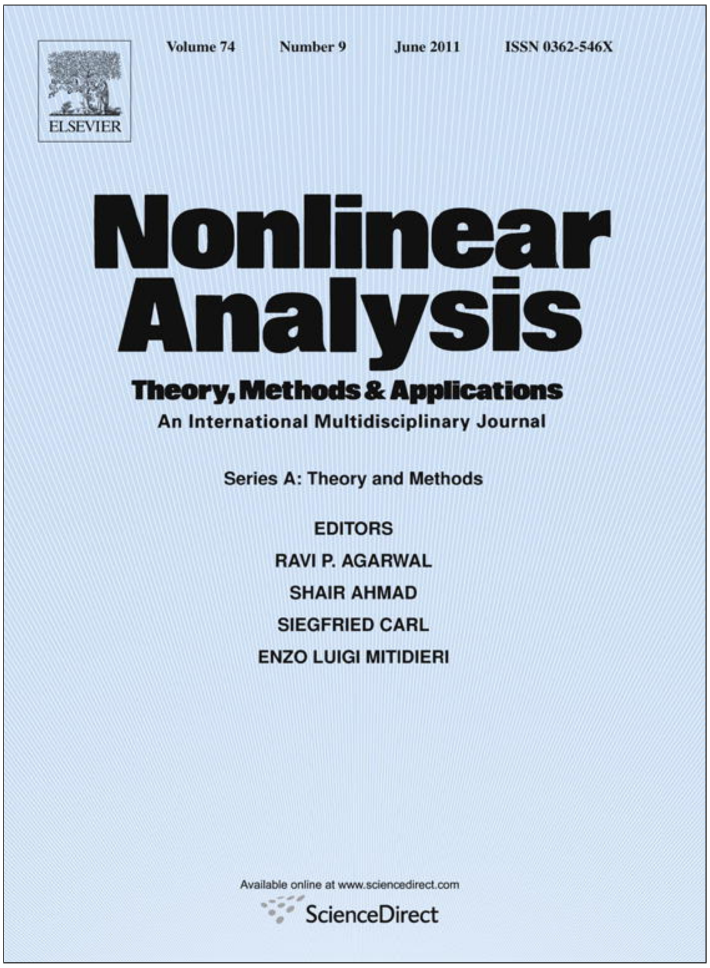
Volume 74 Number 9 June 2011 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITORS
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Available online at www.sciencedirect.com
ScienceDirect

Nonlinear Analysis 74 (2011) 3000–3010

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

New generalizations of basic theorems in the KKM theory

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ARTICLE INFO

Article history:
Received 2 November 2010
Accepted 19 January 2011

MSC:
47H04
47H10
52A99
54H25
58E35
90C47

Keywords:
Intersectionally closed
Unionly open
Abstract convex space
Geometric lemma
Fixed point
Maximal element
Minimax inequality
Variational inequality

ABSTRACT

In the present paper, we obtain a new KKM type theorem for intersectionally closed-valued KKM maps and some useful new basic consequences. Typical examples of them are abstract forms of Fan's matching theorem, Fan's geometric lemma, the Fan–Browder fixed point theorem, maximal element theorems, Fan's minimax inequality, variational inequalities, and others.

1. Introduction

The KKM theory, first called by the author [1], is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

Some of the basic theorems that were applications of the KKM theorem were first obtained by Fan, Browder, and others for convex subsets of topological vector spaces (not necessarily Hausdorff). Later extensions of the theory were due to Lassonde for convex spaces, Horvath for H-spaces, Park for G-convex spaces, and others; see [2] and the references therein.

Recently, the KKM theory has been extended to abstract convex spaces by the author and we obtained a large number of new results in such frame; see [3] and the references therein. In fact, there are very large numbers of equivalent formulations, generalizations, and applications of the KKM theorem.

Note that, in many works on the theory, some authors replaced the closed-valuedness of related KKM maps by more general concepts. We noted several times that such replacements are inessential and useless in many cases. However, more recently, Luc et al. [4] have introduced a meaningful concept of *intersectionally closed-valued* multimaps and applied them to several related problems. This new concept is more general than the quite long-standing one of transfer closed-valuedness.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.01.020

In the present paper, we obtain new KKM type theorems for intersectionally closed-valued KKM maps and some useful basic consequences. Typical examples of them are abstract forms of Fan's matching theorem, Fan's geometric lemma, the Fan–Browder fixed point theorem, maximal element theorems, Fan's minimax inequality, variational inequalities, and others. This paper would be helpful to apply those new basic results and to extend many known results on closed-valued multimaps to the ones on intersectionally closed-valued multimaps.

