The Fixed Point Method versus the KKM Method

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Abstract. In this survey, we compare the fixed point method and the KKM method in nonlinear analysis. Especially, we consider two methods in the proofs of the following important theorems in the chronological order: (1) The von Neumann minimax theorem, (2) The von Neumann intersection lemma, (3) The Nash equilibrium theorem, (4) The social equilibrium existence theorem of Debreu, (5) The Gale-Nikaido-Debreu theorem, (6) The Fan-Browder fixed point theorem, (7) Generalized Fan minimax inequality, and (8) The Himmelberg fixed point theorem.

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1. Introduction

In various fields in mathematical sciences, there are many results which can be proved by some fixed point theorems or some KKM type intersection theorems originated by Knaster, Kuratowski, and Mazurkiewicz (KKM, 1929). For example, in our previous work (Park, 2010a) on generalizations of the Nash equilibrium theorem, we found that there are two major methods, that is, the fixed point method (with or without using any continuous selection theorems) and the KKM method with respect to various abstract convexities.

Later we noticed that, for some of the key results in nonlinear analysis or the game theory, the fixed point method and the KKM method are major tools to deduce them. Examples of such key results are given chronologically as follows:

(1) The von Neumann minimax theorem
(2) The von Neumann intersection lemma
(3) The Nash equilibrium theorem
(4) The social equilibrium existence theorem of Debreu
(5) The Gale-Nikaido-Debreu theorem
(6) The Fan-Browder fixed point theorem
(7) Generalized Fan minimax inequality
(8) The Himmelberg fixed point theorem

Recall that many of the above results were originally proved by the Brouwer or Kakutani fixed point theorems. Nowadays, generalized forms of them can be deduced from the KKM theory with respect to various abstract convexities or from the fixed point theorems for acyclic maps.

Our aim in this survey is to compare the fixed point method and the KKM method in the proofs of the above sample results, and to give some of the most general forms of such results.

Recall that such comparison will also work for lots of results other than the above ones in the KKM theory.

2. Abstract Convex Spaces and the KKM Spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in (Park, 2008a,b,c, 2010b):

Definition 1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \to 2^E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition 3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \to 2^E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Now we have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

We recall the following in (Park, 1999,2001,2008a,b,c):

Definition 4. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

Definition 5. For an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbf{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbf{R}}$.

For the basic theory on KKM spaces, see Park (2010b) and the references therein.

3. The KKM type theorems

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained the following celebrated KKM theorem from the Sperner combinatorial lemma in 1928:

Theorem 1. (KKM, 1929) *Let $A_i\,(0 \leq i \leq n)$ be $n+1$ closed subsets of an n-simplex $p_0 p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

The first application of this KKM theorem was to give a simple proof of the Brouwer fixed point theorem; see (KKM, 1929) and (Park, 1999). Later, it is known that the Brouwer theorem, the Sperner lemma, and the KKM theorem are mutually equivalent.

In 1961, Fan extended the KKM theorem as follows:

Lemma 1. (Fan, 1961) *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

From the partial KKM principle for abstract convex spaces, we have a whole intersection property of the Fan type as follows:

Theorem 2. (Park, 2011b) *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and a map $G : D \to 2^E$ satisfy the following:*

(1) *$\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)}$ (G is intersectionally closed-valued);*

(2) *$\overline{G}$ is a KKM map (that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$); and*

(3) *there exists a nonempty compact subset K of E such that one of the following holds:*

(i) *$E = K$;*

(ii) *$\bigcap \{\overline{G(z)} \mid z \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(iii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$\overline{L_N} \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then $\bigcap \{G(z) \mid z \in D\} \neq \emptyset$.

When G is closed-valued or transfer closed-valued in (1), we obtain the conclusion

$$K \cap \bigcap \{G(z) \mid z \in D\} \neq \emptyset.$$

Theorem 2 subsumes a very large number of particular KKM type theorems in the literature and has a number of equivalent formulations; see (Park, 2011b).

