On the von Neumann Type Minimax Theorems in Abstract Convex Spaces

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추상볼록공간에서의 폰 노이만 형 최대최소정리에 관하여

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ABSTRACT

In this paper, we review generalizations of the von Neumann-Sion minimax theorem mainly due to the present author. We give them based on the fixed point theory or the KKM theory on subsets of topological vector spaces, convex spaces, H-spaces, G-convex spaces, abstract convex spaces, or other spaces.

Key words and phrases. Minimax theory; KKM theory; Fixed point; Convex space; H-space, Generalized convex space; Abstract convex space.

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초 록

이 글에서는 추상볼록공간에서의 폰 노이만 형 최대최소정리의 여러 형태 중에서 주로 본인이 발견한 것들을 다룬다. 실제로 위상벡터공간의 부분집합, 볼록공간, H-공간, G-볼록공간, 추상볼록공간, 그 밖의 공간들에서의 부동점이론과 KKM 이론으로부터 이같은 정리들을 이끌어낸 역사를 체계적으로 다룬다.

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1. Introduction

Since the von Neumann minimax theorem appeared in (von Neumann 1928), more than eight decades elapsed. The minimax theory in its most classical meaning is that, given a real-valued function f defined on a product space $X \times Y$, one tries to find conditions that ensure the validity of the equality

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in Y} f(x, y).$$

More generally, a two function minimax theorem for $f, g : X \times Y \to \mathbb{R}$ with $f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$ means that, under certain conditions, the inequality

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) \leq \inf_{x \in X} \sup_{y \in Y} g(x, y);$$

holds; see (Ricceri and Simons 1998).

It is well-known that there are a large variety of methods and tools that have been used to study the minimax theory. However we are mainly concerned with generalizations of the von Neumann-Sion type minimax theorems by the fixed point method and the KKM method; see (Park 2011b). One of the well-known generalizations of the von Neumann theorem using fixed point method was given by Kakutani (1941). Later Sion (1958) extended the minimax theorem by applying the Knaster-Kuratowski-Mazurkiewitz (simply, KKM) theorem in (KKM 1929). It is the origin of the KKM method in the minimax theory. Since then there have appeared a large number of generalizations of the minimax theorem incorporated with the fixed point theory or the KKM theory on various types of abstract convex spaces.

These theories are well-developed on the classes of convex subsets of topological vector spaces, the Lassonde type convex spaces, the Horvath type H-spaces, the Park type G-convex spaces, abstract convex spaces in the sense of Park, or other spaces; see (Park 1999a, 2010c, 2010e) and the references therein.

In this paper, we review more than twenty generalizations of the von Neumann minimax theorem mainly due to the present author since (Park 1989) appeared. We give them based on the fixed point theory or the KKM theory on various types of abstract convex spaces. In Section 2, we recall the minimax theorem due to von Neumann (1928) and its generalization due to Sion (1958). Section 3 deals with our generalizations on convex subsets of topological vector spaces and on the Lassonde type convex spaces. In Section 4, we are concerned with our minimax theorems on the Horvath type H-spaces or the Park type G-convex spaces. Finally, Section 5 deals with new minimax theorems on recently developed abstract convex spaces in the sense of Park.

2. The von Neumann - Sion minimax theorem

In 1928, J. von Neumann obtained a particular form of the following minimax theorem, which is one of the fundamental theorems in the theory of games developed by himself:

Theorem. (Kakutani 1941) *Let $f(x, y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x \in K} \min_{y \in L} f(x, y) = \min_{y \in L} \max_{x \in K} f(x, y).$$

Originally, von Neumann assumed the bilinearity of f. The first proof of von Neumann made rather involved use of topology, functional calculus, and the Brouwer fixed point theorem. For the history of earlier proofs of the theorem, see (von Neumann 1953) and (Dantzig 1956).

Moreover, von Neumann (1937) obtained a particular form of the following intersection theorem:

Lemma. (Kakutani 1941) *Let K and L be two compact convex sets in the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let us consider their Cartesian product $K \times L$ in $\mathbb{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. We adopted the above formulations of Theorem and Lemma from (Kakutani 1941).

According to Debreu (A commentary on the Kakutani fixed point theorem, in *Collected Works of Kakutani*),

> "Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved elementary means."

Later Fan obtained a fixed point theorem on locally convex Hausdorff topological vector spaces generalizing the Kakutani theorem, and applied it to the following minimax theorem:

Theorem. (Fan 1952) *Let L_1, L_2 be two locally convex Hausdorff topological linear spaces, and K_1, K_2 be two compact convex sets in L_1, L_2, resp. Let f be a real-valued continuous function defined on $K_1 \times K_2$. If, for every $x_0 \in K_1$, $y_0 \in K_2$, the sets $\{x \in K_1 : f(x, y_0) = \max_{\xi \in K_1} f(\xi, y_0)\}$ and $\{y \in K_2 : f(x_0, y) = \min_{\eta \in K_2} f(x_0, \eta)\}$ are convex, then*

$$\max_{x \in K} \min_{y \in L} f(x, y) = \min_{y \in L} \max_{x \in K} f(x, y).$$

Further, von Neumann's minimax theorem was extended for arbitrary topological vector spaces by Sion as follows:

Theorem. (Sion 1958) *Let X be a compact convex set in a topological vector space and Y a convex set in a topological vector space not necessarily the same space for X. Let f be a real-valued function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x, \cdot)$ is a lower semicontinuous, quasiconvex function on Y, and*

(2) *for each fixed $y \in Y$, $f(\cdot, y)$ is an upper semicontinuous, quasiconcave function on X,*
then we have

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).$$

Further, if Y is also compact, then we have

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Here, f is *lower semicontinuous* (l.s.c.) whenever the set $\{y \in Y : f(x, y) > r\}$ is open, and *quasiconcave* whenever $\{x \in X : f(x, y) > r\}$ is convex for each $r \in \mathbb{R}$. Moreover, f is *upper semicontinuous* (u.s.c.) whenever $\{x \in X : f(x, y) < r\}$ is open, and *quasiconvex* whenever $\{y \in Y : f(x, y) < r\}$ is convex for each $r \in \mathbb{R}$.