Section 2 deals with some history of the basic theorems and their pioneering applications due to Fan, Browder, and others. In Section 3, we introduce basic concepts of abstract convex spaces and KKM spaces. Section 4 deals with general KKM type theorems on abstract convex spaces satisfying partial KKM principle for KKM maps having intersectionally closed values. In Section 5, we deduce a Fan type whole intersection theorem, the Fan–Browder type fixed point theorems, and maximal element theorems for abstract convex spaces having partial KKM principle. Section 6 deals with a Fan type matching theorem, the Fan minimax inequality, and some variational inequalities on abstract convex spaces. Moreover, we add certain corrections and improvements of our previous work [3, Theorem 3] and the statements (XII)–(XVII) in [3].

Finally, note that each of all new results in this paper has a large number of known particular cases in the literature and we will not trace them; see [3] and the references therein.

2. The historical basic theorems

In this section, some of the earlier historical basic theorems of the KKM theory are introduced. We follow [2], where we can find a number of earlier related literature.

A milestone on the history of the KKM theory was erected by Fan [5] in 1961. He extended the original KKM theorem to arbitrary topological vector spaces as follows.

Lemma (*Fan [5]*)**.** *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied.*

(i) *The convex hull of any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem. Fan [5] applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space. Fan also obtained the following geometric or section property of convex sets, which is equivalent to the preceding lemma.

Lemma (*Fan [5]*)**.** *Let X be a compact convex set in a topological vector space. Let A be a closed subset of $X \times X$ with the following properties.*

(i) *$(x, x) \in A$ for every $x \in X$.*
(ii) *For any fixed $y \in X$, the set $\{x \in X : (x, y) \notin A\}$ is convex (or empty).*

Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

Fan applied this lemma to give a simple proof [5] of the Tychonoff theorem and to prove two results [6] generalizing the Pontrjagin–Iohvidov–Kreĭn theorem on the existence of invariant subspaces of certain linear operators. Also, Fan [7] applied his KKMF lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff theorem. The main results of Fan [7] were extended by Ma (1969), who obtained a generalization of the Nash equilibrium theorem for infinite case.

Moreover, a theorem concerning sets with convex sections was applied to prove the following results in [8,9].

An intersection theorem (which generalizes the von Neumann lemma (1937).
An analytic formulation (which generalizes the equilibrium theorem of Nash (1951) and the minimax theorem of Sion (1958)).
A theorem on systems of convex inequalities of Fan (1957).
Extremum problems for matrices.
A theorem of Hardy–Littlewood–Pólya concerning doubly stochastic matrices.
A fixed point theorem generalizing Tychonoff (1935) and Iohvidov (1964).
Extensions of monotone sets.
Invariant vector subspaces.
An analogue of Helly's intersection theorem for convex sets.

Moreover, Fan [10] in 1972 established a minimax inequality from the KKMF theorem.

Theorem (*Fan [10]*)**.** *Let X be a compact convex set in a topological vector space. Let f be a real function defined on $X \times X$ such that:*

(a) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X;*
(b) *for each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.*

Then the following minimax inequality holds.

$$\min_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

Fan gave applications of this inequality as follows.

A variational inequality (extending Hartman–Stampacchia (1966) and Browder (1967)).
A geometric formulation of the inequality (equivalent to the Fan–Browder theorem).
Separation properties of upper demicontinuous multimaps, coincidence and fixed point theorems.
Properties of sets with convex sections (Fan [8]).
A fundamental existence theorem in potential theory.

Furthermore, Fan [11,12] introduced a KKM theorem with a coercivity (or compactness) condition for noncompact convex sets and, from this, extended many known results to noncompact cases.

On the other hand, Browder [13] in 1968 restated Fan's geometric lemma [5] in the convenient form of a fixed point theorem by means of the Brouwer fixed point theorem and the partition of unity argument. Since then the following is known as the Fan–Browder fixed point theorem.

Theorem (*Browder [13]*)**.** *Let K be a nonempty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^{-1}(y) = \{x \in K \mid y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later this is also known to be equivalent to the Brouwer theorem. Browder [13] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This is also applied by Borglin and Keiding (1976) and Yannelis and Frabhakar (1983), to the existence of maximal elements in mathematical economics. For further developments on generalizations and applications of the Fan–Browder theorem, we refer to Park [14,15].

3. Abstract convex spaces and the KKM spaces

Let A be a subset of a topological space X. We denote by $\overline{A}$ the closure of A in X and, by $\mathrm{Int}\,A$ the interior of A. Let Δ_n be the standard n-dimensional simplex in $\mathbb{R}^{n+1}$. Let $\langle D\rangle$ be the set of all nonempty finite subsets of a set D.

A multimap or simply a map $F : X \multimap Y$ is a function $F : X \to 2^Y$ to the power set of Y and $F^- : Y \multimap X$ is defined by $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [3].

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any nonempty $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$ in $(E, D; \Gamma)$, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

If E is compact, then $(E, D; \Gamma)$ is called a *compact* abstract convex space.

Example. Known examples of abstract convex spaces are given in [3] and the references therein.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space. If a map $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

We have abstract convex subspaces as the following simple observation.

Proposition 3.1 (*[16]*)**.** *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D'\rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \quad \text{for } A \in \langle D'\rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace relative to D'.