4. The von Neumann type minimax theorem

In 1928, J. von Neumann obtained the following minimax theorem, which is one of the fundamental results in the theory of games developed by himself. We adopt Kakutani's formulation in (Kakutani, 1941):

Theorem 3. (von Neumann, 1928) *Let $f(x, y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x\in K} \min_{y\in L} f(x, y) = \min_{y\in L} \max_{x\in K} f(x, y).$$

In order to give simple proofs of von Neumann's Lemma in 1937 and the above minimax theorem, Kakutani obtained the following generalization of the Brouwer fixed point theorem to multimaps:

Theorem 4. (Kakutani, 1941) *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of nonempty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary 1. (Kakutani, 1941) *Theorem 4.2 is also valid even if S is an arbitrary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary 1 readily implies von Neumann's Lemma, and later Nikaido noted that those two results are directly equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

In 1958, von Neumann's minimax theorem was extended by Sion to arbitrary topological vector spaces as follows:

Theorem 5. (Sion, 1958) *Let X, Y be a compact convex set in a topological vector space. Let f be a real-valued function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous, quasiconvex function on Y, and*

(2) *for each fixed $y \in Y$, $f(x, y)$ is an upper semicontinuous, quasiconcave function on X,*

then we have

$$\min_{y\in Y} \max_{x\in X} f(x, y) = \max_{x\in X} \min_{y\in Y} f(x, y).$$

Sion's proof was based on the KKM theorem and this is the first application of the theorem after (KKM, 1929).

Recently, by the KKM method, we obtained a very general form of the von Neumann minimax theorem as follows; see (Park, 2010a).

Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces. For their product, we can define $\Gamma_{X\times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y\rangle$.

Theorem 6. (Park, 2010a) *Let* $(E;\Gamma) := (X \times Y; \Gamma_{X\times Y})$ *be the product abstract convex space,* $f, s, t, g : X \times Y \to \overline{\mathbf{R}}$ *be four functions,*

$$\mu := \inf_{y\in Y} \sup_{x\in X} f(x,y) \quad \textit{and} \quad \nu := \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Suppose that

(1) $f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ *for each* $(x,y) \in X \times Y$*;*

(2) *for each* $r < \mu$ *and* $y \in Y$, $\{x \in X \mid s(x,y) > r\}$ *is* Γ_1*-convex; for each* $r > \nu$ *and* $x \in X$, $\{y \in Y \mid t(x,y) < r\}$ *is* Γ_2*-convex;*

(3) *for each* $r > \nu$*, there exists a finite set* $\{x_i\}_{i=1}^m \subset X$ *such that*

$$Y = \bigcup_{i=1}^{m} \mathrm{Int}\,\{y \in Y \mid f(x_i,y) > r\}; \text{ and}$$

(4) *for each* $r < \mu$*, there exists a finite set* $\{y_j\}_{j=1}^n \subset Y$ *such that*

$$X = \bigcup_{j=1}^{n} \mathrm{Int}\,\{x \in X \mid g(x,y_j) < r\}.$$

If $(E;\Gamma)$ *satisfies the partial KKM principle, then we have*

$$\mu = \inf_{y\in Y} \sup_{x\in X} f(x,y) \le \sup_{x\in X} \inf_{y\in Y} g(x,y) = \nu.$$

5. The von Neumann type intersection theorems

The minimax theorem was later extended by von Neumann in 1937 to the following intersection lemma. We also adopt Kakutani's formulation:

Lemma 2. (von Neumann, 1937) *Let* K *and* L *be two bounded closed convex sets in the Euclidean spaces* $\mathbf{R}^m$ *and* $\mathbf{R}^n$ *respectively, and let us consider their Cartesian product* $K \times L$ *in* $\mathbf{R}^{m+n}$*. Let* U *and* V *be two closed subsets of* $K \times L$ *such that for any* $x_0 \in K$ *the set* U_{x_0}*, of* $y \in L$ *such that* $(x_0,y) \in U$*, is nonempty, closed and convex and such that for any* $y_0 \in L$ *the set* V_{y_0}*, of all* $x \in K$ *such that* $(x,y_0) \in V$*, is nonempty, closed and convex. Under these assumptions,* U *and* V *have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. Kakutani gave a simple proof by applying his fixed point theorem. Lemma 2 was generalized by Fan, Ma, and others to various Fan type intersection theorems for sets with convex sections.