Sion noted that, until that time, there seemed to be essentially two types of argument: one uses some form of separation of disjoint convex sets by a hyperplane and the other uses a fixed point theorem. In order to overcome the difficulty in those two types, Sion adopted new argument based on the KKM theorem (1929).

There are other types of generalizations of the von Neumann theorem by many authors by different methods; see (Ricceri and Simons 1998). We give only one example due to H. Tuy (1974, Corollary 2) as follows:

Theorem. (Tuy 1974) *Let C and D be convex subsets of Hausdorff topological vector spaces X and Y, and let D be compact, $f(x, y)$ be a real* *function defined on $C \times D$ which is quasiconvex in x and quasiconcave in y, l.s.c. in x (or u.s.c. in x), and u.s.c. in y.*

Then the following equality is satisfied:

$$\inf_{x \in C} \sup_{y \in D} f(x, y) = \sup_{y \in D} \inf_{x \in C} f(x, y).$$

So far, we found that numerous generalizations of the von Neumann minimax theorem can be divided into three classes: the one using fixed point theorems as in (Kakutani 1941) and (Fan 1952), the one based on the KKM theory as in (Sion 1958), and the one based on other methods as in (Tuy 1974). Note that in order to apply any fixed point theorem, we need some rich structures like the local convexity or certain generalized concepts. On the other hand, one of the main aims to generalize the minimax theorem seems to eliminate as much as possible the convexity related to the theorem. This is accomplished in the recent development in the KKM theory.

In this paper, we are mainly concerned with generalizations of the von Neumann - Sion type minimax theorems by the fixed point method and the KKM method on the classes of convex subsets of topological vector spaces, the Lassonde type convex spaces, the Horvath type H-spaces, the Park type G-convex spaces, abstract convex spaces in the sense of Park, or other spaces; see (Park 1999a, 2010c, 2010e).

3. On convex spaces

The concept of convex subsets of a topological vector space was extended to the following by Lassonde (1983):

Definition. A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex halls of its finite subsets. A nonempty subset L of X is called a *c-compact set* if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

In this section, we recall mainly our minimax theorems on convex spaces or convex subsets of topological vector spaces.

(I) In 1989, as in Fan (1966) and Lassonde (1983), from a geometric theorem concerning sets with convex sections in the KKM theory, we obtained the following (Park 1989, Theorem 21):

Theorem A. *Let X and Y be convex spaces, and $f, g : X \times Y \to \overline{\mathbb{R}} = [-\infty, +\infty]$ functions satisfying the following conditions:*

(1) *$f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,*

(2) *$g(\cdot, y)$ is quasiconvex on X for each $y \in Y$,*

(3) *$g(x, \cdot)$ is u.s.c. on Y for each $x \in X$,*

(4) *$f(\cdot, y)$ is l.s.c. on X for each $y \in Y$,*

(5) *$f(x, \cdot)$ is quasiconcave on Y for each $x \in X$,*

(6) *there exists a c-compact set $L \subset X$ such that if $\mu > \sup_{y \in Y} \inf_{x \in X} g(x, y)$ for some $\mu \in \mathbb{R}$, then $\{y \in Y \mid g(x, y) \geq \mu$ for all $x \in L\}$ is compact.*

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

Remark. 1. If $f \equiv g$, then the equality holds in the conclusion.

2. Condition (3) is originally given as follows:

(3)′ *$g(x, \cdot)$ is u.s.c. on compact subsets of Y for each $x \in X$.*

This is redundant since we can adopt the compactly generated extension of the original topology of Y; see (Park 2011a).

The same remark also holds for Condition (4).

3. If Y is compact, Condition (6) is automatically satisfied; and if X is compact the conclusion is actually

$$\min_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y)$$

since $\sup_{y \in Y} f(x, y)$ is l.s.c. on X.

This is due to Liu (1978, Theorem 1). In fact, he assumed

(3)″ *$g(x, \cdot)$ is u.s.c. on convex hulls of finite subsets of Y for each $x \in X$*

instead of (3). This is also redundant since we adopted the finitely generated topology of the convex space Y; see (Park 2011a).

4. A slightly weaker form of 3. is given by Granas and Liu (1984, Lemma), (1986, Theorem 5.3). They used their result to obtain a generalization of Fan's theorem on systems of inequalities, a 'sup inf sup' inequality, Browder's variational inequality, the Tychonoff fixed point theorem, and a generalization of Kneser's minimax theorem.

5. If X and Y are compact and if $f \equiv g$, then Theorem A reduces to the von Neumann - Sion minimax theorem.

(II) In 1994, we applied a geometric property in the KKM theory to the following minimax theorem (Park et al. 1994, Theorem 4.1):

Theorem B. *Let X and Y be convex spaces and $f : X \times Y \to \mathbb{R}$ an l.s.c. function such that*

(1) *for each* $x \in X$, $y \mapsto f(x,y)$ *is quasiconcave on* Y*;*

(2) *for each* $y \in Y$, $x \mapsto f(x,y)$ *is quasiconvex on* X*; and*

(3) *there is a nonempty compact subset* K *of* X *and, for each finite subset* N *of* X*, there is a compact convex subset* $L_N \subset X$ *containing* N *such that, for each* $x \in L_N \setminus K$ *and each* $y \in Y$*, we have*

$$\inf_{z \in L_N} f(z,y) < f(x,y).$$

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \inf_{x \in X} f(x,y).$$

Remark. Even the particular case when X and Y are compact, Theorem B differs from Sion's minimax theorem (1958) in the continuity conditions. Instead of the lower semicontinuity of f on $X \times Y$, Sion requires that, for each $x \in X$, $y \mapsto f(x,y)$ is u.s.c. and for each $y \in Y$, $x \mapsto f(x,y)$ is l.s.c.