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Example. Known examples of KKM spaces are given in [3] and the references therein.

Proposition 3.2. *Let $(E, D; \Gamma)$ be an abstract convex space and $(X, D'; \Gamma')$ a subspace. If $(E, D; \Gamma)$ satisfies the partial KKM principle, then so does $(X, D'; \Gamma')$.*

Proof. Suppose that a closed-valued map $G' : D' \multimap X$ satisfies

$$\Gamma'_A \subset G'(A) \quad \text{for all } A \in \langle D' \rangle.$$

Define a map $G : D \multimap E$ by

$$G(y) := \begin{cases} G'(y) & \text{for } y \in D' \\ \overline{X} & \text{otherwise.} \end{cases}$$

Then

$$\Gamma_A = \Gamma'_A \subset G'(A) = G(A) \quad \text{for } A \in \langle D' \rangle; \quad \text{and}$$
$$\Gamma_A \subset \overline{X} = G(A) \quad \text{for } A \in \langle D \rangle \quad \text{with } A \cap (D \setminus D') \neq \emptyset.$$

Since $(E, D; \Gamma)$ satisfies the partial KKM principle and G has closed values, the family $\{G(y)\}_{y\in D}$ has the finite intersection property, and hence so does its subfamily $\{G'(y)\}_{y\in D'}$. Therefore, $(X, D'; \Gamma')$ satisfies the partial KKM principle. □

4. General KKM type theorems

The KKM type theorems give the whole intersection property for the map values of a KKM map. The following is a standard form equivalent to [3, Theorem 3].

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $G : D \multimap E$ a multimap satisfying*

(1) $\Gamma_N \subset \overline{G}(N)$ for any $N \in \langle D \rangle$ (that is, $\overline{G}$ is a KKM map).

Then $\{\overline{G(z)}\}_{z\in D}$ has the finite intersection property.

Further, if

(2) there exists a nonempty compact subset K of E such that either

(i) $K \supset \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D \rangle$; or

(ii) for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$K \supset L_N \cap \bigcap_{z\in D'} \overline{G(z)} \neq \emptyset.$$

then $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

Proof. The first part is a simple consequence of definition.

Case (i) of the second part: Since $\{\overline{G(z)} \mid z \in D\}$ has the finite intersection property, so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property.

Case (ii) of the second part: Suppose that $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} = \emptyset$. Then $K \subset \bigcup\{E \setminus \overline{G(z)} \mid z \in N\}$ for some $N \in \langle D \rangle$ since K is compact. Let L_N be the compact Γ-convex subset of E in (ii). Define $G' : D' \multimap L_N$ by $G'(z) := \overline{G(z)} \cap L_N$ for $z \in D'$. Then $A \in \langle D' \rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset \overline{G}(A) \cap L_N = G'(A)$ by (2), and hence $G' : D' \multimap L_N$ is a KKM map on $(L_N, D'; \Gamma')$ with closed values. Since $(E, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$ by Proposition 3.2. Hence, $\{G'(z) \mid z \in D'\}$ has the finite intersection property. Since L_N is compact, $\bigcap\{G'(z) \mid z \in D'\} \neq \emptyset$. For any $y \in \bigcap\{G'(z) \mid z \in D'\} = \bigcap\{\overline{G(z)} \cap L_N \mid z \in D'\}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{X \setminus \overline{G(z)} \mid z \in N\}$, we have $y \notin \overline{G(z)}$ for some $z \in N \subset D'$. This is a contradiction. □

Recall that conditions (i) and (ii) in Theorem 4.1 are usually called the *compactness conditions* or the *coercivity conditions*, which hold trivially when $E = K$, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that, as the above proof shows, Theorem 4.1(ii) can be deduced from the compact case of Theorem 4.1, and hence it seems to be not a big problem to treat the case (ii).

The essence of Theorem 4.1 for case (ii) can be stated as follows.

Corollary 4.2. *Let E be a topological space, K a nonempty compact subset of E, D a nonempty set, and $G : D \multimap E$ a map. Suppose that, for each $N \in \langle D\rangle$, there exist a compact subset L_N of E, a nonempty subset D' of D, and a map $\Gamma : \langle D'\rangle \multimap L_N$ such that $N \subset D'$, $(L_N, D'; \Gamma)$ is an abstract convex space satisfying the partial KKM principle, and*

$$K \supset L_N \cap \bigcap_{z\in D'} \overline{G(z)} \neq \emptyset.$$

Then $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

Proof. By Theorem 4.1, we have

$$K \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \neq \emptyset.$$

Suppose the conclusion does not hold, that is, $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} = \emptyset$. Then $K \subset \bigcup\{E \setminus \overline{G(z)} \mid z \in N\}$ for some $N \in \langle D\rangle$ since K is compact. Then for this N, we have $K \cap \bigcap\{\overline{G(z)} \mid z \in N\} \neq \emptyset$ with $N \subset D'$. Hence, there exists $y \in K \cap \overline{G(z)}$ for all $z \in N$. However, $y \in K \subset \bigcup\{E \setminus \overline{G(z)} \mid z \in N\}$ and we have $y \notin \overline{G(z)}$ for some $z \in N \subset D'$. This is a contradiction. □

Among the numerous consequences of Theorem 4.1, we introduce only one as follows.

