{i=1}^n X_i \to \mathbb{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Suppose that $X_{-i} = \prod_{j \in I \setminus \{i\}} X_j$ and denote by x and x_{-i} an element of X and X_{-i}, respectively. A strategy n-tuple $(y_1^*, \dots, y_n^*) \in X$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(y_i^*, y_{-i}^*) \ge u_i(x_i, y_{-i}^*) \quad \text{for all } x_i \in X_i \text{ and } i \in I.$$

The following is known:

Lemma 3. *Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $X := \prod_{i \in I} X_i$ be equipped with the product topology and $D := \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.*

Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be a family of G-convex spaces. Then $(X, D; \Gamma)$ is a G-convex space.

The following is a generalization of the Nash theorem in [6, Theorem 4]:

Theorem 4. *Let $\Lambda := \{X_1, \dots, X_n; u_1, \dots, u_n\}$ be a game where each $(X_i; \Gamma_i)$ is a compact abstract convex space such that $(X; \Gamma) := (\prod_{i=1}^n X_i; \Gamma)$, where Γ is given as above, and satisfies the partial KKM principle, and each $u_i : X \to \mathbb{R}$ is continuous. If for each i and for any given point $x_{-i} \in X_{-i}$, $x_i \mapsto u_i(x_i, x_{-i})$ is a quasiconcave function on X_i, then there exists a Nash equilibrium for Λ.*

Proof. For each i, let $e_i : X_i \hookrightarrow X$ be an embedding for some $a = (a_1, \dots, a_n)$ such that $e_i : x_i \in X_i \mapsto (x_i, a_{-i}) \in X$. Suppose that $D_i := e_i(X_i) \subset X$ and $\Gamma^i := \Gamma|_{\langle D_i \rangle}$. Then $(X \supset D_i; \Gamma^i)$ is an abstract convex space and easily seen to satisfy the partial KKM principle. Note that D_i is a Γ^i-convex subset, and $z \in D_i$ implies $z = (z_i, a_{-i}) \in X$.

For $u_i : X \to \mathbb{R}$, define $f_i : D_i \times X \to \mathbb{R}$ and $g_i : X \times X \to \mathbb{R}$ by

$$f_i(z, y) := u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) \quad \text{and} \quad g_i(x, y) := u_i(x_i, y_{-i}) - u_i(y_i, y_{-i}),$$

respectively. Then $f_i(z, y) = g_i(z, y)$ on $D_i \times X$ and $g_i(x, x) = 0$ for all $x \in X$.

Now we apply Theorem 2 for the abstract convex space $(X, D_i; \Gamma^i)$ with $\alpha = \beta = 0$.

(1) Since each u_i is continuous, for each $z \in D_i$, the set

$$\{y \in X \mid f_i(z, y) > 0\} = \{y \in X \mid u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) > 0\}$$

is open.

(2) For each $y \in X$, $z \mapsto u_i(z_i, y_{-i})$ is quasiconcave. Therefore $\{z \in D_i \mid u_i(z_i, y_{-i}) > r\}$ is Γ^i-convex for each $r \in \mathbb{R}$ and hence

$$\{z \in D_i \mid f_i(z, y) = u_i(z_i, y_{-i}) - u_i(y_i, y_{-i}) > 0\}$$

is convex and contained in $\{x \in X \mid g_i(x, y) > 0\}$.

(3) X is compact. Consequently, all requirements (1)–(3) of Theorem 2 are satisfied. Moreover, the conclusion (ii) does not hold since $g_i(x, x) = 0$ for all $x \in X$. Therefore, we have:

(i) there exists a $y^i \in X$ such that $f_i(z, y^i) \le 0$ for all $z \in D$; that is,

$$u_i(y^i_i, y^i_{-i}) \ge u_i(z_i, y^i_{-i}) \quad \text{for all } z_i \in X_i \quad \text{and} \quad i \in I.$$

Then $y^* := (y^1_1, \ldots, y^n_n)$ is the required Nash equilibrium. □

Remark. (1) Ziad [20] indicated that the Nash theorem follows from the Fan inequality. The above proof completes this matter in a general form.

(2) Since the Nash theorem follows from the Fan inequality and the latter has a large number of generalizations for various abstract convex spaces, our argument works for corresponding generalizations of the Nash theorem. More precisely, since all convex subsets of topological vector spaces, Lassonde type convex spaces, Horvath type H-spaces (e.g., hyperconvex metric spaces), ϕ_A-spaces, G-convex spaces, and spaces of many other types are abstract convex spaces satisfying the partial KKM principle, Theorem 4 can be applied to all of them; see [2,12–14].

For example, the following is the Nash theorem of [6, Theorem 4]:

Corollary 4.1. *Let $\Lambda := \{X_1, \ldots, X_n; u_1, \ldots, u_n\}$ be a game where each X_i is a nonempty compact convex subset of a topological vector space and each u_i is continuous. If for each $i \in I$ and for any given point $x_{-i} \in X_{-i}$, $x_i \mapsto u_i(x_i, x_{-i})$ is a quasiconcave function on X_i, then there exists a Nash equilibrium for Λ.*

Remark. (1) In 2006, Torres-Martínez [21] showed that a particular type of the Nash equilibrium theorem [3,4], and hence Theorem 4, implies the Brouwer theorem. Therefore, all results in this work are all equivalent to the Brouwer theorem.

(2) For generalizations of the Nash theorem of other types, see [2,7] and the references therein.

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