Let $\{X_i\}_{i\in I}$ be a family of sets, and let $i \in I$ be fixed. Let

$$X = \prod_{j\in I} X_j \quad \text{and} \quad X^i = \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: Its ith coordinate is x_i and, for $j \ne i$, its jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

Some of the most general forms of the von Neumann intersection lemma are the following in the KKM theory:

Theorem 7. (Park, 2010c) *Let* $\{(X_i; \Gamma_i)\}_{i=1}^n$ *be a finite family of compact abstract convex spaces such that* $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ *satisfies the partial KKM principle and, for each* i*, let* A_i *and* B_i *are subsets of* X *satisfying*

(1) *for each* $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A_i\}$*; and*

(2) *for each* $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ *is open in* X^i.

Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

Theorem 8. (Park, 2001) *Let* $\{(X_i; \Gamma_i)\}_{i\in I}$ *be a family of Hausdorff compact G-convex spaces and, for each* $i \in I$*, let* A_i *and* B_i *are subsets of* $X = \prod_{i\in I} X_i$ *satisfying the following:*

$(1)'$ *for each* $x^i \in X^i$, $\emptyset \neq \mathrm{co}_{\Gamma_i} B_i(x^i) \subset A_i(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A_i\}$*; and*

$(2)'$ *for each* $y_i \in X_i$, $B_i(y_i) := \{x^i \in X^i \mid [x^i, y_i] \in B_i\}$ *is open in* X^i.

Then we have $\bigcap_{i\in I} A_i \neq \emptyset$.

6. The Nash equilibrium theorem

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem (Nash, 1950, 1951) on equilibrium points of non-cooperative games. The following formulation is given by (Fan, 1966):

Theorem 9. (Nash) *Let* $X_1, X_2, \cdots, X_n$ *be* n (≥ 2) *nonempty compact convex sets each in a real Hausdorff topological vector space. Let* $f_1, f_2, \cdots, f_n$ *be* n *real-valued continuous functions defined on* $\prod_{i=1}^n X_i$*. If for each* $i = 1, 2, \cdots, n$ *and for any given point* $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j\neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ *is a quasi-concave function on* X_i*, then there exists a point* $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ *such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

The original form of this theorem in (Nash, 1950, 1951) was for Euclidean spaces and its proofs were based on the Brouwer or Kakutani fixed point theorem.

Moreover, based on a generalization of the Kakutani fixed point theorem due to Fan (1952) and Glicksberg (1952), certain generalizations of the Nash theorem were obtained; for example, see (Aliprantis et al., 2006) and (Becker et al., 2006).

The first proof of the Nash theorem by the KKM method was given by Fan (1966). Applying the KKM method or the fixed point method, we obtained some of the most general forms of the Nash theorem as follows:

Theorem 10. (Park, 2010c) *Let* $\{(X_i; \Gamma_i)\}_{i=1}^n$ *be a finite family of compact abstract convex spaces such that* $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ *satisfies the partial KKM principle and, for each* i*, let* $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ *be real functions such that*

(0) $f_i(x) \leq g_i(x)$ *for each* $x \in X$*;*

(1) *for each* $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ *is quasiconcave on* X_i*;*

(2) *for each* $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ *is u.s.c. on* X_i*; and*

(3) *for each* $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ *is l.s.c. on* X^i.

Then there exists a point $\widehat{x} \in X$ *such that*

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all } i = 1, 2, \ldots, n.$$

Theorem 11. (Park, 2001) *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ be real functions satisfying* (0)-(3) *in Theorem 10. Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \geq \max_{y_i\in X_i} f_i[\widehat{x}^i, y_i] \quad \textit{for all} \quad i \in I.$$

7. The social equilibrium existence theorem of Debreu

Acyclic versions of the social equilibrium existence theorem of Debreu (1952) are obtained in (Park, 1998, 2011a) as follows:

A *polyhedron* is a set in $\mathbf{R^n}$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R^n}$. The product of two polyhedra is a polyhedron.

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery or, more generally, to Begle (1950):

Lemma 3. (Eilenberg and Montgomery, 1946) *Let Z be an acyclic polyhedron and $T : Z \to 2^Z$ an acyclic map* (*that is, u.s.c. with acyclic values*). *Then T has a fixed point $\widehat{x} \in Z$; that is, $\hat{x} \in T(\widehat{x})$.*

Recently, we obtained the following new *collectively fixed point theorem* equivalent to Lemma 3:

Theorem 12. (Park, 2011a) *Let $\{X_i\}_{i\in I}$ be any family of acyclic polyhedra, and $T_i : X \to 2^{X_i}$ an acyclic map for each $i \in I$. Then there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each $i \in I$.*

From this, we have the following acyclic version of the *social equilibrium existence theorem* of Debreu (1952):

Theorem 13. (Park, 2011a) *Let $\{X_i\}_{i\in I}$ be any family of acyclic polyhedra, $A_i : X^i \to 2^{X_i}$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \leq f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) := \max_{y\in A_i(x^i)} g_i[x^i, y]$ is an l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) := \{x_i \in A_i(x^i) \mid f_i[x^i, x_i] \geq \varphi_i(x^i)\}$$

is acyclic.