Corollary 4.3 ([3]). *An abstract convex space $(E, D; \Gamma)$ satisfies the partial KKM principle if and only if the following Fan–Browder fixed point property holds. Let $S : E \multimap D$, $T : E \multimap E$ be maps satisfying*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
(2) *$S^-(z)$ is open for each $z \in D$;*
(3) *$E = \bigcup_{z\in M} S^-(z)$ for some $M \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$.

Recall that the main conclusions of most of KKM type theorems follow from the form

$$\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$$

for a multimap $G : D \multimap E$.

Consider the following related four conditions.

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.
(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [4]).
(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

Definition. For an abstract convex space $(E, D; \Gamma)$, a subset X of E is said to be *intersectionally closed* (resp., *transfer closed*) if there is an intersectionally (resp., transfer) closed-valued map $G : D \multimap E$ such that $X = G(z)$ for some $z \in D$.

Luc et al. [4] noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c) as follows. Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

Example. The following maps G are intersectionally closed-valued, but not transfer closed-valued.

(1) $G(z) = (0, 1)$ for every $z \in [0, 1]$ is a constant multimap from $D = [0, 1]$ to $E = [0, 1]$; see [4].
(2) $G(z)$ is a convex set in a Euclidean space having a relative interior point in common; see Rockafellar [17, Theorem 6.5].
(3) For a given subset E of a topological vector space with $x^* \in E$, each $G(z)$, $z \in D$, is nicely star shaped at x^*; see [4].

From Theorem 4.1, we have the following KKM type theorem.

Theorem 4.4. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $G : D \multimap E$ a map satisfying conditions (1) and (2) of Theorem 4.1.*

(α) *If G is transfer closed-valued, then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*
(β) *If G is intersectionally closed-valued, then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Proof. Since $\overline{G}$ is a KKM map with closed values, by Theorem 4.1, we have $K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset$.

(α) Since G is transfer closed-valued, we have

$$K \cap \bigcap_{z\in D} G(z) = K \cap \bigcap_{z\in D} \overline{G(z)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued, we have

$$\overline{\bigcap_{z\in D} G(z)} = \bigcap_{z\in D} \overline{G(z)} \neq \emptyset.$$

This implies the conclusion. □

When G is closed-valued or transfer closed-valued in Theorem 4.4, it subsumes a very large number of particular KKM type theorems in the literature and has a number of equivalent formulations as in [18].

5. Fixed points and maximal elements

Note that Theorem 4.4 can be reformulated to the equivalent forms of fixed point theorems, matching theorems, analytic alternatives, minimax inequalities, variational inequalities, geometric and section properties as in Section 2 or in [18–20], more generally. In this section, we deal with only some of them and obtain generalized forms of known results.

From Theorem 4.4, we have another whole intersection property as follows.

Theorem 5.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $G : D \multimap E$ a map satisfying*

(1) *there exists a nonempty compact subset K of E such that either*
 (i) *$K \supset \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K \supset L_N \cap \bigcap_{z\in D'} \overline{G(z)} \neq \emptyset.$$

Suppose that there exists a map $H : E \multimap E$ satisfying

(2) *for each $x \in E$, $x \in H(x)$;*
(3) *for each $x \in E$, $\mathrm{co}_\Gamma(D \setminus G^-(x)) \subset X \setminus H^-(x)$.*
 (α) *If G is transfer closed-valued, then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*
 (β) *If G is intersectionally closed-valued, then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.*

Proof. In view of Theorem 4.4, it suffices to show that (2) and (3) imply that G is a KKM map. Suppose that there exists an $N \in \langle D\rangle$ such that $\Gamma_N \not\subset G(N)$; that is, there exist an $x \in \Gamma_N$ such that $x \notin G(y)$ for all $y \in N$. In other words, $N \in \langle D \setminus G^-(x)\rangle$. By (3), $\Gamma_N \subset X \setminus H^-(x)$. Since $x \in \Gamma_N$, we have $x \notin H^-(x)$ or $x \notin H(x)$. This contradicts (2). □

The following is our version of Fan's geometric lemma [5, Lemma 4].

Corollary 5.2. *Let $(X, D; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle and $A \subset X \times X$, $C \subset D \times X$. Suppose that*

(1) *$(x, x) \in A$ for every $x \in X$;*
(2) *for each $z \in D$, $\{y \in X \mid (z, y) \in C\}$ is intersectionally closed;*
(3) *for any fixed $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid (z, y) \notin C\} \subset \{x \in X \mid (x, y) \notin A\}$.*

Then there exists a point $y_0 \in X$ such that $D \times \{y_0\} \subset C$.