Recall that Simons (1986) gave a two function version of Sion's theorem. Analogously, we gave a two function version of Theorem B as follows (Park et al. 1994, Theorem 4.2):

Theorem C. *Let* X *and* Y *be convex spaces and* $f, g : X \times Y \to \mathbb{R}$ *such that* $f \leq g$*. Suppose that*

(1) *for each* $x \in X$, $y \mapsto g(x,y)$ *is quasiconcave and, for each* $y \in Y$, $x \mapsto f(x,y)$ *is l.s.c.;*

(2) g *is l.s.c. on* $X \times Y$ *and, for each* $y \in Y$, $x \mapsto f(x,y)$ *is quasiconvex; and*

(3) *there is a nonempty compact subset* K *of* X *and, for each finite subset* N *of* X*, there is a compact convex subset* $L_N \subset X$ *containing* N *such that, for each* $x \in L_N \setminus K$ *and each* $y \in Y$*,*

$$g(x,y) \leq \inf_{z \in L_N} g(z,y) \text{ implies } f(x,y) \leq \sup_{v \in Y} \inf_{u \in X} g(u,v).$$

Then we have

$$\inf_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \inf_{x \in X} g(x,y).$$

Corollary. *Let* X *and* Y *be convex spaces and* $f, g : X \times Y \to \mathbb{R}$ *functions satisfying (1) and (2) of Theorem C, and* $f \leq g$*. If* X *is compact, then we have*

$$\min_{x \in X} \sup_{y \in Y} f(x,y) \leq \sup_{y \in Y} \min_{x \in X} g(x,y).$$

(III) A subset K of a t.v.s. E is said to be *convexly totally bounded* (c.t.b. for short) in the sense of Idzik (Idzik and Park 1998) if for every neighborhood V of $0 \in E$ there exist a finite subset $\{x_i : i \in I\} \subset K$ (I is finite) and a finite family of convex sets $\{C_i : i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $K \subset \bigcup_{i \in I}(x_i + C_i)$.

In 1998, based on fixed point theory, we obtained a saddle point theorem and a minimax theorem (Idzik and Park 1998, Theorems 3.3 and 3.4) as follows:

Theorem D. *Let X, Y be two compact convex c.t.b. subsets, each in a Hausdorff topological vector space, and $f : X \times Y \to \overline{\mathbb{R}}$ a continuous function. Suppose that for each $x_0 \in x$ and $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are convex. Then

(i) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0),$$

(ii) *we have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

(IV) Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

In 1998, based on a fixed point theorem of Eilenberg-Montgomery or Begle, we obtained acyclic versions of a saddle point theorem and a minimax theorem (Park 1998a, Corollaries 1, 2):

Theorem E. *Let X, Y be two acyclic polyhedra, and $f : X \times Y \to \overline{\mathbb{R}}$ a continuous function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are acyclic. Then

(i) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0),$$

(ii) *we have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

(V) A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

In 2000, based on a fixed point theorem due to Park, we obtained the following von Neumann type minimax theorem (Park 2000a, Theorem 4):

Theorem F. *Let X be a compact topological space and Y an admissible (in the sense of Klee) compact convex subset of a Hausdorff topological vector space, and $f : X \times Y \to \mathbb{R}$ a continuous real function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y : f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are acyclic. Then

(1) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, \eta) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0),$$

(2) *we have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Remark. For Euclidean spaces or locally convex topological vector spaces, if acyclicity is replaced by convexity, then Theorem F reduces to the von Neumann minimax theorem (1928) or (Fan 1972, Theorem 3), resp.

The following generalization of the von Neumann minimax theorem is a simple consequence of Theorem F:

Theorem G. *Let X, Y, and f be the same as in Theorem F. Suppose that*

(1) *for every $x \in X$ and $\alpha \in \mathbb{R}$, $\{y \in Y : f(x,y) \le \alpha\}$ is acyclic; and*

(2) *for every $y \in Y$ and $\beta \in \mathbb{R}$, $\{x \in X : f(x,y) \ge \beta\}$ is acyclic.*

Then we have

$$\max_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \max_{x \in X} f(x,y).$$

Park (2002, 2008a) gave the same result by slightly different procedures.

(VI) It is well-known that a subset X of a topological vector space E is said to be *almost convex* if for any neighborhood V of 0 and for any finite subset $A := \{x_1, x_2, \cdots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \cdots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \cdots, n$ and $\mathrm{co}\, B \subset X$.

Park (2007, Theorem 6.5) deduced the following from fixed point theorems due to himself:

Theorem H. *Let Y_1, Y_2 be compact admissible subsets of Hausdorff topological vector spaces E_1, E_2, resp., and X_1, X_2 be almost convex dense subsets of Y_1, Y_2, resp. Let f be a continuous real function defined on $Y_1 \times Y_2$ such that for any $x_1 \in X_1$, $y_2 \in X_2$, the sets*

$$\{x \in Y_1 : f(x, y_2) = \max_{\xi \in Y_1} f(\xi, y_2)\}$$

and

$$\{y \in Y_2 : f(x_1, y) = \min_{\eta \in Y_2} f(x_1, \eta)\}$$

are acyclic (resp., have trivial shapes). Then

$$\max_{x \in Y_1} \min_{y \in Y_2} f(x,y) = \min_{y \in Y_2} \max_{x \in Y_1} f(x,y).$$

(VII) For convex subsets X of a topological vector space E, Park (2010d) showed that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies generalized forms of the Sion minimax theorem as follows:

Theorem I. *Let X and Y be nonempty convex subsets of two topological vector spaces, and $f, s, t, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be four functions,*

$$\mu := \inf_{y \in Y} \sup_{x \in X} f(x,y) \text{ and } \nu := \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