Then there exists an equilibrium point $\widehat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\widehat{a}^i) \quad \textit{and} \quad f_i(\widehat{a}) = \max_{a_i\in A(\widehat{a}^i)} g_i[\widehat{a}^i, a_i] \quad \textit{for all} \quad i \in I.$$

Debreu's theorem is the contractible case in (3) for a finite I.

In Park (1998, 2011a), this is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the following *Nash equilibrium theorem*:

Theorem 14. (Park, 2011a) *Let* $\{X_i\}_{i\in I}$ *be a family of acyclic polyhedra,* $X = \prod_{i=1}^n X_i$*, and for each* i*,* $f_i : X \to \overline{\mathbf{R}}$ *a continuous function such that*

(0) *for each* $x^i \in X^i$ *and each* $\alpha \in \overline{\mathbf{R}}$*, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

Then there exists a point $\widehat{a} \in X$ *such that*

$$f_i(\widehat{a}) = \max_{y_i \in X_i} f_i[\widehat{a}^i, y_i] \quad \textit{for all} \quad i \in I.$$

In our previous work (Park, 1998), Theorems 12-14 were obtained for a finite index set I.

8. The Gale-Nikaido-Debreu theorem

A *convex space* (in the sense of Lassonde) is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*. For details, see (Park, 1999) and references therein.

In 1994, we introduced an *admissible* class $\mathcal{A}_c^\kappa(X, Y)$ of maps $T : X \to 2^Y$ between topological spaces X and Y as the one such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathcal{A}_c(K, Y)$ satisfying $\Gamma(x) \subset T(x)$ for all $x \in K$; where $\mathcal{A}_c$ is consisting of finite composites of maps in $\mathcal{A}$, and $\mathcal{A}$ is a class of maps satisfying the following properties:

(i) $\mathcal{A}$ contains the class of (single-valued) continuous functions;

(ii) each $F \in \mathcal{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathcal{A}_c(P, P)$ has a fixed point.

Let X be a convex space and Y a Hausdorff space. In 1997, we introduced a more general "better" admissible class $\mathcal{B}$ of multimaps as follows:

$F \in \mathcal{B}(X, Y) \iff F : X \to 2^Y$ such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P)$ has a fixed point.

Here, $\mathcal{A}$ and $\mathcal{B}$ should be denoted by "Fraktur A" and "Fraktur B", resp.

In 1997, we obtained a KKM type theorem related to the better admissible class $\mathcal{B}$ of multimaps. From the theorem, we deduced the following generalization of the so-called Walras excess demand theorem:

Theorem 15. (Park, 1997) *Let* X *be a convex space,* Y *a Hausdorff space,* $T \in \mathcal{B}(X, Y)$ *a compact map,* $c \in \mathbf{R}$*, and* $\phi, \psi : X \times Y \to \overline{\mathbf{R}}$ *two extended real-valued functions such that*

(1) $\phi(x, y) \leq \psi(x, y)$ *for each* $(x, y) \in X \times Y$*;*

(2) *for each* $x \in X$*,* $y \mapsto \psi(x, y)$ *is u.s.c. on* Y*;*

(3) *for each* $y \in Y$*,* $x \mapsto \phi(x, y)$ *is quasiconvex on* X*; and*

(4) $\phi(x, y) \geq c$ *for all* $(x, y) \in T$ *(Walras law).*

Then there exists a Walras equilibrium; that is, there exists a $y_0 \in Y$ *such that*

$$c \leq \phi(x, y_0) \quad \textit{for all} \quad x \in X.$$

This generalize a result due to Granas and Liu (1986).

