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Remarks on some basic concepts in the KKM theory

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ABSTRACT

In the KKM theory, some authors adopt the concepts of the compact closure (ccl), compact interior (cint), transfer compactly closed-valued multimap, transfer compactly l.s.c. multimap, and transfer compactly local intersection property, respectively, instead of the closure, interior, closed-valued multimap, l.s.c. multimap, and possession of a finite open cover property. In this paper, we show that such adoption is inappropriate and artificial. In fact, any theorem with a term with "transfer" attached is equivalent to the corresponding one without "transfer". Moreover, we can invalidate terms with "compactly" attached by giving a finer topology on the underlying space. In such ways, we obtain simpler formulations of KKM type theorems, Fan–Browder type fixed point theorems, and other results in the KKM theory on abstract convex spaces.

1. Introduction

The KKM theory, first called this by the author [1], is the study of applications of various equivalent or generalized formulations of the KKM theorem given by Knaster, Kuratowski, and Mazurkiewicz in 1929. At the beginning, the theory was mainly devoted to the study of convex subsets of topological vector spaces (t.v.s.). Later, it was extended to convex spaces by Lassonde, and to C-spaces (or H-spaces) by Horvath, and others. In 1993–2006, the KKM theory was further extended to generalized convex (G-convex) spaces in a sequence of papers by the present author and others; see [2,3] and references therein. Furthermore, since 2006, we have introduced a new concept of abstract convex spaces which are adequate for establishing the KKM theory; see [3–7].

The "closed" version of the following is the origin of the KKM theory; see [2].

Theorem 1.1 (*KKM*). *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is,* $\operatorname{co} A \subset G(A)$ *for each $A \subset D$) with closed (or open) values. Then $\bigcap_{z\in D} G(z) \neq \emptyset$.*

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Since 1995, there have appeared the concepts of the compact closure (ccl), compact interior (cint), transfer compactly closed-valued multimap, transfer compactly l.s.c. multimap, and transfer compactly local intersection property, respectively, instead of the basic concepts of the closure, interior, closed-valued multimap, l.s.c. multimap, and possession of a finite open cover property. The aim of introducing the above unfamiliar terms is to claim theorems of utmost generality, but, unfortunately, such terms are very difficult to check in practice and, in many cases, have no proper examples. Therefore they are artificial, impractical, and useless.

In our previous work [8] in 2000, we showed that we can invalidate many such terms on G-convex spaces by replacing the original topology of the underlying space by a finer topology. However, in the last decade, a number of authors continued to use the inappropriate terms. Even in 2009, there were authors who were still using such obsolete terminology; we cite just a few examples: [9–13].

In this paper, we show that such adoption of terms is inappropriate and artificial in the KKM theory of abstract convex spaces. In fact, any theorem with a term with "transfer" attached is equivalent to the corresponding one without "transfer". Moreover, we can invalidate terms with "compactly" attached by giving a finer topology on the underlying space. In such ways, we obtain simpler formulations of KKM type theorems, Fan–Browder type fixed point theorems, and other results in the KKM theory on abstract convex spaces. Consequently, we can upgrade the theory by eliminating the above-mentioned inappropriate terms and by stating theorems in more elegant forms.

2. KKM type theorems

Multimaps are also called simply maps. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following from [3–7]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

For any $D' \subset D$, the *Γ-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such a case, a subset X of E is said to be *Γ-convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. If $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example 2.1. There are plenty of examples of abstract convex spaces; see [3–8]. Here we give only two classes of them:

(I) A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ in the sense of Park is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J its face corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

The standard examples of G-convex spaces are any convex subset of a t.v.s., the Lassonde type convex spaces, and the Horvath type H-spaces; see [2] and references therein. A G-convex space $(X; \Gamma)$ was called an L-space by Ben-El-Mechaiekh et al. There are many imitations or modifications of G-convex spaces; for example, other L-spaces, FC-spaces, ϕ_A-spaces, GFC-spaces, and many others which can be made into G-convex spaces; see [4,7,14,15].

(II) A *ϕ_A-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D\rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [4,14]. The so-called FC-spaces $(X; \{\varphi_A\})$ due to Ding in [10–12] are particular kinds of ϕ_A-spaces for the case $X = D$. Later, ϕ_A-spaces were called GFC-spaces in [13].