Suppose that

(1) $f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ *for each* $(x,y) \in X \times Y$;

(2) *for each* $r < \mu$ *and* $y \in Y$, $\{x \in X \mid s(x,y) > r\}$ *is convex; for each* $r > \nu$ *and* $x \in X$, $\{y \in Y \mid t(x,y) < r\}$ *is convex;*

(3) *for each* $r > \nu$, *there exists a finite subset* $\{x_i\}_{i=1}^m$ *of* X *such that*

$$Y = \bigcup_{i=1}^{m} \mathrm{Int}\{y \in Y : f(x_i, y) > r\}; \text{ and}$$

(4) *for each* $r < \mu$, *there exists a finite subset* $\{y_j\}_{j=1}^n$ *of* Y *such that*

$$X = \bigcup_{j=1}^{n} \mathrm{Int}\{x \in X : g(x, y_j) < r\}.$$

Then we have $\mu \le \nu$, *that is,*

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \le \sup_{x \in X} \inf_{y \in Y} g(x,y).$$

This extends various forms of the Sion type minimax theorem.

4. On generalized convex spaces

The following generalization of convex spaces are well-known; see (Horvath 1991), (Park 1993, 1999a, 2010c) and the references therein:

Definition. A pair $(X;\Gamma)$ is called an *H-space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle X \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle X \rangle$.

Here $\langle X \rangle$ denotes the set of all nonempty finite subsets of X. Originally, an H-space is called a c-space by Horvath (1991).

More generally, we have the following (Park 1999a,b, 2000b, 2002, 2003, 2010a):

Definition. A *generalized convex space* or a *G-convex space* $(X, D;\Gamma)$ due to Park is a triple consisting of a topological space X, a nonempty set D and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$.

For a G-convex space $(X \supset D;\Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

From now on, we are mainly concerned with $(X, \Gamma) = (X, X; \Gamma)$.

In this section, we recall generalizations of the minimax theorems on H-spaces or G-convex spaces.

(I) In 1993, we deduced several generalizations of the von Neumann-Sion minimax theorem and the saddle-point theorem from a Fan-Browder type fixed point theorem in the KKM theory of H-spaces as follows (Park 1993, Theorems 8 and 9):

Theorem J. *Let X and Y be H-spaces and $f, g, s, t : X \times Y \to \overline{\mathbb{R}}$ functions such that*

(1) $f \leq s \leq t \leq g$ *on* $X \times Y$*;*

(2) *for each* $x \in X$, $f(x, \cdot)$ *is l.s.c. on* Y*;*

(3) *for each* $y \in Y$, $g(\cdot, y)$ *is u.s.c. on* X*;*

(4) *for each* $y \in Y$ *and* $c \in \mathbb{R}$, $\{x \in X : s(x, y) > c\}$ *is* Γ*-convex; and*

(5) *for each* $x \in X$ *and* $c \in \mathbb{R}$, $\{y \in Y : t(x, y) < c\}$ *is* Γ*-convex.*

Suppose that there exists a nonempty compact subset K of $X \times Y$ such that for each $c \in \mathbb{R}$, either

(i) *there exists an* $M \in \langle X \times Y \rangle$ *satisfying*

$$(X \times Y) \setminus K \subset \bigcup_{(x_i, y_i) \in M} \{\bar{x} \in X : g(\bar{x}, y_i) < c\} \times \{\bar{y} \in Y : f(x_i, \bar{y}) > c\}; \text{ or}$$

(ii) *for each* $N \in \langle X \times Y \rangle$*, there exists a compact* Γ*-convex subset* L_N *of* $X \times Y$ *containing* N *such that*

$$L_N \setminus K \subset \bigcup_{(x,y) \in L_N} \{\bar{x} \in X : g(\bar{x}, y) < c\} \times \{\bar{y} \in Y : f(x, \bar{y}) > c\}.$$

Then we have

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Theorem K. *Let $(X; \Gamma)$ be an H-space, $\phi : X \times X \to \overline{\mathbb{R}}$ a function, and $\gamma \in \overline{\mathbb{R}}$ such that*

(1) *for each* $x \in X$, $\phi(x, \cdot)$ *is l.s.c. on* X*;*

(2) *for each* $A \in \langle X \rangle$ *and each* $y \in \Gamma_A$, $\min_{x \in A} \phi(x, y) \leq \gamma$*;*

(3) *there exists a nonempty compact subset K of X satisfying either*

(i) *there exists an* $M \in \langle X \rangle$ *such that*

$$\{y \in X : \phi(x, y) \leq \gamma \text{ for all } x \in M\} \subset K; \text{ or}$$

(ii) *for each* $N \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that for each* $y \in L_N \setminus K$ *there exists an* $x \in L_N$ *satisfying* $\phi(x,y) > \gamma$;

$(1)'$ *for each* $y \in X$, $\phi(\cdot, y)$ *is u.s.c. on* X;

$(2)'$ *for each* $B \in \langle X \rangle$ *and each* $x \in \Gamma_B$, $\max_{y \in B} \phi(x,y) \geq \gamma$; *and*

$(3)'$ *there exists a nonempty compact subset* K' *of* X *satisfying either*

(i)$'$ *there exists an* $M \in \langle X \rangle$ *such that*

$$\{x \in X : \phi(x,y) \geq \gamma \text{ for all } y \in M\} \subset K'; \text{ or}$$

(ii)$'$ *for each* $N \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* X *containing* N *such that for each* $x \in L_N \setminus K'$ *there exists a* $y \in L_N$ *satisfying* $\phi(x,y) < \gamma$.

Then we have

$$\max_{x \in K'} \inf_{y \in X} \phi(x,y) = \min_{y \in K} \sup_{x \in X} \phi(x,y) = \gamma.$$

(II) Park (1999b) obtained several von Neumann - Sion type minimax theorems, which show that the KKM method is definitely better than the fixed point method in certain sense.