The following is a different version of Theorem 15:

Theorem 16. (Park, 1997) *Let X be a compact convex space, Y a Hausdorff space, and $T \in \mathcal{A}_c^\kappa(X,Y)$. Let $\phi : X \times Y \to \mathbf{R}$ be a continuous function and $c \in \mathbf{R}$ such that*

(1) *for each $y \in Y$, $x \mapsto \phi(x,y)$ is quasiconvex on X; and*

(2) *$\phi(x,y) \geq c$ for all $(x,y) \in T$ (Walras law).*

Then there exists a Warlas equilibrium; that is, there exists an $(x_0, y_0) \in T$ such that

$$c \leq \phi(x_0, y_0) \leq \phi(x, y_0) \quad forall \quad x \in X.$$

This extends results due to Granas-Liu (1986), Gwinner (1981), and Zeidler (1986).

From Theorem 16, we deduced the following generalization of the Gale-Nikaido-Debreu theorem:

Theorem 17. (Park, 1997) *Let $(E, F, \langle\ ,\ \rangle)$ be a dual system of Hausdorff topological vector spaces E and F, where the real bilinear form $\langle\ ,\ \rangle$ is continuous on compact subsets of $E \times F$. Let X be a nonempty compact convex subset of E, P the convex cone $\bigcup\{rX \mid r \geq 0\}$, and $P^+ = \{y \in F \mid \langle p, y\rangle \geq 0,\ p \in P\}$ its positive dual cone. Then for any map $T \in \mathcal{A}_c^\kappa(X,F)$ satisfying $\langle x, y\rangle \geq 0$ for $(x,y) \in T$, there exists an $\overline{x} \in X$ such that $T\overline{x} \cap P^+ \neq \emptyset$.*

This generalizes a result of Gwinner (1981). The Gale-Nikaido-Debreu theorem is the case $P = \{x \in \mathbf{R}^n \mid x_i \geq 0,\ 1 \leq i \leq n\}$, $X = \{x \in P \mid x_1 + \cdots + x_n = 1\}$, the standard $(n-1)$-simplex, and $T \in \mathcal{K}(X, \mathbf{R}^n)$, where $\mathcal{K}$ denotes the class of Kakutani maps (that is, u.s.c. maps with closed convex values). For the references, see (Gwinner, 1981).

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0,1)$, H^p for $0 < p < 1$, and many others.

In the paper (Park, 2009), from a fixed point theorem for acyclic maps due to Park, we deduced the following generalization of Gwinner's extension of the Walras theorem in Gwinner (1981):

Theorem 18. (Park, 2009) *Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible. Let $c \in \mathbf{R}$, $f : K \times L \to \mathbf{R}$ a continuous function, and $T : K \to 2^L$ a multimap. Suppose*

(1) *for each $y \in L$, $f(\cdot, y)$ is quasiconvex;*

(2) *T is an acyclic map; and*

(3) *for each $x \in K$ and $y \in T(x)$, we have $f(x,y) \geq c$. (the Walras law)*

Then there exists a Walras equilibrium, that is, there exist $\overline{x} \in K$, $\overline{y} \in L$ such that

$$\overline{y} \in T(\overline{x}) \ \ and \ \ f(x, \overline{y}) \geq c \ \ for\ all \ \ x \in K.$$

Since every convex subset of a locally convex t.v.s. is admissible, from Theorem 18, we immediately have the following:

Corollary 2. *Theorem 18 is also valid even if E is a locally convex t.v.s.*

For a Kakutani map T instead of an acyclic map, Corollary 2 reduces to (Park, 2010d, Theorem 8], where local convexity of F is redundant. To specialize Theorem 18 towards the Gale-Nikaido-Debreu theorem, we boil down the function f to a bilinear form $\langle \cdot, \cdot \rangle$ for a dual system $(E, F, \langle \cdot, \cdot \rangle)$ of t.v.s. E and F.

For a convex cone P of E, the *dual cone* is defined by

$$P^+ := \{y \in F \mid \langle p, y \rangle \geq 0,\ p \in P\}.$$

Theorem 19. (Park, 2010d) *Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual system of t.v.s. E and F such that the bilinear form $\langle \cdot, \cdot \rangle$ is continuous on compact subsets of $E \times F$. Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible; and P the convex cone $\bigcup\{rK \mid r \geq 0\}$. Let $T : K \to 2^L$ be an acyclic map such that $\langle x, y \rangle \geq 0$ for all $x \in K$ and $y \in T(x)$. Then there exists $\overline{x} \in K$ such that $T(\overline{x}) \cap P^+ \neq \emptyset$.*

Corollary 3. (Park, 2010d) *Theorem 18 is also valid even if E is a locally convex t.v.s. instead of the admissibility of K.*

For a Kakutani map T instead of an acyclic map, Corollary 3 reduces to (Gwinner, 1981, Corollary to Theorem 8), where local convexity of F is redundant.