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property *also* holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

From the partial KKM principle we have a whole intersection property of the Fan type as follows [7, Theorem 2.9]:

Theorem 2.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $G : D \multimap E$ a map such that:*

(1) *G is closed-valued;*
(2) *G is a KKM map; and*
(3) *there exists a nonempty compact subset K of E such that one of the following holds:*
 (i) *$K = E$;*
 (ii) *$K = \bigcap\{G(z) \mid z \in M\}$ for some $M \in \langle D\rangle$; or*
 (iii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap_{z\in D'} G(z) \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. *Case* (i). In this case every $G(y)$ is compact. Hence Case (i) reduces to (ii).

Case (ii). Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap G(z) \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property.

Case (iii). Suppose that $K \cap \bigcap\{G(z) \mid z \in D\} = \emptyset$. Then $K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$ for some $N \in \langle D\rangle$ since K is compact. Let L_N be the compact Γ-convex subset of E in (iii). Define $G' : D' \multimap L_N$ by $G'(z) := G(z) \cap L_N$ for $z \in D'$. Then $A \in \langle D'\rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset G(A) \cap L_N = G'(A)$ by (2); and hence $G' : D' \multimap L_N$ is a KKM map on $(L_N, D'; \Gamma')$ with closed values. Since $(X, D; \Gamma)$ satisfies the partial KKM principle, so does $(L_N, D'; \Gamma')$; see [5, Lemma 2]. Hence, $\{G'(z) \mid z \in D'\}$ has the finite intersection property. Since L_N is compact, $\bigcap\{G'(z) \mid z \in D'\} \neq \emptyset$ by Case (i). For any $y \in \bigcap\{G'(z) \mid z \in D'\}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{X \setminus G(z) \mid z \in N\}$, we have $y \notin G(z)$ for some $z \in N \subset D'$. This is a contradiction.

Therefore, we must have $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$. □

Recall that conditions (i)–(iii) in Theorem 2.1 are usually called the compactness conditions or the coercivity conditions. Theorem 2.1 is a particular form of [7, Theorem 2.12] and its proof is given here for completeness. Case (iii) has numerous variations or particular forms appearing in a very large number of literature entries, but, as the above proof shows, it can be easily deduced from the compact Case (i), and hence it seems to be not a big problem to treat Case (iii).

For a multimap $F : D \multimap X$, we define a multimap $\overline{F} : D \multimap X$ by $\overline{F}(z) := \overline{F(z)}$ for all $z \in D$, where $\overline{}$ denotes the closure operator.

From Theorem 2.1, we can deduce an equivalent form of [6, Theorem 8.2]:

Corollary 2.1. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $G : D \multimap E$ a map such that:*

(1) *$\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ (that is, G is transfer closed-valued);*
(2) *$\overline{G}$ is a KKM map; and*
(3) *there exists a nonempty compact subset K of E such that either:*
 (i) *$K \supset \bigcap\{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K \supset L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Note that (1) implies

$$K \cap \bigcap\{\overline{G(z)} \mid z \in D\} \neq \emptyset \Longrightarrow K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset.$$

Therefore, Theorem 2.1 works. □

Note that condition (1) of Theorem 2.1 implies the condition (1) of Corollary 2.1, and not conversely. However Corollary 2.1 reduces to Theorem 2.1 if G has closed values. Therefore, Theorem 2.1 and Corollary 2.1 are equivalent.

In view of this equivalency of Theorem 2.1 and Corollary 2.1, any "transfer" versions of KKM type theorems can be easily deduced from the corresponding non-"transfer" version even when they are practically or urgently necessary. Since every theorem in the KKM theory is based on Theorem 1.1 or the partial KKM principle, it is enough to treat only the non-"transfer" version and this will not lose any generality.

Example 2.2. (I) Corollary 2.1 originates from Tian [16, Theorem 2]. He gave a definition as follows: Let X be a set and Y a topological space. A map $G : X \multimap Y$ is said to be *transfer open-valued* [resp., *transfer closed-valued*] if for each $x \in X$, $y \in G(x)$ [resp., $y \notin G(x)$] implies that there exists a point $x' \in X$ such that $y \in \operatorname{Int} G(x')$ [resp., $y \notin \overline{G(x')}$].