The following (Park 1999b, Theorem 2) is based on a coincidence theorem and a continuous selection theorem in the fixed point theory:

Theorem L. *Let* $(X;\Gamma)$ *and* $(Y;\Gamma')$ *be G-convex spaces,* Y *a Hausdorff compact space,* $f : X \times Y \to \overline{\mathbb{R}}$ *an extended real function, and* $\mu := \sup_{x \in X} \inf_{y \in Y} f(x,y)$. *Suppose that*

(1) $f(x,\cdot)$ *is l.s.c. on* Y *and* $\{y \in Y : f(x,y) < r\}$ *is* Γ'*-convex for each* $x \in X$ *and* $r > \mu$; *and*

(2) $f(\cdot,y)$ *is u.s.c. on* X *and* $\{x \in X : f(x,y) > r\}$ *is* Γ*-convex for each* $y \in Y$ *and* $r > \mu$.

Then

$$\sup_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \sup_{x \in X} f(x,y).$$

Corollary. *Under the hypothesis of Theorem L, further if* X *is compact, then* f *has a saddle point* $(x_0,y_0) \in X \times Y$ *such that*

$$\max_{x \in X} f(x,y_0) = f(x_0,y_0) = \min_{y \in Y} f(x_0,y).$$

Particular Forms. We list historically well-known particular forms of Theorem L and Corollary in chronological order:

1. von Neumann (1928), Kakutani (1941): X and Y are compact convex subsets of Euclidean spaces and f is continuous.

2. Nikaidô (1954): Euclidean spaces in the above are replaced by Hausdorff topological vector spaces, and f is continuous in each variable.

3. Sion (1958): X and Y are convex sets in Theorem L and Corollary.

4. Komiya (1981, Theorem 3): X and Y are compact convex spaces in the sense of Komiya.

5. Bielawski (1987, Theorem (4.13)): X and Y are compact spaces having certain simplicial convexities.

6. Horvath (1991, Prop. 5.2): X and Y are H-spaces with Y compact.

Remark. In 4 and 6 above, Hausdorffness of Y is assumed since their authors used the partition of unity argument. However, 3 and 5 were based on the corresponding KKM theorems which need not the Hausdorffness of Y; see Theorem N below.

From the same coincidence theorem, we had another minimax theorem (Park 1999b, Theorem 3):

Theorem M. *Let* $(X;\Gamma)$ *and* $(Y;\Gamma')$ *be* G*-convex spaces,* Y *Hausdorff compact,* $f : X \times Y \to \overline{\mathbb{R}}$ *an l.s.c. function, and* $\mu := \sup_{x\in X} \inf_{y\in X} f(x,y)$. *Suppose that*

(1) *for each* $r > \mu$ *and* $y \in Y$, $\{x \in X : f(x,y) > r\}$ *is* Γ*-convex; and*

(2) *for each* $r > \mu$ *and* $x \in X$, $\{y \in X : f(x,y) \leq r\}$ *is acyclic.*

Then

$$\sup_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \sup_{x\in X} f(x,y).$$

Particular Forms. 1. von Neumann (1928), Kakutani (1941): X and Y are compact convex subsets of Euclidean spaces, f is continuous, and Γ-convexity and acyclicity are replaced by convexity.

2. Nikaidô (1954): Euclidean spaces were replaced by Hausdorff topological vector spaces in the above.

From a Fan type intersection theorem in the KKM theory, we deduced the following improved version (Park 1999b, Theorem 3) of Corollary to Theorem L:

Theorem N. *Let* $(X;\Gamma)$ *and* $(Y;\Gamma')$ *be compact* G*-convex spaces and* $f : X \times Y \to \overline{\mathbb{R}}$ *a function satisfying conditions (1) and (2) of Theorem M. Then*

(i) f *has a saddle point* $(x_0, y_0) \in X \times Y$*; and*

(ii) *we have*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y).$$

Particular Forms. All of the examples given for Theorem L and Corollary follow from Theorem N. Especially, (Sion 1958, Theorem 3.4) is a particular form of Theorem N, and (Sion 1958, Corollary 3.5) is a non-Hausdorff version of Theorem L and can be obtained from Theorem N by following his own method.

(III) Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X := \prod_{i\in I} X_i$ be equipped with the product topology. For each $i \in I$, let $\pi_i : X \to X_i$ be the projection. For each $A \in \langle X\rangle$, define $\Gamma(A) := \prod_{i\in I} \Gamma_i(\pi_i(A))$. Then $(X;\Gamma)$ is a G-convex space.

Note also that a product of G-convex subsets is also G-convex in the product G-convex space.

For a G-convex space $(X;\Gamma)$, a function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp., $\{x \in X : f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

In 2000, we showed that a typical classical application of the KKM theorem can be extended to G-convex spaces. For example, we had the following generalization (Park 2000b, Theorem 18) of the von Neumann - Sion minimax theorem:

Theorem O. *Let $(X;\Gamma)$ and $(Y;\Gamma')$ be compact G-convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(1) *$f(x,y) \le g(x,y)$ for each $(x,y) \in X \times Y$;*

(2) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and $g(x,\cdot)$ is quasiconvex on Y; and*

(3) *for each $y \in Y$, $f(\cdot,y)$ is quasiconcave and $g(\cdot,y)$ is u.s.c. on X. Then we have*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \max_{x\in X}\inf_{y\in Y} g(x,y).$$

Remark. If $f = g$ and if X is a convex space, Theorem O reduces to Sion's generalization (1958) of the von Neumann minimax theorem:

$$\min_x \max_y f(x,y) = \max_y \min_x f(x,y).$$

(IV) Park (2003, Section 6) generalized the contents of (Park 1998b) to G-convex spaces as follows:

Let Z be a *complete linearly ordered space*; that is, a linearly ordered set whose every subset has a least upper bound. Examples are the extended real line $\overline{\mathbb{R}}$, the extended Euclidean space $\overline{\mathbb{R}}^n$, and any compact (in the Euclidean topology) subset of $\mathbb{R}^n$ with respect to the lexicographic order.

