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Remarks on Fixed Points of Generalized Upper Hemicontinuous Maps

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Abstract

In this short note, we give some variants of the fixed point theorems in [7] on generalized upper hemicontinuous (g.u.h.c.) multimaps whose domains and ranges may have different topologies. Our new theorems refine our previous results in [7] and simply generalize Balaj's two map versions [1] of Halpern's fixed point theorems.

Key words: Convex space, Generalized upper hemicontinuous (g.u.h.c.), Algebraic boundary, Fixed point, Coincidence point.

AMS Subject Classification: 47H10, 54H25.

1 Introduction

The celebrated Kakutani fixed point theorem in 1941 for convex-valued upper semicontinuous multimaps initiated the study of fixed points of multimaps in the last seven decades. The Kakutani theorem and its numerous generalizations were applied to game theory, mathematical economics, systems and control theory, coincidence theory, minimax theory, variational inequalities, convex analysis, and many equilibrium theorems.

In our earlier works [4,5], we unified, improved, and generalized a lot of fixed point theorems on Kakutani maps or acyclic maps defined on convex subsets of topological vector spaces.

One of the main fixed point theorems in [5] concerns with convex-valued generalized upper hemicontinuous maps whose domains and ranges may have different topologies. Moreover, in our previous work [7], we obtained some refined and generalized versions of main theorems in [4,5] with slightly different proofs. Consequently, we gave there new fixed point theorems on generalized upper hemicontinuous multimaps whose domains and ranges may have different topologies and these include known theorems appeared in almost 50 published works.

Almost same time to [7], Balaj [1] obtained a unified generalization of two fixed point theorems of Halpern [3] and applied it to a coincidence theorem and a minimax inequality. Note that Balaj's theorem is a two map version and, on the surface, seems to be not related to the results in [7].

Our aim in this paper is to reformulate the main results of [7] to include Balaj's results for two maps. Actually, these results are simple refined versions of the corresponding ones in [7] and simply generalize Balaj's two map versons [1] of Halpern's fixed point theorems.

In this paper, we assume all topological spaces are Hausdorff. Moreover, for simplicity, all topological vector spaces (t.v.s.) are real, contrary to our previous work [7].

2 Preliminaries

A *convex space* X is a nonempty convex set equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$.

Any multimap (simply, map) is assumed to be nonempty valued. For a topological space X and a t.v.s. E with its topological dual E^*, a multimap $F : X \to 2^E$ is said to be

(i) *upper semicontinuous* (*u.s.c.*) if for each $x \in X$ and open neighborhood V of $F(x)$ in E, there exists an open neighborhood U of x in X such that $F(U) \subset V$;

(ii) *upper demicontinuous* (*u.d.c.*) if for each $x \in X$ and open half-space H in E containing $F(x)$, there exists an open neighborhood U of x in X such that $F(U) \subset H$;

(iii) *upper hemicontinuous* (*u.h.c.*) if for each $p \in E^*$ and for any real α, the set $\{x \in X : \sup p(F(x)) < \alpha\}$ is open in X; and

(iv) *generalized u.h.c.* (*g.u.h.c.*) if for each $p \in E^*$, the set $\{x \in X : \sup p(F(x)) \geq p(x)\}$ is closed in X.

Note that (i)⇒(ii)⇒(iii)⇒(iv). In our earlier works [4,7], we unified a large number of generalizations of the Kakutani theorem to maps of the above-mentioned types.

Recall that a real function $g : X \to \mathbf{R}$ on a topological space X is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X : g(x) > r\}$ [resp., $\{x \in X : g(x) < r\}$] is open for each $r \in \mathbf{R}$. If X is a convex set, then g is *quasiconcave* [resp., *quasiconvex*] if $\{x \in X : g(x) > r\}$ [resp., $\{x \in X : g(x) < r\}$] is convex for each $r \in \mathbf{R}$.

We use the following form of the existence theorem of maximizable quasiconcave functions on convex spaces due to Park and Bae [9].

Theorem 0. *Let X be a convex space and $\widehat{X}$ the set of all u.s.c. quasiconcave real functions on X. Suppose that*

(0.1) *for each $x \in X$, $T(x) \supset S(x) \neq \emptyset$ and $T(x)$ is a convex subset of $\widehat{X}$;*

(0.2) *for each $g \in \widehat{X}$, $S^{-}(g)$ is open in X; and*

(0.3) *there exists a c-compact set $L \subset X$ and a nonempty compact set $K \subset X$ such that for every $x \in X \setminus K$ and $g \in T(x)$, $g(x) < \max g[x, L]$.*

Then there exist an $\overline{x} \in K$ and a $g \in T(\overline{x})$ such that $g(\overline{x}) = \max g(X)$.