Note that G is transfer open-valued iff $G(X) := \bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{Int} G(x)$; and G is transfer closed-valued iff $\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \overline{G(x)}$. These are simpler definitions than the original one.

(II) Notions more refined than the above definitions are due to Wu and Zhang [17] as follows: A family $\{D_\alpha \mid \alpha \in I\}$ of some subsets of a topological space X is called *closed* [resp., *open*] *transfer complete* if for each $x \in X$ such that $x \notin D_{\alpha_0}$ [resp., $x \in D_{\alpha_0}$] for some $\alpha_0 \in I$, there exists $\alpha' \in I$ such that $x \notin \overline{D_{\alpha'}}$ [resp., $x \in \operatorname{Int} D_{\alpha'}$]. Obviously, if $\{D_\alpha \mid \alpha \in I\}$ is a family of some closed [resp., open] subset of X, then it is closed [resp., open] transfer complete. A multimap $T : Y \multimap X$ is said to be *transfer closed-valued* if the family $\{T(y) \mid y \in Y\}$ is closed transfer complete.

These notions have disappeared recently in the sense that nobody seems to use them any more.

Now is the proper time to discard such notions.

3. Compact interiors and compact closures

It should be noted that the conclusions of any KKM type theorems are certain set-theoretical intersection properties of multimap values of the KKM maps, and hence it is possible to change the topologies of the underlying spaces without loss of generality. For this purpose, we recall the following:

A topological space X is called a *compactly generated space* (or a *k-space*) iff the following condition holds:

(K) $A \subset X$ is open [resp., closed] iff $A \cap K$ is open [resp., closed] in K for each compact set K in X (that is, iff A is compactly open [resp., compactly closed]).

There are lots of examples of compactly generated spaces; see [18].

An abstract convex space $(E, D; \Gamma)$ is said to have *finitely generated topology* iff the following condition holds:

(F) $A \subset E$ is open [resp., closed] iff $A \cap \Gamma_N$ is open [resp., closed] in Γ_N for each $N \in \langle D \rangle$ (that is, iff A is finitely open [resp., finitely closed]).

A convex subset of a topological vector space with the finite topology has the finitely generated topology, and any convex space in the sense of Lassonde [19] has the finitely generated topology.

The following shows that any G-convex space is a KKM space [8,20], and its proof is given here for completeness:

Theorem 3.1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that:*

(1) *F has closed [resp., open] values; and*
(2) *F is a KKM map.*
Then $\{F(z) \mid z \in D\}$ has the finite intersection property.
Furthermore, if
(3) *$\bigcap_{z\in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z\in D} \overline{F(z)} \neq \emptyset.$$

Proof. Let $N := \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$. Then there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that, for any $0 \le i_0 < i_1 < \cdots < i_k \le n$, we have

$$\phi_N(\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subset \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n).$$

Since F is a KKM map, it follows that

$$\mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \phi_N^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n)) \subset \bigcup_{j=0}^{k} \phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n)).$$

Since $F(a_{i_j}) \cap \phi_N(\Delta_n)$ is closed [resp., open] in the compact subset $\phi_N(\Delta_n)$ of Γ_N, $\phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n))$ is closed [resp., open] in Δ_n. Note that $e_i \multimap \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n))$ is a KKM map on $\{e_0, e_1, \ldots, e_n\}$. Hence, by the KKM Theorem 1.1, we have

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \Gamma_N) \supset \bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n)) \neq \emptyset.$$

This readily implies $\Gamma_N \cap \bigcap_{z\in N} F(z) \neq \emptyset$. The second conclusion is clear. □

Note that any ϕ_A-space satisfies Theorem 3.1 and its proof with $\Gamma_A := \phi_A(\Delta_n)$.

By closely examining the proof of Theorem 3.1, we see that the values $F(z)$ might be *compactly closed* [resp., *compactly open*] in X. Moreover, the proof works if we assume that, for each $z \in D$ and $N \in \langle D \rangle$, $F(z) \cap \Gamma_N$ is closed [resp., open] in Γ_N, instead of (1); that is, F has *finitely closed* [resp., *finitely open*] values; see [21].