For a topological space X, a function $f : X \to Z$ is said to be *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] whenever $\{x \in X : f(x) > z\}$ [resp., $\{x \in X : f(x) < z\}$] is open in X for each $z \in Z$.

If X is compact and $f : X \to Z$ is l.s.c., then there exists an $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. For any family $\{f_i\}_{i\in I}$ of l.s.c. functions $f_i : X \to Z$, the function $\sup_{i\in I} f_i$ is also l.s.c.

The following is (Park 2003, Theorem 6.1), which reduces to (Park 1998a, Theorem 2) for convex spaces:

Theorem P. *Let $(X;\Gamma)$ be a G-convex space, $(Y;\Gamma')$ a Hausdorff compact G-convex space, and $f : X \times Y \to Z$ a function. Suppose that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap (a, b) \neq \emptyset$;*

(2) *$f(x, \cdot)$ is l.s.c. on Y and $\{y \in Y : f(x, y) < s\}$ is Γ'-convex for each $x \in X$ and $s \in U$; and*

(3) *$f(\cdot, y)$ is u.s.c. on X and $\{x \in X : f(x, y) > s\}$ is Γ-convex for each $y \in Y$ and $s \in U$.*

Then

$$\sup_{x\in X} \min_{y\in Y} f(x, y) = \min_{y\in Y} \sup_{x\in X} f(x, y).$$

Remark. For convex spaces, if $U = Z$, then Theorem P is a consequence of Komornik (1982, Theorem 2) for interval spaces with different proof. For convex spaces, Theorem P reduces to (Park 1998b, Theorem 3) which includes a lot of known results.

Corollary. *Under the hypothesis of Theorem P, further if X is compact, then f has a saddle point.*

Remark. For convex spaces, Corollary reduces to Arandjelović (1992, Theorem 3), which extends the Sion minimax theorem; see (Park 1998b).

The following minimax theorem (Park 2003, Theorem 6.2) is a variant of Theorem P and generalizes (Park 1998b, Theorem 3):

Theorem Q. *Let $(X;\Gamma)$ be a G-convex space, Y a Hausdorff compact space, and $f : X \times Y \to Z$ an l.s.c. function such that*

(1) *there is a subset $U \subset Z$ such that $a, b \in f(X \times Y)$ with $a < b$ implies $U \cap [a, b) \neq \emptyset$;*

(2) *for each $s \in U$ and $y \in Y$, $\{x \in X : f(x, y) > s\}$ is Γ-convex; and*

(3) *for each* $s \in U$ *and* $x \in X$, $\{y \in X : f(x,y) \leq s\}$ *is acyclic.*
Then

$$\sup_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \sup_{x \in X} f(x,y).$$

Remark. For convex spaces, related results were obtained by Ha (1980, Theroem 4), Komiya (1981, Theorem 3), and Komornik (1982, Theorem 3). For convex spaces, Theorem Q reduces to (Park 1998b, Theorem 4).

5. On abstract convex spaces

Since 2006, we introduced the concept of abstract convex spaces generalizing G-convex spaces; see (Park 2010c) and the references therein.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any nonempty $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In this section, we are mainly concerned with the case $E = D$, and let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : E \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all} \quad A \in \langle E \rangle,$$

then G is called a *KKM map* with respect to F. When $E = Z$, a *KKM map* $G : E \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$-*map* if, for any KKM map $G : E \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in E}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical.

Definition. The *partial KKM principle* for an abstract convex space $(E;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E,E)$; that is, for any closed-valued KKM map $G : E \multimap E$, the family $\{G(y)\}_{y\in E}$ has the finite intersection property. The *KKM principle* is the statement that the same property *also* holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

Note that the product of any family of abstract convex spaces is also an abstract convex space. In fact, we clearly have the following:

Lemma. *Let $\{(E_i;\Gamma_i)\}_{i\in I}$ be any family of abstract convex spaces. Let $E := \prod_{i\in I} E_i$ For each $i \in I$, let $\pi_i : E \to E_i$ be the projection. For each $A \in \langle E\rangle$, define $\Gamma(A) := \prod_{i\in I} \Gamma_i(\pi_i(A))$. Then $(E;\Gamma)$ is an abstract convex space.*

Note also that the product of abstract convex subsets is also abstract convex in the product abstract convex space.

Let $(X;\Gamma)$ and $(Y;\Gamma')$ be abstract convex spaces. For their product, as in Lemma, we can define $\Gamma_{X\times Y}(A) := \Gamma(\pi_1(A)) \times \Gamma'(\pi_2(A))$ for $A \in \langle X \times Y\rangle$.

For an abstract convex space $(X;\Gamma)$, a function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

An abstract convex space $(X;\Gamma)$ is said to be *compact* if X is a compact topological space.

(I) In 2008, as a direct application of a coincidence theorem in the KKM theory, we obtained the following generalization (Park 2008b, Theorem 8.2) of the von Neumann - Sion minimax theorem:

Theorem R. *Let $(X;\Gamma)$ and $(Y;\Gamma')$ be compact abstract convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(1) *$f(x,y) \le g(x,y)$ for each $(x,y) \in X \times Y$;*

(2) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and $g(x,\cdot)$ is quasiconvex on Y; and*

(3) *for each $y \in Y$, $f(\cdot,y)$ is quasiconcave and $g(\cdot,y)$ is u.s.c. on X.*

If $(X \times Y; \Gamma_{X\times Y})$ *is a KKM space, where* $\Gamma_{X\times Y}$ *is the product convexity defined as above, then we have*

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \le \max_{x\in X} \inf_{y\in Y} g(x,y).$$

Remark. 1. If $f = g$ and if X is a convex space, Theorem R reduces to Sion's generalization (1958) of the von Neumann minimax theorem (1928):

$$\min_x \max_y f(x,y) = \max_y \min_x f(x,y).$$

2. If $(X;\Gamma)$ and $(Y;\Gamma')$ are G-convex spaces, $X \times Y$ is a KKM space. Therefore, Theorem R works for G-convex spaces and reduces to Theorem O.