Let X be a subset of a t.v.s. E and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined by Halpern [3] as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For any $p \in E^*$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| \ : \ u \in U, \ v \in V\}.$$

Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, resp., with respect to E.

3 Main results

We had the following [7, Theorem 1] by applying Theorem 0. We give its proof for completeness.

Theorem 1. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and $F : X \to 2^E$. Suppose that, for each $p \in E^*$,*

(1.0) *$p|_X$ is continuous on X;*

(1.1) *$X_p = \{x \in X : \sup p(F(x)) \geq p(x)\}$ is closed in X;*

(1.2) *$x \in K$ and $p(x) = \max p(X)$ implies $x \in X_p$; and*

(1.3) *$x \in X \setminus K$ and $p(x) = \max p[x, L]$ implies $x \in X_p$.*

Then there exists an $x \in \bigcap\{X_p : p \in E^*\}$.

Proof. Note that $E^* \subset \widehat{X}$ by (1.0). For each $x \in X$, define

$$S(x) = \{p|_X : p \in E^* \text{and } \sup p(F(x)) < p(x)\}.$$

Then $S(x)$ is a convex subset of $\widehat{X}$. Suppose that $S(x) \neq \emptyset$ for each $x \in X$; that is, for each $x \in X$, there exists a $p \in E^*$ such that $x \notin X_p$. Note that, for each $g \in \widehat{X}$,

$$S^{-}(g) = \{x \in X : \sup p(F(x)) < p(x)\} = X \setminus X_p$$

if $g = p|_X$ for some $p \in E^*$ and

$$S^{-}(g) = \emptyset \quad \text{if} \quad g \notin \{p|_X : p \in E^*\}.$$

Then $S^{-}(g)$ is open in X for each $g \in \widehat{X}$ by (1.1). Therefore, (0.1) and (0.2) are satisfied. Further, (1.3) implies (0.3). In fact, for every $x \in X \setminus K$ and $p \in E^*$ satisfying $\sup p(F(x)) <$

$p(x)$, we have $x \notin X_p$. Therefore, $p(x) < \max p[x, L]$ by (1.3). Now, by applying Theorem 0, there exist an $\overline{x} \in K$ and a $p \in E^*$ such that $p|_X \in S(\overline{x})$ and $p(\overline{x}) = \max p(X)$. Note that $p|_X \in S(\overline{x})$ implies $\overline{x} \notin X_p$. This contradicts (1.2).

Remarks. 1. As we noted in [5], in Theorem 1, we do not require any concrete connection between topologies of X and E except

(1.0) $p|_X \in \widehat{X}$ (since $p|_X$ is continuous on X) for all $p \in E^*$.

In order to assure the continuity of $p|_X$ for all $p \in E^*$, it is sufficient to assume that
(i) as a convex space, X has any topology finer than the relative weak topology with respect to E, and
(ii) E has any topology finer than its weak topology.
This is why there have appeared fixed point theorems on maps whose domains and ranges have different topologies.

2. If F is u.h.c., then F satisfies the "continuity" condition (1.1) for all $p \in E^*$, but not conversely; see [5]. So, any map F satisfying (1.1) is called to be *generalized u.h.c.*

3. The "boundary" condition (1.2) is equivalent to the following:

(1.2)′ $x \in K$ and $p(x) = \max p(\overline{I_X(x)})$ implies $x \in X_p$.

In fact, $p(x) = \max p(X)$ is equivalent to $p(x) = \max p(\overline{I_X(x)})$.

Let X be a nonempty convex subset of a vector space E. Following Fan [2], the *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* Bd $X = \mathrm{Bd}_E X$ of X is the complement of $\mathrm{Int}_E X$ in $\overline{X}$. It is known that $\delta_E(X) \subset$ Bd X and in general $\delta_E(X) \neq$ Bd X.
Moreover, the "boundary" condition (1.2)′ is equivalent to the following:

(1.2)″ $x \in K \cap \delta_E(X)$ and $p(x) = \max p(\overline{I_X(x)})$ implies $x \in X_p$.

In fact, if $x \in K \setminus \delta_E(X)$ and $p(x) = \max p(\overline{I_X(x)})$, then for any $y \in E$, there exists an $r > 0$ such that $x + ry \in X$ and hence $p(x) \geq p(x + ry)$, which readily implies $p(y) \leq 0$ or $p = 0$. This contradicts the arbitrariness of p. Therefore, (1.2) is trivially satisfied.

4. The "coercivity" or "compactness" condition (1.3) is equivalent to the following:

(1.3)′ $x \in X \setminus K$ and $p(x) = \max p(\overline{I_L(x)})$ implies $x \in X_p$.

In fact, $p(x) = \max p[x, L]$ is equivalent to $p(x) = \max p(\overline{I_L(x)})$. Note that if X itself is compact (that is, if $X = K$), then (1.3)′ holds trivially.