With the choice $P := \{x \in \mathbf{R}^n \mid x_i \geq 0;\ i = 1, 2, \ldots, n\}$ and $K = L := \{x \in P \mid x_1 + x_2 + \cdots + x_n = 1\}$ (the standard simplex), the Gale-Nikaido-Debreu theorem can be immediately obtained from Corollary 3.

9. The Fan-Browder type fixed point theorem

In 1968, Browder obtained an equivalent form of Fan's geometric lemma (Fan, 1961). Since then the following is known as the Fan-Browder fixed point theorem:

Theorem 20. (Browder, 1968) *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K, T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^-(y) = \{x \in K \mid y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Browder proved this by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem.

Later the Hausdorffness in the Fan lemma and Browder's theorem was known to be redundant by Lassonde in 1983. Moreover, the Fan lemma and Browder's theorem are known to be equivalent to the KKM theorem. Consequently the Browder theorem can be obtained by a simple KKM proof.

There are several scores of generalizations of Theorem 20. The following are some of recent ones:

Theorem 21. (Park, 2008c, 2010b) *An abstract convex space $(X, D; \Gamma)$ is a KKM space iff for any maps $S : D \to 2^X,\ T : X \to 2^X$ satisfying*

(1) *$S(z)$ is open [resp., closed] for each $z \in D$;*
(2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$X = \bigcup_{z \in M} S(z)$ for some $M \in \langle D \rangle$,*

T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

From Theorem 2, we have the following Fan-Browder type fixed fixed point theorem:

Theorem 22. *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, X is compact, and $S : D \to 2^X$, $T : X \to 2^X$ maps. Suppose that*

(1) *S is unionly open-valued (that is, S^c is intersectionally closed-valued);*

(2) *for each $x \in X$, $M \in \langle S^-(x)\rangle$ implies $\Gamma_M \subset T^-(x)$; and*

(3) *$X = S(D)$.*

Then T has a fixed point.

10. Generalized Fan minimax inequality

One of the most remarkable equivalent formulations of the KKM theorem is the following *minimax inequality* established by Ky Fan from his KKM lemma:

Theorem 23. (Fan, 1972) *Let X be a compact convex set in a t.v.s. Let f be a real-valued function defined on $X \times X$ such that* :

(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X.*

(b) *For each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.*

Then the minimax inequality

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} f(x, x)$$

holds.

Fan actually assumed the Hausdorffness of the topological vector space in his KKM lemma and his minimax inequality, but it was known to be redundant later. However, the inequality became a crucial tool in proving many existence problems in nonlinear analysis, especially, in various variational inequality problems.

The compactness, convexity, lower semicontinuity, and quasiconcavity in the inequality are extended or modified by a large number of authors. For example, the quasicocavity is extended to γ-DQCV by (Zhou and Chen, 1988). Further, (Lin and Tian, 1993, Theorem 3) defined γ-DQCV in slightly more general form: Let Y be a convex subset of a Hausdorff t.v.s. E and let $\emptyset \neq X \subset Y$. A functional $\varphi(x, y) : X \times Y \to \overline{\mathbf{R}}$ is said to be *γ-diagonally quasi-concave* (*γ-DQCV*) in x if, for any finite subset $\{x_1, \ldots, x_m\} \subset X$ and any $x_\lambda \in \mathrm{co}\,\{x_1, \ldots, x_m\}$, we have $\min_{1\leq j\leq m} \varphi(x_j, x_\lambda) \leq \gamma$.

Theorem 24. (Lin and Tian, 1993) *Let Y be a nonempty convex subset of a Hausdorff t.v.s. E, let $\emptyset \neq X \subset Y$, and let $\varphi : X \times Y \to \overline{\mathbf{R}}$ be a functional such that*

(i) *$(x, y) \mapsto \varphi(x, y)$ is l.s.c. in y;*

(ii) *$(x, y) \mapsto \varphi(x, y)$ is γ-DQCV in x;*

(iii) *there exists a nonempty subset C of X such that $\bigcap_{x\in C}\{y \in Y \mid \varphi(x, y) \leq \gamma\}$ is compact and C is contained in a compact convex subset B of Y.*

Then there exists a point $y^* \in Y$ such that $\varphi(x, y^*) \leq \gamma$ for all $x \in X$.