In fact, condition (1) of Theorem 3.1 can be replaced by the following:

(1)′ *F has compactly closed [resp., compactly open] values; or*
(1)″ *F has finitely closed [resp., finitely open] values.*

However, by adopting the compactly generated topology on X or the finitely generated topology on $(X, D; \Gamma)$, it is easily seen that $(1)'$ or $(1)''$ simply implies (1).

For a topological space $(X, \mathcal{T})$, the *compactly generated extension* of the topology $\mathcal{T}$ is the new topology consisting of all compactly closed [resp., open] subsets. Similarly, for an abstract convex space $(E, D; \Gamma)$, the *finitely generated extension* of the original topology $\mathcal{T}$ of E is the new topology consisting of all finitely closed [resp., open] subsets.

In this way, we have the following modified form of Theorem 3.1:

Theorem 3.2. *Let $(X, D; \Gamma)$ be a G-convex space and $G : D \multimap X$ a map such that:*

(1) *G is compactly [resp., finitely] closed-valued;*
(2) *G is a KKM map; and*
(3) *the condition* (3) *of Theorem* 3.1 *holds.*

Then $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Let us replace the topology of X by its compactly [resp., finitely] generated extension. Then $(X, D; \Gamma)$ with this new topology is another G-convex space and satisfies the partial KKM principle. Moreover, G becomes a closed-valued KKM map. Then, Theorem 3.1 works. □

Note that Theorems 3.1 and 3.2 are equivalent. Therefore, we can eliminate the term "compactly" or "finitely" in Theorem 3.2 without any loss of generality.

Such inappropriate terminology has a quite long history and appears even in the most recent literature:

Example 3.1. (I) For any given subset A of a topological space X, Ding [22,23] defined *the compact closure* and *the compact interior* of A, denoted by $\mathrm{ccl}(A)$ and $\mathrm{cint}(A)$, as

$\mathrm{ccl}(A) = \bigcap\{B \subset X \mid A \subset B$ and B is compactly closed in $X\}$, and

$\mathrm{cint}(A) = \bigcup\{B \subset X \mid B \subset A$ and B is compactly open in $X\}$, respectively.

These do not mean the compactness of closure and interior, respectively. However, these are simply the closure and the interior of A whenever X has the compactly generated topology.

Similarly, "*finitely* metrically closed sets" in [24] could be simplified.

(II) Moreover, Ding [22,23] extended the above definition as follows: Let X and Y be two topological spaces. A map $G : X \multimap Y$ is said to be *transfer compactly open-valued* [resp., *transfer compactly closed-valued*] on X if for $x \in X$ and for each nonempty compact subset K of Y, $y \in G(x) \cap K$ [resp., $y \notin G(x) \cap K$] implies that there exists a point $x' \in X$ such that $y \in \mathrm{Int}_K(G(x') \cap K)$ [resp., $y \notin \mathrm{cl}_K(G(x') \cap K)$].

However, this concept reduces to that of being transfer open-valued [resp., transfer closed-valued] if we give Y the compactly generated topology.

Therefore, Ding's use of "compactly" open [resp., closed] sets does not generalize anything and is not practical. Recall that Ding misled many naive readers or followers for a long period by using ccl or cint, without giving any proper example. Recently he seems to be not using ccl and cint sometimes. But some naive followers are still using them. We give only a few examples that appeared in 2009.

(III) In 2009 [12], its authors' definition of the extension of Park's better admissible class $\mathfrak{B}$ of multimaps is incorrectly stated and, hence, results in [12] cannot generalize anything. Moreover, in [12] inappropriate terminology like *FC*-spaces, cint, ccl, and others due to Ding are adopted. Its authors claimed that their results generalize corresponding results due to Ding for *G*-convex spaces.

(IV) In 2009 [13], its authors defined *GFC*-space as exactly the same as ϕ_A-spaces and obtained the same KKM theorem. So their *GFC*-spaces can be made into *G*-convex spaces. Moreover, they adopted inappropriate terminology like cint, ccl, and transfer compactly open-valued maps due to Ding.

(V) In 2009 [9], there appeared several false statements; see [25]. Moreover, its author used *transfer compactly lower semicontinuous real-valued functions.* Further, let X and E be two topological spaces and $Y \subset E$. A map $H : X \multimap E$ is said to be *transfer compactly open-valued with respect to* Y if, for each $(x, y) \in X \times E$ with $y \in H(x) \cap Y$, there exist $x \in X$ and a compactly open neighborhood $O(y)$ of y in Y such that $O(y) \subset H(x') \cap Y$.