Let $cc(E)$ denote the set of nonempty closed convex subsets of E and $kc(E)$ the set of nonempty compact convex subsets of E.
From Theorem 1, we have the following basic fixed point theorem:

Theorem 2. *Under the hypothesis of Theorem 1, further suppose that there is a map* G :

$X \to 2^E$ such that $F \subset G$ and either

(A) *E^* separates points of E and $G : X \to kc(E)$; or*

(B) *E is locally convex and $G : X \to cc(E)$.*

Then there exists an $x \in X$ such that $x \in G(x)$.

Proof. By Theorem 1, there exists an $x \in \bigcap\{X_p : p \in E^*\}$. Suppose that $x \notin G(x)$. Then under the assumptions (A) or (B), the standard separation theorems on a t.v.s. assure the existence of a $p \in E^*$ satisfying $\sup p(G(x)) < p(x)$. Since $F \subset G$ implies $F(x) \subset G(x)$, we have $\sup p(F(x)) \leq \sup p(G(x)) < p(x)$. Hence, $x \notin X_p$, which is a contradiction.

Remarks 1. Using the method in [5], we can reformulate Theorem 2 to a coincidence theorem and an existence theorem for critical points or zeros of multimaps.

2. In case $F = G$, note that $x \in F(x)$ if and only if $x \in \bigcap\{X_p : p \in E^*\}$. This is a useful information on the location of a fixed point.

3. In case $F = G$, Theorem 2 reduces to [7, Theorem 2].

From Theorem 2, we obtain the following more visualizable geometric form of a fixed point and surjectivity theorem, which generalizes [4, Theorem 6] and refines [5, Theorem 2], [7, Theorem 3]:

Theorem 3. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and $F, G : X \to 2^E$ such that $F \subset G$ maps satisfying either*

(A) *E^* separates points of E and $G : X \to kc(E)$, or*

(B) *E is locally convex and $G : X \to cc(E)$.*

(I) *Suppose that for each $p \in E^*$,*

(1.0) *$p|_X$ is continuous on X;*

(3.1) *$X_p = \{x \in X : \inf p(F(x)) \leq p(x)\}$ is closed in X;*

(3.2) *$d_p(F(x), \overline{I_X(x)}) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3.3) *$d_p(F(x), \overline{I_L(x)}) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in G(x)$.

(II) *Suppose that for each $p \in E^*$,*

(1.0) *$p|_X$ is continuous on X;*

(3.1)′ *$X_p = \{x \in X : \sup p(F(x)) \geq p(x)\}$ is closed in X;*

(3.2)′ *$d_p(F(x), \overline{O_X(x)}) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3.3)′ *$d_p(F(x), \overline{O_L(x)}) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in G(x)$. Further, if $F = G$ is u.h.c., then $F(X) \supset X$.

Proof. In order to apply Theorem 2, we first show that (3.2) $\Longrightarrow$ (1.2). Let $x \in K \cap \delta_E(X)$ such that $p(x) = \max p(X)$. Suppose that $\inf p(F(x)) > p(x)$. Then for any $v \in F(x)$, $u \in X$, $z = x + r(u - x) \in I_X(x)$, and $r > 0$, we have

$$|p(v - z)| = p(v - x) + rp(x - u) \geq p(v - x) = p(v) - p(x)$$

and hence

$$d_p(F(x), \overline{I_X(x)}) = d_p(F(x), I_X(x)) \geq \inf p(F(x)) - p(x) > 0.$$

This contradicts (3.2). Therefore, we should have $\inf p(F(x)) \leq p(x)$ or $x \in X_p$. Hence, (1.2)$''$ holds.

Similarly, we can show that (3.3) $\Longrightarrow$ (1.3). Note that "(3.1) holds for all p" is equivalent to "(1.1) = (3.1)$'$ holds for all p". Therefore, all of the requirements of Theorem 2 are satisfied. Now by Theorem 2, Case (I) follows.

For (II) consider $2x - G(x)$ instead of $G(x)$ in (I) as in [5], we can conclude that G has a fixed point. For the surjectivity result for u.h.c. $F = G$, let $y \in X$. Consider $x \mapsto F(x) + x - y$ instead of $F(x)$ and $[y, L]$ instead of L in Case (II). Then there exists an $x \in X$ such that $x \in F(x) + x - y$; that is, $y \in F(x)$.

Remarks 1. (3.1) and (3.1)$'$ are actually the same.