Lin and Tian (1993) proved Theorem 24 by applying the partition of unity argument [this is why Hausdorffness is assumed] and the Brouwer fixed point theorem. Moreover, they showed that Theorem 24 is equivalent to the Fan KKM lemma [where Hausdorffness is redundant].

Comparing these two proofs, *the KKM method is simpler than the Fixed Point method assuming the redundant Hausdorffness.*

Recently, the $\mathcal{C}$-quasiconcavity due to (Hou, 2009) unifies the diagonal transfer quasiconcavity (weaker than quasiconcavity) and the $\mathcal{C}$-concavity (weaker than concavity). However, S.-Y. Chang (2010) extended the $\mathcal{C}$-quasiconcavity to 0-pair-concavity and obtained a new Fan type inequality.

From the partial KKM principle we can deduce a very general version of the Fan minimax inequality:

Theorem 25. (Park, 2010a,2011b) *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, $f : D \times X \to \overline{\mathbf{R}}$, $g : X \times X \to \overline{\mathbf{R}}$ extended real functions, and $\gamma \in \overline{\mathbf{R}}$ such that*

(1) *for each $z \in D$, $G(z) := \{y \in X \mid f(z, y) \leq \gamma\}$ is intersectionally closed;*

(2) *for each $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in X \mid g(x, y) > \gamma\}$; and*

(3) *the compactness condition (3) in Theorem 2 holds.*

Then either

(i) *there exists a $\widehat{x} \in X$ such that $f(z, \widehat{x}) \leq \gamma$ for all $z \in D$; or*

(ii) *there exists an $x_0 \in X$ such that $g(x_0, x_0) > \gamma$.*

We found that Chang's Fan type inequality follows from this.

11. The Himmelberg fixed point theorem

In 1972, Himmelberg defined that a subset A of a t.v.s. E is said to be *almost convex* if for any neighborhood $V \in \mathcal{V}$ of the origin 0 in E and for any finite set $\{w_1, \ldots, w_n\}$ of points of A, there exist $z_1, \ldots, z_n \in A$ such that $z_i - w_i \in V$ for all i, and $\mathrm{co}\{z_1, \ldots, z_n\} \subset A$.

Himmelberg derived the following from the Kakutani fixed point theorem:

Theorem 26. (Himmelberg, 1972) *Let K be a nonvoid compact subset of a separated locally convex space L and $G : K \to K$ be a u.s.c. multifunction such that $G(x)$ is closed for all x in K and convex for all x in some dense almost convex subset A of K. Then G has a fixed point.*

Theorem 27. (Himmelberg, 1972) *Let T be a nonvoid convex subset of a separated locally convex space L. Let $F : T \to T$ be a u.s.c. multifunction such that $F(x)$ is closed and convex for all $x \in T$, and $F(T)$ is contained in some compact subset C of T. Then F has a fixed point.*

We stated their original forms. Usually Theorem 27 is called the Himmelberg fixed point theorem which unifies and generalizes historically well-known theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, and others. There have appeared a very large number of generalizations of the theorem even within the category of topological vector spaces, see (Park, 2008). Especially, there are generalizations of Theorems 26 and 27 which can be deduced from the KKM theorem and its open version; see also (Park, 2008).

Finally, in this section, we show the open version of the KKM principle can be also applied to fixed point theorems for KKM spaces.

Definition 6. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is the one with a basis $\mathcal{U}$ of a uniform structure of E.

A *KKM uniform space* $(E, D; \Gamma; \mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a uniform structure of E.

A KKM uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *LΓ-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Theorem 28. (Park, 2009) *Let $(X \supset D; \Gamma; \mathcal{U})$ be a Hausdorff LΓ-space and $T : X \to 2^X$ a compact u.s.c. map with closed Γ-convex values. Then T has a fixed point $x_0 \in X$.*

This is an example of generalizations of the Himmelberg theorem for abstract convex spaces. For more generalizations, see Park (2008d, 2009).

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