Proof. For each $z \in D$, let $G(z) := \{y \in X \mid (z, y) \in C\}$. Then $G(z)$ is intersectionally closed by (2). Since X is compact, the compactness condition holds trivially. For each $x \in X$, let $H(x) := \{y \in X \mid (x, y) \in A\}$. Then (1) and (3) imply conditions (2) and (3) of Theorem 5.1, resp. Therefore, by Theorem 5.1, $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$, which is to be proved. □

If $X = D$ is a compact convex set in a topological vector space and $A = C$ is a closed subset of $X \times X$, then Corollary 5.2 reduces to Fan [5, Lemma 4]. Fan applied this Lemma to give a simple proof [5] of the Tychonoff fixed point theorem and to many results; see Section 2.

For a multimap $G : D \multimap E$, consider the following related four conditions.

(a) $\bigcup_{z\in D} G(z) = E$ implies $\bigcup_{z\in D} \mathrm{Int}\, G(z) = E$.
(b) $\mathrm{Int} \bigcup_{z\in D} G(z) = \bigcup_{z\in D} \mathrm{Int}\, G(z)$ (G is *unionly open-valued* [4]).
(c) $\bigcup_{z\in D} G(z) = \bigcup_{z\in D} \mathrm{Int}\, G(z)$ (G is *transfer open-valued*).
(d) G is open-valued.

As for intersectionally (resp., transfer) closed sets, we can define *unionly* (resp., *transfer*) *open* sets.

Proposition 5.3 ([4]). *The multimap G is intersectionally closed-valued (resp., transfer closed-valued) if and only if its complement G^c is unionly open-valued (resp., transfer open-valued).*

In view of this proposition, we have proper examples of unionly open-valued maps.

From Theorem 5.1, we have the following fixed point theorem.

Theorem 5.4. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $S : D \multimap E$, $T : E \multimap E$ maps. Suppose that*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S^-(x) \subset T^-(x)$;*
(2) *$E = S(D)$;*
(3) *there exists a nonempty compact subset K of E such that either*
 (i) *$K \supset \bigcap_{z\in M} \overline{E \setminus S(z)}$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex compact subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, and*
$$K \supset L_N \cap \bigcap_{z\in D'} \overline{E \setminus S(z)} \neq \emptyset.$$
 (α) *If S is transfer open-valued, then there exists an $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x})$.*
 (β) *If S is unionly open-valued, then there exists an $\bar{x} \in E$ such that $\bar{x} \in T(\bar{x})$.*

Proof. Suppose that $x \notin T(x)$ for all $x \in E$. Let $G(z) := E \setminus S(z)$ for $z \in D$ and $H(x) := E \setminus T(x)$ for $x \in E$. Then all of the requirements of Theorem 5.1 are satisfied.

(α) Therefore, by Theorem 5.1, there exists a $y_0 \in K \cap \bigcap_{z\in D} G(z)$; that is, $y_0 \in K$ such that $y_0 \notin S(z)$ for all $z \in D$. This contradicts (2).
(β) This is similar to (α). □

From Theorem 5.4, we have the following Fan–Browder type fixed point theorems.

Corollary 5.5. *Let $(X, D; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle, and $S : D \multimap X$, $T : X \multimap X$ maps. Suppose that*

(1) *S is unionly open-valued;*
(2) *for each $x \in X$, $\mathrm{co}_\Gamma S^-(x) \subset T^-(x)$;*
(3) *$X = S(D)$.*

Then T has a fixed point.

Proof 1. Put $X = E = K$ in Theorem 5.4. Then the compactness condition holds trivially. □

We have another proof using Corollary 4.3.

Proof 2. Since S is unionly open-valued, (3) implies $X = \bigcup_{z\in D} \mathrm{Int}\, S(z)$. Since X is compact, we have $X = \bigcup_{z\in M} \mathrm{Int}\, S(z)$ for some $M \in \langle D\rangle$. Now, by applying Corollary 4.3 with S^-, T^- instead of S, T, resp., we have the conclusion. □

In case $X = D$ and $S = T$, we have the following corollary.

Corollary 5.6. *Let $(X; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle, and $T : X \multimap X$ a map. Suppose that*

(1) *T is unionly open-valued;*
(2) *for each $x \in X$, $T^-(x)$ is nonempty and Γ-convex.*

Then T has a fixed point.

When T is open-valued, Corollary 5.6 reduces to the Fan–Browder fixed point theorem.
From Theorem 5.4, we have the following equivalent maximal element theorem.