2. Note that the map $x \mapsto F(x) + x - y$ in the proof of Case (II) is u.h.c.

3. In case $F = G$, Theorem 3 reduces to [7, Theorem 3].

Actually, Theorem 3 is equivalent to [7, Theorem 3] in view of the following:

Lemma. *Let X be a convex space and E is a t.v.s. If $F : X \to 2^E$ is g.u.h.c. [resp., u.h.c.], then so is $G := \overline{\text{co}}\, F : X \to 2^E$.*

Proof. For the g.u.h.c. case, let $p \in E^*$. We have to show $\{x \in X : \sup\, p(\overline{\text{co}}\, F(x)) \geq p(x)\}$ is closed whenever $X_p = \{x \in X : \sup\, p(F(x)) \geq p(x)\}$ is closed. This suffices to show that $\sup\, p(\overline{\text{co}}\, F(x)) = \sup\, p(F(x))$.

In fact, any $y \in \text{co}\, F(x)$ can be expressed as $y = \sum_{i=1}^n r_i y_i$ for some n with $y_i \in F(x)$ and $r_i > 0$, $\sum_{i=1}^n r_i = 1$. Then

$$p(y) = \sum_{i=1}^n r_i p(y_i) \leq \sum_{i=1}^n r_i \sup\, p(F(x)) = \sup\, p(F(x))$$

and hence $\sup\, p(\overline{\text{co}}\, F(x)) = \sup\, p(\text{co}\, F(x)) = \sup\, p(F(x))$.

Similarly, for the u.h.c. case, we can show that $\{x \in X : \sup\, p(\overline{\text{co}}\, F(x)) \geq \alpha\}$ for any real α is closed whenever $X_p = \{x \in X : \sup\, p(F(x)) \geq \alpha\}$ is closed.

4 Generalizations of Balaj's results

From Theorem 3 and Lemma, we immediately obtain the following:

Theorem 4. *Let K be a nonempty compact convex subset of a t.v.s. E on which E^* separates points and let $F, G : K \to 2^E$ be two maps such that*

(a) *$F \subset G$;*

(b) *F is g.u.h.c.;*

(c) *G has compact convex values;*

(d) *$F(y) \cap \overline{I_K(y)} \neq \emptyset$ [resp., $F(y) \cap \overline{O_K(y)} \neq \emptyset$] for each $y \in K$.*

Then G has a fixed point $y_0 \in K$.

Balaj's unification [1, Theorem 3] of Halpern's theorems is a similar form of Theorem 4 where F is u.h.c. and E is locally convex (when G has closed convex values).

Corollary 4.1. *Let K be a nonempty compact convex subset of a t.v.s. E on which E^* separates points and $F : K \to 2^K$ a g.u.h.c. map. Then $G := \overline{\mathrm{co}}\, F : K \to 2^K$ has a fixed point.*

Proof 1. Since $G : K \to kc(K)$ and $F \subset G$, Theorem 2(A) with $K = X$ works.
Proof 2. By Lemma, G is also g.u.h.c. Hence Theorem 4 works.

In case $F = G$, Corollary 4.1 reduces to the following, which can be regarded the 2011 version of the Fan-Glicksberg fixed point theorem; see [6]:

Corollary 4.2. *Let K be a nonempty compact convex subset of a t.v.s. E on which E^* separates points and $F : K \to kc(K)$ a g.u.h.c. map. Then F has a fixed point.*

From Theorem 4, we have the following coincidence result:

Theorem 5. *Let C be a convex space, K a nonempty compact convex subset of a t.v.s. E on which E^* separates points, $S : C \to 2^K$ a map such that S^- has a continuous selection $f : K \to C$ and let $F, G : C \to 2^E$ be two maps such that*

(a) *$F \subset G$;*
(b) *$F \circ f : K \to 2^E$ is g.u.h.c.;*
(c) *G has compact convex values;*
(d) *$F(f(y)) \cap \overline{I_K(y)} \neq \emptyset$ [resp., $F(f(y)) \cap \overline{O_K(y)} \neq \emptyset$] for each $y \in K$.*

Then S and G have a coincidence point $x_0 \in C$, that is, $S(x_0) \cap G(x_0) \neq \emptyset$.

Proof. Apply Theorem 4 with $F \circ f$, $G \circ f$ instead of F, G, resp. Then the requirements (a)-(d) of Theorem 4 are satisfied. Hence, there exists an $y_0 \in K$ such that $y_0 \in G(f(y_0))$. Let $x_0 = f(y_0) \in S^-(y_0)$. Then $y_0 \in S(x_0) \cap G(x_0)$.

Note that Balaj [1, Theorem 4] follows from Theorem 5, where S has convex fibers and and a companion map $T : C \to 2^K$ such that $T \subset S$, T has open values in K and nonempty fibers. It is well-known that such S^- has a continuous selection; see [8]. Moreover, he assumed that (1) F is u.h.c. and hence so is $F \circ f$; and (2) E is locally convex and G has closed convex values.

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