(II) Park (2010b) obtained another minimax theorem for abstract convex spaces as follows:

Theorem S. *Let* $(E;\Gamma) := (X \times Y; \Gamma_{X\times Y})$ *be the product abstract convex space,* $f, s, t, g : X \times Y \to \overline{\mathbb{R}}$ *be four functions,*

$$\mu := \inf_{y\in Y} \sup_{x\in X} f(x,y) \quad \textit{and} \quad \nu := \sup_{x\in X} \inf_{y\in Y} g(x,y).$$

Suppose that

(1) $f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ *for each* $(x,y) \in X \times Y$*;*

(2) *for each* $r < \mu$ *and* $y \in Y$*,* $\{x \in X : s(x,y) > r\}$ *is* Γ*-convex; for each* $r > \nu$ *and* $x \in X$*,* $\{y \in Y : t(x,y) < r\}$ *is* Γ'*-convex;*

(3) *for each* $r > \nu$*, there exists a finite set* $\{x_i\}_{i=1}^m \subset X$ *such that*

$$Y = \bigcup_{i=1}^m \mathrm{Int}\,\{y \in Y : f(x_i,y) > r\}; \textit{ and}$$

(4) *for each* $r < \mu$*, there exists a finite set* $\{y_j\}_{j=1}^n \subset Y$ *such that*

$$X = \bigcup_{j=1}^n \mathrm{Int}\,\{x \in X : g(x,y_j) < r\}.$$

If $(E;\Gamma)$ *satisfies the partial KKM principle, then we have*

$$\mu = \inf_{y\in Y} \sup_{x\in X} f(x,y) \le \sup_{x\in X} \inf_{y\in Y} g(x,y) = \nu.$$

Example. This reduces to Theorem I when X and Y are convex sets. For convex spaces X, Y and $f = s = t = g$, Theorem P reduces to Cho-Kim-Lee (2009, Theorem 8).

Corollary 1. *Let* $(X;\Gamma)$ *and* $(Y;\Gamma')$ *be compact abstract convex spaces,* $(E;\Gamma) := (X \times Y; \Gamma_{X\times Y})$ *the product abstract convex space, and* $f, g : X \times Y \to \overline{\mathbb{R}}$ *functions satisfying*

(1) $f(x,y) \leq s(x,y) \leq t(x,y) \leq g(x,y)$ *for each* $(x,y) \in X \times Y$*;*

(2) *for each* $x \in X$, $f(x,\cdot)$ *is l.s.c. and* $t(x,\cdot)$ *is quasiconvex on* Y*; and*

(3) *for each* $y \in Y$, $s(\cdot,y)$ *is quasiconcave and* $g(\cdot,y)$ *is u.s.c. on* X.

If $(E;\Gamma)$ *satisfies the partial KKM principle, then we have*

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \leq \max_{x\in X} \inf_{y\in Y} g(x,y).$$

Example. 1. Particular or slightly different versions of Corollary 1 were obtained by Liu (1978) and Granas (1990, Théorèmes 3.1 et 3.2) for convex subsets of topological vector spaces.

2. For $f = s$, $g = t$, if X and Y are G-convex spaces, Corollary 1 reduces to Theorem O.

2. For $f = s$, $g = t$, Corollary 1 reduces to (Park 2010a, Theorem 3) and Theorem R.

For the case $f = s = t = g$, Corollary 1 reduces to the following in (Park, 2010a):

Corollary 2. *Let* $(X;\Gamma)$ *and* $(Y;\Gamma')$ *be compact abstract convex spaces and* $f : X \times Y \to \overline{\mathbb{R}}$ *an extended real function such that*

(1) *for each* $x \in X$, $f(x,\cdot)$ *is l.s.c. and quasiconvex on* Y*; and*

(2) *for each* $y \in Y$, $f(\cdot,y)$ *is u.s.c. and quasiconcave on* X.

If $(X \times Y; \Gamma_{X\times Y})$ *satisfies the partial KKM principle, then*

(i) f *has a saddle point* $(x_0,y_0) \in X \times Y$*; and*

(ii) *we have*

$$\max_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \max_{x\in X} f(x,y).$$

Example. All of the particular forms of Theorem L and its Corollary follow from Corollary 2.

Remark. In (Park 2010c), Theorem S and its Corollaries are restated in a systematic investigation of the KKM theory. Moreover, in (Park 2010e), we showed that the partial KKM principle is equivalent to a Fan-Browder type fixed point theorem in an abstract convex space $(E, D; \Gamma)$, and that this theorem implies Theorem S and its Corollaries.

(III) Park (2011c) obtained the following general concept of 0-pair concavity due to S.-Y. Chang (2010):

Definition. Let X be a nonempty set and Y be a topological space, and $D \subset X$. A function $f : X \times Y \to \mathbb{R}$ is said to be *generally 0-pair-concave* on D, if for any $\{x^0, \ldots, x^n\} \in \langle X \rangle$, there is a multimap $\Phi_n \in \mathfrak{KC}(\Delta_n, Y)$ such that

$$\min_{i \in I(\lambda)} f(x^i, y) \leq 0$$

for all $\lambda = \{\lambda_0, \ldots, \lambda_n\} \in \Delta_n$ and $y \in \Phi_n(\lambda)$, where $I(\lambda) = \{i : \lambda_i \neq 0\}$.