Again, its author did not give any proper example of these definitions.

4. The local intersection property

The (partial) KKM principle is equivalent to the following Fan–Browder type fixed point theorem [6,26]:

Theorem 4.1. *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*

(1) *$S(z)$ is open for each $z \in D$;*
(2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$X = \bigcup_{z \in N} S(z)$ for some $N \in \langle D \rangle$,*

T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

An abstract convex space $(X, D; \Gamma)$ is a KKM space iff the above condition also holds for any map $S : D \multimap X$ satisfying

(1)′ *$S(z)$ is closed for each $z \in D$*

instead of (1).

Corollary 4.1. *Let $(X; \Gamma)$ be an abstract convex space satisfying the partial KKM principle and $S : X \multimap X$ a map such that*

(4) *$X = \bigcup\{\text{Int}\, S^-(z) \mid z \in N\}$ for some $N \in \langle X \rangle$.*

Then there exists a point $x_0 \in X$ such that $x_0 \in \text{co}_\Gamma\, S(x_0)$.

Proof. In Theorem 4.1, replace S by $\text{Int}\, S^-$. Then (1) holds. Define $T : X \multimap X$ by $T^-(y) := \text{co}_\Gamma\, S(y)$ for $y \in X$. Then $\text{co}_\Gamma(\text{Int}\, S^-)^-(y) \subset \text{co}_\Gamma S(y) = T^-(y)$, and hence (2) follows. Moreover, (4) implies (3). Therefore, by Theorem 4.1, T has a fixed point $x_0 \in X$, that is, $x_0 \in T^-(x_0) = \text{co}_\Gamma S(x_0)$. □

Corollary 4.1 implies the original Fan–Browder fixed point theorem when X is a compact convex subset of a topological vector space.

From Corollary 4.1, we have the following equivalent form:

Corollary 4.2. *If we replace* (4) *in Corollary 4.1 by:*

(4)′ *$\bigcup\{S^-(z) \mid z \in X\} = \bigcup\{\text{Int}\, S^-(z) \mid z \in X\}$ (or S^- is transfer open-valued) and if X is compact, then either:*

(a) *there exists an $x_0 \in X$ such that $S(x_0) = \emptyset$; or*
(b) *there exists an $x_1 \in X$ such that $x_1 \in \text{co}_\Gamma\, S(x_1)$.*

Proof. Suppose that for any $x \in X$, there exists a $z \in X$ such that $z \in S(x)$ or $x \in S^-(z)$. Then (4)′ implies $X = \bigcup\{\text{Int}\, S^-(z) \mid z \in X\}$. If X is compact, this implies (4), and hence (b) holds by Corollary 4.1. □

The point x_0 in (a) is called a *maximal element* of S; see [27].

Example 4.1. (I) The origins of Theorem 4.1 are due to Fan and Browder. Browder's method is based on a continuous selection theorem and the Brouwer fixed point theorem. Sonnenschein weakened the openness assumption of Fan as in (4)′; see [27].

(II) Note that condition (4) originates from Tarafdar [28]. It is clear that (4) implies that S^- is transfer open-valued, and not conversely.

(III) In 1996, Wu and Shen [29] gave the following definition: For topological spaces X and Y, a map $G : X \multimap Y$ is said to have *the local intersection property* on X if for each $x \in X$ with $G(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x in X such that $\bigcap_{z \in N(x)} G(z) \neq \emptyset$. Note that if G has open inverse images $G^-(y)$, then G has the local intersection property. In [29], Browder's selection method is incorrectly stated there as due to Yannelis and Prabhakar.

Since then a large number of authors have investigated almost trivial equivalent conditions of the local intersection property or more general artificial properties.

It is easily seen that if G has nonempty values, then G has the local intersection property [29] if and only if G^- is transfer open-valued [14] and if and only if condition (4)′ holds for $G = S$. It is also stated that G is *transfer open inversed valued* in [24].

(IV) In 1999, Ding [30] refined the above notion as follows: A map $G : X \multimap Y$ is said to have the *compactly local intersection property* on X if $G|_K$ has the local intersection property for any nonempty compact subset K of X.