Theorem 5.7. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $S : E \multimap D$, $T : E \multimap E$ maps. Suppose that*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
(2) *for each $x \in E$, $x \notin T(x)$;*
(3) *there exists a nonempty compact subset K of E such that either*
 (i) *$K \supset \bigcap_{z\in M} \overline{E \setminus S^-(z)}$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex compact subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, and*
$$K \supset L_N \cap \bigcap_{z\in D'} \overline{E \setminus S^-(z)} \neq \emptyset.$$
 (α) *If $S^- : D \multimap E$ is transfer open-valued, then S has a maximal element $\bar{x} \in K$, that is, $S(\bar{x}) = \emptyset$.*
 (β) *If $S^- : D \multimap E$ is unionly open-valued, then S has a maximal element $\bar{x} \in E$, that is, $S(\bar{x}) = \emptyset$.*

Proof. Replace S and T in Theorem 5.4 by S^- and T^-, resp. □

From Theorem 5.7 or Corollary 5.5, we have the following maximal element theorem.

Corollary 5.8. *Let $(X, D; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle, and $S : X \multimap D$, $T : X \multimap X$ maps. Suppose that*

(1) *$S^- : D \multimap X$ is unionly open-valued;*
(2) *for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$;*
(3) *for each $x \in X$, $x \notin T(x)$.*

Then S has a maximal element $x_0 \in X$; that is, $S(x_0) = \emptyset$.

Proof. Replace S and T in Corollary 5.5 by S^- and T^-, resp. □

For $X = D$ and $S = T$, Corollary 5.8 reduces to the following.

Corollary 5.9. *Let $(X; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle, and $T : X \multimap X$ a map. Suppose that*

(1) *$T^- : X \multimap X$ is unionly open-valued;*
(2) *for each $x \in X$, $T(x)$ is Γ-convex;*
(3) *for each $x \in X$, $x \notin T(x)$.*

Then T has a maximal element $x_0 \in X$; that is, $T(x_0) = \emptyset$.

6. Matching theorems, minimax inequalities, and variational inequalities

From Theorem 5.4 or, equivalently, Theorem 5.7, we obtain the following Fan type matching theorem for unionly open covers.

Theorem 6.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $S : D \multimap E$. Suppose that*

(1) *S is unionly open-valued;*
(2) *$E = S(D)$;*
(3) *there exists a nonempty compact subset K of E such that either*
   (i) *$K \supset \bigcap_{z \in M} \overline{E \setminus S(z)}$ for some $M \in \langle D \rangle$; or*
   (ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex compact subspace L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*
$$K \supset L_N \cap \bigcap_{z \in D'} \overline{E \setminus S(z)} \neq \emptyset.$$

Then there exists an $M \in \langle D \rangle$ such that $\Gamma_M \cap \bigcap \{S(z) \mid z \in M\} \neq \emptyset$.

Proof. Let $H : E \multimap E$ and $T : E \multimap E$ be defined by $H(y) := \bigcup \{\Gamma_M \mid M \in \langle S^-(y) \rangle\}$ for $y \in E$ and $T(x) = H^-(x)$ for $x \in E$. Then all of the requirements of Theorem 5.4 are satisfied, and hence T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$. Consequently, we have $x_0 \in T^-(x_0) = \bigcup \{\Gamma_M \mid M \in \langle S^-(x_0) \rangle\}$, and hence there exists a finite set $M \subset S^-(x_0) \subset D$ such that $x_0 \in \Gamma_M$. Since $M \in \langle S^-(x_0) \rangle$ implies $x_0 \in S(z)$ for all $z \in M$, we have $x_0 \in \Gamma_M \cap \bigcap \{S(z) \mid z \in M\}$. □

Here, for abstract convex spaces satisfying the partial KKM principle, note that Theorems 4.4, 5.1, 5.4 and 5.7 for the case (β) and 6.1 are equivalent, and the following suffices to show the fact.

Proof of Theorem 4.4(β) using Theorem 6.1. Suppose that its conclusion $\bigcap \{G(z) \mid z \in D\} \neq \emptyset$ does not hold. Since G is intersectionally closed-valued, we have

$$\bigcap_{z \in D} \overline{G(z)} = \overline{\bigcap_{z \in D} G(z)} = \emptyset.$$

Then $E = S(D)$ where $S(z) = E \setminus \overline{G(z)}$ for $z \in D$. Then, by Theorem 6.1, there exists an $M \in \langle D \rangle$ such that $\Gamma_M \cap \bigcap \{S(z) \mid z \in M\} \neq \emptyset$; that is, $\Gamma_M \not\subset \overline{G}(M)$. This contradicts condition (1) of Theorem 4.1. □

The following particular tautology of Theorem 4.1 follows from any of 4.1, 4.4, 5.1, 5.4, 5.7 and 6.1.