As we mentioned several times, Ding's notion reduces to the usual one if we adopt the compactly generated topology.

(V) In 2009 [12], its authors adopt the peculiar concept cint, ccl, the transfer compactly closedness, and the compact local intersection property. Ding's routine claim that his *FC*-spaces contain *L*-convex spaces and *G*-convex spaces is repeated here. They give an incorrect example of an *FC*-space which is not an *L*-convex space. Here *L*-convex spaces mean *L*-spaces due to Ben-El-Mechaiekh et al.

5. Transfer lower semicontinuity

In order to discuss other terminology, we consider the following generalized form of Tan et al. [31, Theorem 2.1]:

Theorem 5.1. *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, X compact and $f : X \times D \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ a function such that for each $r \in \mathbb{R}$:*

(1) *$\bigcup_{z \in D}\{x \in X \mid f(x, z) > r\} = \bigcup_{z \in D} \text{Int}\{x \in X \mid f(x, z) > r\}$; and*
(2) *the map $z \longmapsto \overline{\{x \in X \mid f(x, z) \leq r\}}$ is KKM on D.*

Then there exists $x^* \in X$ such that $f(x^*, z) \leq r$ for all $z \in D$.

Proof. Let $F : D \multimap X$ be defined by $F(z) := \{x \in X \mid f(x,z) \leq r\}$ for $z \in D$. Then $\overline{F} : D \multimap X$ is a closed-valued KKM map on a compact abstract convex space $(X, D; \Gamma)$ satisfying the partial KKM principle. Note that (1) is equivalent to $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)}$. Therefore, by Corollary 2.1, we have $x^* \in \bigcap_{z\in D} F(z)$. This implies the conclusion. □

Example 5.1. (I) Note that (1) simply tells us that F is transfer closed-valued.

This fact is defined by Wu and Zhang [17] as follows: For a set X, a topological space Y, and $f : X \times Y \to \mathbb{R}$, $f(x,y)$ is said to be *W-lower* [resp., *W-upper*] *semicontinuous* in y if for each $y \in Y$ and each $r \in \mathbb{R}$ with $\{x \in X \mid f(x,y) > r\} \neq \emptyset$ [resp., $\{x \in X \mid f(x,y) < r\} \neq \emptyset$], there exists $x' \in X$ such that $y \in \mathrm{Int}\{z \in Y \mid f(x',z) > r\}$ [resp., $y \in \mathrm{Int}\{z \in Y \mid f(x',z) > r\}$].

Obviously, if $f(x,y)$ is lower (upper) semicontinuous in y, then $f(x,y)$ is W-lower (W-upper) semicontinuous in y.

(II) Moreover, Tian [16] and Wu and Li [32] defined the following: For a topological space X and a set Y, a function $f : X \times Y \to \mathbb{R}$ is said to be *transfer lower* [resp., *transfer upper*] *semicontinuous* about x on X if for each $r \in \mathbb{R}$, and each $x \in X$ and $y \in Y$, $f(x,y) > r$[resp., $f(x,y) < r$] implies that there exists an open neighborhood $N(x)$ of x and a point $y' \in Y$ such that $f(z,y') > r$[resp.,$f(z,y') < r$] for all $z \in N(x)$.

(III) Wu and Li [32] also noted that: If f is lower [resp., upper] semicontinuous in x, then f is transfer lower [resp., transfer upper] semicontinuous about x on X. If a function $f : X \times Y \to \mathbb{R}$ is transfer lower [resp., transfer upper] semicontinuous about x on X, then the marginal function $x \mapsto \sup_{y\in Y} f(x,y)$ is lower semicontinuous [resp., $x \mapsto \inf_{y\in Y} f(x,y)$ is upper semicontinuous]. The converse is not true.