Corollary 6.2. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $H : D \multimap E$ be a KKM map. Then the family $\{\overline{H(z)} \mid z \in D\}$ has the finite intersection property.*

Proof. Suppose that $\bigcap \{\overline{H(z)} \mid z \in M\} = \emptyset$ for some $M \in \langle D \rangle$. Put $S(z) = E \setminus \overline{H(z)}$ for each $z \in D$. Then $S(z)$ satisfies all of the requirements of Theorem 6.1. [Since $S(M) = E$, the compactness condition (i) holds for any nonempty compact subset K.] By the conclusion of Theorem 6.1, $\Gamma_M \cap \bigcap \{S(z) \mid z \in M\} \neq \emptyset$; that is, $\Gamma_M \not\subset \bigcup \{\overline{H(z)} \mid z \in M\}$, hence $\Gamma_M \not\subset H(M)$, a contradiction. □

Since Corollary 6.2 implies Theorem 4.1, all Theorems 4.1, 4.4, 5.1, 5.4, 5.7 and 6.1, and Theorem 6.1 are mutually equivalent in a wide sense (especially, for closed-valued KKM maps).

There exist another equivalent formulations of Theorem 4.4 analogous to the results due to the author in [3,19].

Theorem 4.4 can be reformulated as for the case (0)′–(XI)′ in [3] and, note that Theorems 5.1, 5.4, 5.7 and 6.1 are only some of them in a dozen equivalent statements. We give another example corresponding to (XI)′ in [3]. The following corrects [3, Theorem 4].

Theorem 6.3 (*Minimax Inequality*). *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, $f : D \times E \to \overline{\mathbb{R}}$, $g : E \times E \to \overline{\mathbb{R}}$ extended real-valued functions, and $\gamma \in \overline{\mathbb{R}}$ such that*

(0) *for each $x \in E$, $g(x, x) \leq \gamma$;*
(1) *for each $z \in D$, $G(z) := \{y \in E \mid f(z, y) \leq \gamma\}$ is intersectionally closed [resp., transfer closed];*
(2) *for each $y \in E$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in E \mid g(x, y) > \gamma\}$;*
(3) *the map $G : D \multimap E$ has a coercivity condition (1) in Theorem 5.1.*

Then

(a) *there exists a $y_0 \in E$ [resp., $y_0 \in K$] such that $f(z, y_0) \leq \gamma$ for all $z \in D$;*
(b) *if $\gamma := \sup_{x\in E} g(x, x)$, then we have*

$$\inf_{y\in E} \sup_{z\in D} f(z, y) \leq \sup_{x\in E} g(x, x).$$

Proof. Let $H(y) := \{x \in E \mid g(x, y) \leq \gamma\}$ for $y \in E$. Then $x \in H(x)$ for each $x \in E$ by (0) and hence, condition (2) of Theorem 5.1 holds. Moreover, condition (2) implies condition (3) of Theorem 5.1. Therefore, by Theorem 5.1, we have $\bigcap_{z\in D} G(z) \neq \emptyset$ [resp., $K \cap \bigcap_{z\in D} G(z) \neq \emptyset$]. The conclusion follows. □

Recall that, for an abstract convex space $(E \supset D; \Gamma)$, a function $f : E \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

Note that condition (2) in Theorem 6.3 generalizes these concepts.

We define new concepts as follows.

Definition. An extended real-valued function $f : D \times E \to \overline{\mathbb{R}}$ is said to be *generally lower* [resp., *upper*] *semicontinuous* (g.l.s.c.) [resp., (g.u.s.c.)] on E whenever, for each $z \in D$, $\{y \in E \mid f(z, y) \leq r\}$ [resp., $\{y \in E \mid f(z, y) \geq r\}$] is intersectionally closed for each $r \in \overline{\mathbb{R}}$.

Example. 1. If the intersectionally closedness are replaced by transfer closedness, then f is said to be transfer l.s.c. [resp., transfer u.s.c.].

2. If the intersectionally closed sets are replaced by transfer closed sets for a particular $\gamma \in \overline{\mathbb{R}}$ instead of arbitrary r, then f is said to be γ-transfer l.s.c. in y [that is, for each $x \in D$, $\{y \in E \mid f(x, y) \leq \gamma\}$ is transfer closed]; see Tian [21].

3. Instead of transfer closedness, some authors also adopt compactly closedness, transfer compactly closedness, or even the finitely closedness. The l.s.c. can be replaced by compactly l.s.c., transfer compactly l.s.c. or finitely l.s.c. This kind of terminology is not essential and generalizes nothing important. We will not adopt such artificial terminology any more. For details, see [22].

4. Note that these concepts can be extended w.r.t. any simply ordered set $\mathbb{S}$ instead of $\overline{\mathbb{R}}$.

Corollary 6.4 (*Minimax Inequality*). *Let $(X, D; \Gamma)$ be a compact abstract convex space satisfying the partial KKM principle, $f : D \times X \to \overline{\mathbb{R}}$, $g : X \times X \to \overline{\mathbb{R}}$ extended real-valued functions, and $\gamma = \sup_{x\in X} g(x, x) \in \overline{\mathbb{R}}$ such that*

(1) *for each $z \in D$, $\{y \in X \mid f(z, y) \leq \gamma\}$ is intersectionally closed;*
(2) *for each $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \gamma\} \subset \{x \in X \mid g(x, y) > \gamma\}$.*

Then

(a) *there exists a $y_0 \in X$ such that $f(z, y_0) \leq \gamma$ for all $z \in D$;*
(b) *we have*

$$\inf_{y\in X} \sup_{z\in D} f(z, y) \leq \sup_{x\in X} g(x, x).$$

Furthermore, if

(3) *$y \mapsto \sup_{z\in D} f(z, y)$ is l.s.c. on X,*

then we have

$$\min_{y\in X} \sup_{z\in D} f(z, y) \leq \sup_{x\in X} g(x, x).$$

Note that if for each $z \in D$, $y \mapsto f(z, y)$ is l.s.c., then (1) and (3) are satisfied.

From now on, for simplicity, we are mainly concerned with compact abstract convex spaces $(X; \Gamma)$ satisfying the partial KKM principle. For example, any compact G-convex space, any compact H-space, any compact Lassonde type convex space, and any compact convex subset of a t.v.s. is such a space. Of course, all results in this section can be generalized to noncompact case by assuming proper compactness conditions for relevant KKM maps.

Consider the following statements for compact abstract convex spaces $(X; \Gamma)$ satisfying the partial KKM principle.

(XII) Minimax inequality. *Corollary 6.4 for the particular case when $X = D$.*

(XIII) Minimax inequality. *Let $f, g : X \times X \to \mathbb{R}$ be functions and $\gamma \in \mathbb{R}$ such that*
(13.1) *for any $x, y \in X$, $f(x, y) \leq g(x, y)$ and $g(x, x) \leq \gamma$;*
(13.2) *for each $x \in X$, $\{y \in X \mid f(x, y) > \gamma\}$ is unionly open in X;*
(13.3) *for each $y \in X$, $\{x \in X \mid g(x, y) > \gamma\}$ is Γ-convex on X.*

Then (i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \leq \gamma \quad \text{for all } x \in X;$$

(ii) *if $\gamma := \sup_{x\in X} g(x, x)$, then*

$$\inf_{y\in X} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} g(x, x).$$

(XIV) Minimax inequality. *Let $f, g : X \times X \to \mathbb{R}$ be functions such that*
(14.1) *$f(x, y) \leq g(x, y)$ for each $(x, y) \in X \times X$,*
(14.2) *for each $x \in X$, $f(x, \cdot)$ is g.l.s.c. on X;*
(14.3) *for each $y \in X$, $g(\cdot, y)$ is quasiconcave on E.*

Then we have

$$\inf_{y\in X} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} g(x, x).$$

(XV) Variational inequality. *Let $p, q : X \times X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ be functions satisfying*
(15.1) *$p(x, y) \leq q(x, y)$ for each $(x, y) \in X \times X$, and $q(x, x) \leq 0$ for all $x \in X$;*
(15.2) *for each $x \in X$, $p(x, \cdot) + h(\cdot)$ is g.l.s.c. on X;*
(15.3) *for each $y \in X$, $q(\cdot, y) - h(\cdot)$ is quasiconcave on X.*

Then there exists a $y_0 \in X$ such that

$$p(x, y_0) + h(y_0) \leq h(x) \quad \text{for all } x \in X.$$

(XVI) Variational inequality. *Let $f, g : X \times X \to \mathbb{R}$ be a function satisfying*
(16.1) *for any $x, y \in X$, $f(x, y) \leq g(x, y)$;*
(16.2) *for each $x \in X$, $\{y \in X \mid f(x, y) < f(y, y)\}$ is unionly open;*
(16.3) *for each $y \in X$, $\{x \in X \mid g(x, y) < g(y, y)\}$ is Γ-convex.*

Then (i) *there exists a $y_0 \in X$ such that*

$$f(x, y_0) \geq f(y_0, y_0) \quad \text{for all } x \in X; \quad \text{and}$$

(ii) *we have*

$$\sup_{y\in X} \inf_{x\in X} f(x, y) \geq \inf_{x\in X} f(x, x).$$

(XVII) Variational inequality. *Let $f, g : X \times X \to \mathbb{R}$ be functions satisfying*
(17.1) *$f \leq g$ on the diagonal $\Delta := \{(x, x) \mid x \in X\}$ and $g \leq f$ on $(X \times X) \setminus \Delta$;*
(17.2) *for each $x \in X$, $y \mapsto g(y, y) - g(x, y)$ is g.l.s.c. on X;*
(17.3) *for each $y \in X$, $x \mapsto f(x, y)$ is quasiconcave on X.*

Then there exists a $y_0 \in X$ such that

$$f(y_0, y_0) \geq f(x, y_0) \quad \text{for all } x \in X.$$

Note that each of (XII)–(XVII) generalizes the corresponding ones in [3] and the following generalization of [3, Theorem 5] holds as in [3].

Theorem 6.5. *For a compact abstract convex space $(X; \Gamma)$ satisfying the partial KKM principle, the statements* (XII)–(XVII) *hold.*

Remark. Previously known particular forms of the KKM theorems, fixed point theorems, and maximal element theorems in this paper and their applications will be introduced elsewhere.

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