(IV) Further, Ding [30] obtained more general terminology as follows: Let X and Y be two topological spaces, $\lambda \in \mathbb{R}$, and $T : X \multimap Y$ a multimap. A function $\phi : X \times Y \times X \to \mathbb{R}$ is said to be λ-*transfer compactly upper semicontinuous* in x with respect to T if for each compact subset K of X, $\{z \in X \mid \sup_{y\in T(x)} \phi(x,y,z) < \lambda\} \neq \emptyset$ implies that there exists an open neighborhood $N(x)$ of x and a point $z' \in X$ such that $\sup_{y\in T(u)} \phi(u,y,z') < \lambda$ for all $u \in N(x) \cap K$. If we define a multimap $G : X \multimap Y$ by $G(x) = \{z \in X \mid \sup_{y\in T(x)} \phi(x,y,z) < \lambda\}$, then G has the compactly local intersection property if and only if $\phi(x,y,z)$ is λ-transfer compactly upper semicontinuous in x with respect to T.

Therefore, ϕ is λ-transfer compactly upper semicontinuous in x with respect to T if and only if

$$\bigcup_{x\in X}\left\{z \in X \mid \sup_{y\in T(x)} \phi(x,y,z) < \lambda\right\} = \bigcup_{x\in X}\mathrm{Int}\left\{z \in X \mid \sup_{y\in T(x)} \phi(x,y,z) < \lambda\right\},$$

or if and only if the map $F : X \multimap Y$ defined by

$$F(x) = \left\{z \in X \mid \sup_{y\in T(x)} \phi(x,y,z) \geq \lambda\right\} \quad \text{for } x \in X$$

is transfer closed-valued, whenever we adopt the compactly generated topology on X.

It should be emphasized that, sometimes, the analytical expression is more informative than any artificial terminology.

Note that Theorem 5.1 is an example of equilibrium theorems. There are lots of generalized forms of Theorem 5.1 with various compactness conditions.

6. For abstract convex spaces $(X \supset D; \Gamma)$

For a G-convex space $(X \supset D; \Gamma)$, a subset Y of X is called a *G-convex subspace* of $(X \supset D; \Gamma)$ if $(Y, Y \cap D; \Gamma')$ is a G-convex space where $\Gamma'_A := \Gamma_A \cap Y$ for $A \in \langle Y \cap D\rangle$. For H-spaces, many authors use the name weakly H-convex subsets for G-convex subspaces. We will not list any such examples. Recall that such concepts are extended to a *Γ-convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$.

For an abstract convex space $(E \supset D; \Gamma)$, there have appeared many formulations of the following KKM theorem:

Theorem 6.1. *Let $(X \supset D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, K a nonempty compact subset of E, and $F : D \multimap X$ a multimap such that:*

(1) $\bigcap_{z\in D} F(z) = \bigcap_{z\in D} \overline{F(z)}$;
(2) *$\overline{F}$ is a KKM map; and*
(3) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$L_N \cap \bigcap\{\overline{F(z)} \mid z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap\{F(z) \mid z \in D\} \neq \emptyset$.

Proof. Choose $D' := L_N \cap D$ in Theorem 2.1 for Case (iii). Then Theorem 2.1 works. □

Note that theorems of this type can be invalidated in view of Theorem 2.1:

Example 6.1. (I) For a convex subset X of a Hausdorff topological vector space and $\emptyset \neq D \subset X$, Tian [16, Theorems 2 and 3] obtained particular forms of Theorem 6.1 adopting one of the following conditions instead of (3):

(2c) There is a nonempty subset D_0 of D such that the intersection $\bigcap_{z\in D_0} \overline{F(z)}$ is compact and D_0 is contained in a compact convex subset of X.

(3c) There is a nonempty subset $D_0 \subset D$ such that for each $x \in X \setminus D_0$ there exists a point $z \in D_0$ with $x \notin \overline{F(z)}$ and D_0 is contained in a compact convex subset of X.

Tian [16] applied his theorems to obtain a Fan type minimax inequality, existence of maximal elements, existence of a price equilibrium, and solutions of the complementarity problems.

(II) Ding [22, Corollaries 3.1 and 3.2] restated Tian's results by adopting terminology like transfer compactly closed sets and compact closure, and claimed that his results improve on the corresponding ones of Tian.

(III) Chang et al. [33, Lemma 2.2] obtained a particular form of Theorem 6.1 for an H-space $(X; \Gamma)$.

(IV) Lin and Park [34, Lemma 1] obtained Theorem 6.1 for G-convex spaces with $X = D$.

As in our previous works [8,35], a form of Theorem 6.1 for G-convex spaces can have more than a dozen equivalent formulations. Here we give just one example, as follows.

The following popular form of the generalized Fan–Browder fixed point theorem is equivalent to Theorem 6.1:

Theorem 6.2. *Let $(X \supset D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, K a nonempty compact subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that:*

(1) *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$;*

(2) *$K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$; and*

(3) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in L_N \cap D\}.$$

Then T has a fixed point.

Example 6.2. (I) There are more general formulations of Theorem 6.2 mainly due to the author, and many particular forms are still arising. Moreover, the origin of conditions (3) in Theorems 6.1 and 6.2 is due to Chang [36]; they are generalizations of Ky Fan's original conditions, and have been adopted by the present author since 1992. However, still many authors are using particular forms of conditions (3) in Theorems 6.1 and 6.2.

In the following, we list just a few particular forms of Theorem 6.2.

(II) Tan and Zhang [37, Theorems 3.1–3.3] obtained particular forms of Theorem 6.2 for the case where $X = D = K$ is compact. Moreover, they had to assume the Hausdorffness of X contrary to their claim.

(III) Let X be a convex subset of a topological vector space and $G : X \multimap X$ a map such that $G^-(y)$ is compactly open for each $y \in X$. Ding and Yuan [38, Theorem 2.2] showed that there exists $x \in X$ such that $x \in \mathrm{co}\, G(x)$ under the assumption that

(c) *there exist a nonempty compact convex subset X_0 and a nonempty compact subset K of X such that, for each $x \in X \setminus K$, we have that* $\mathrm{co}\,(X_0 \cup \{x\}) \cap \mathrm{co}\, G(x) \neq \emptyset$.

We show that their result follows from Theorem 6.2 under the assumption that $G(x) \neq \emptyset$ for all $x \in X$. Let $X = D$ and $S = T = \mathrm{co}\, G$. Then $T(x)$ is nonempty convex for each $x \in X$, and $S^-(y) = (\mathrm{co}\, G)^-(y)$ is easily seen to be open as $G^-(y)$ is open for each $y \in X$. This shows that the conditions (1) and (2) of Theorem 6.2 hold.

We claim that (c) implies condition (3) of Theorem 6.2. For any $N \in \langle X\rangle$, let $L_N := \mathrm{co}\,(X_0 \cup N)$. Then L_N is a compact convex subset of X as so is X_0. For any $x \in L_N \setminus K$, by (c) there exists a $z \in \mathrm{co}\,(X_0 \cup \{x\}) \cap \mathrm{co}\, G(x)$ and hence $z \in L_N = L_N \cap D$ and $z \in \mathrm{co}\, G(x) = S(x)$. Therefore, $x \in S^-(z) = (\mathrm{co}\, G)^-(z) = \mathrm{Int}(\mathrm{co}\, G)^-(z) = \mathrm{Int}\, S^-(z)$. Therefore, (3) holds.

Now, by Theorem 6.2, there exists an $x \in X$ such that $x \in T(x) = \mathrm{co}\, G(x)$.

Finally, we obtain a simpler formulation of Theorem 6.2 as follows:

Theorem 6.3. *Let $(X \supset D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, K a nonempty subset of X, and $S : X \multimap D$, $T : X \multimap X$ multimaps. Suppose that:*

(1)′ *for each $x \in X$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$;*

(2)′ *$K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in N\}$ for some $N \in \langle D\rangle$; and*

(3)′ *there exists a Γ-convex subset L_N of X containing N such that*

$$L_N \setminus K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in M\}$$

for some $M \in \langle L_N \cap D\rangle$.

Then T has a fixed point in L_N.

Proof. Note that $K \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in N\}$ for some $N \in \langle D\rangle$. For this N, we have an abstract convex space $(L_N, L_N \cap D; \Gamma')$ as in (3)′, where $\Gamma' := \Gamma|_{\langle L_N \cap D\rangle}$. In this subspace L_N, we have

$$L_N = (L_N \setminus K) \cup (L_N \cap K) \subset \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in M \cup N\},$$

where $M \cup N \in \langle L_N \cap D\rangle$. Therefore, by Theorem 4.1 with $S|_{L_N} : L_N \multimap L_N \cap D$ and $T|_{L_N} := \mathrm{co}_{\Gamma'}(S|_{L_N})$ instead of S and T, respectively, the conclusion follows. □

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