Note that the equilibrium for a two-person zero-sum game is a minimax problem. We obtained the following generalization of (Chang 2010, Theorem 3.3):

Theorem T. *Let X and Y be topological spaces, D, C be nonempty compact subsets of X, Y, resp., $f, g : X \times Y \to \mathbb{R}$, and $U : (D \times C) \times (X \times Y) \to \mathbb{R}$ be defined by $U((x, y), (u, v)) = f(u, y) - g(x, v)$. Assume that:*

(1) *the function $x \mapsto \sup_{y \in Y} f(x, y)$ is l.s.c. on D;*
(2) *the function $y \mapsto \inf_{x \in X} g(x, y)$ is u.s.c. on C;*
(3) *U is generally 0-pair-concave on $X \times Y$.*

Then the minimax inequality

$$\min_{x \in D} \sup_{y \in Y} f(x, y) \leq \max_{y \in C} \inf_{x \in X} g(x, y)$$

holds. Furthermore, if $f = g$, then

$$\inf_{x \in X_1} \sup_{y \in Y_1} f(x, y) = \sup_{y \in Y_2} \inf_{x \in X_2} f(x, y);$$

where X_i is either D or X and Y_i is either C or Y for $i = 1, 2$.

Remark. 1. When U is 0-pair-concave on $X \times Y$ in (3), Theorem T reduces to (Chang 2010, Theorem 3.3).

2. Theorem T also generalizes Theorem 3 in (Kim and Lee 2007, pp.1211–1214).

REFERENCES

Arandjelović, I. 1992. An extension of the Sion's minimax theorem. Zb. rad. Fil. fak. u Nišu, Ser. Mat. 6: 1–3.

Bielawski, R. 1987. Simplicial convexity and its applications. J. Math. Anal. Appl. 127: 155–171.

Chang, S.-Y. 2010. Inequalities and Nash equilibria. Nonlinear Anal. 73: 2933–2940.

Cho, Y. J., Kim, J. K. and Lee, B. S. 2009. Remarks on KKM maps and applications. J. Adv. Res. Appl. Math. 1(1): 1–8.

Danzig, G. B. 1956. Constructive proof of the min-max theorem. Pacific J. Math. 6: 25–33.

Fan, K. 1952. Fixed-point and minimax theorems in locally convex topological linear spaces. Proc. Nat. Acad. Sci. U.S.A. 38: 121–126.

Fan, K. 1966. Applications of a theorem concerning sets with convex sections. Math. Ann. 163: 189–203.

Fan, K. 1972. A minimax inequality and applications. In: "Inequalities III" (O.Shisha, Ed.), Academic Press, New York, pp. 103–113.

Granas, A. 1990. Quelques méthodes topologique èn analyse convexe. In: Méthdes topologiques en analyse convexe, Sém. Math. Supér. 110, Press. Univ. Montréal, pp. 11–77.

Granas, A. and Liu, F.-C. 1984. Théorèmes du minimax. C. R. Acad. Sc. Paris 298: 329–332.

Granas, A. and Liu, F.-C. 1986. Coincidences for set-valued maps and minimax inequalities. J. Math. Pures et Appl. 65: 119–148.

Ha, C. W. 1980. Minimax and fixed point theorems. Math. Ann. 248: 73–77.

Horvath, C. D. 1991. Contractibility and generalized convexity. J. Math. Anal. Appl. 156: 341–357.

Idzik, A. and Park, S. 1998. Leray-Schauder type theorems and equilibrium existence theorems. In: Differential Inclusions and Optimal Control, Lect. Notes in Nonlinear Anal. 2: 191–197.

Kakutani, S. 1941. A generalization of Brouwer's fixed-point theorem. Duke Math. J. 8: 457–459.

Knaster, B., Kuratowski, K., und Mazurkiewicz, S. 1929. Ein Beweis des Fixpunktsatzes für n-Dimensionale Simplexe. Fund. Math. 14: 132–137.

Kim, W. K. and Lee, K. H. 2007. Nash equilibrium and minimax theorem with $\mathcal{C}$-concavity. J. Math. Anal. Appl. 328: 1206–1216.

Komiya, H. 1981. Convexity in topological space. Fund. Math. 111: 107–113.

Komornik, V. 1982. Minimax theorems for upper semicontinuous functions. Acta Math. Acad. Sci. Hungar. 40: 159–163.

Lassonde, M. 1983. On the use of KKM multifunctions in fixed point theory and related topics. J. Math. Anal. Appl. 97: 151–201.

Liu, F.-C. 1978. A note on the von Neumann - Sion minimax principle. Bull. Inst. Math. Acad. Sinica 6: 517–524.

Nikaido, H. 1954. On von Neumann's minimax theorem. Pacific J. Math. 4: 65–72.

Park, S. 1989. Generalized Fan-Browder fixed point theorems and their applications. Collec. of Papers Dedicated to J. G. Park, Chonpook Nat. University, pp. 51–77.

Park, S. 1993. On minimax inequalities on spaces having certain contractible subsets. Bull. Austral. Math. Soc. 47: 25–40.

Park, S. 1997. Coincidence theorems for the better admissible multimaps and their applications. Proc. WCNA96, Nonlinear Anal. 30: 4183-4191

Park, S. 1998a. Remarks on a social equilibrium existence theorem of G. Debreu. Appl. Math. Letters 11(5): 51–54.

Park, S. 1998b. Minimax theorems in convex spaces. Novi Sad J. Math. 28: 1–8.

Park, S. 1999a. Ninety years of the Brouwer fixed point theorem, Vietnam J. Math. 27: 193–232.

Park, S. 1999b. Minimax theorems and the Nash equilibria on generalized convex spaces. Josai Math. Monograph 1: 33–46.

Park, S. 2000a. Acyclic versions of the von Neumann and Nash equilibrium theorems. J. Comp. Appl. Math. 113: 83–91.

Park, S. 2000b. Elements of the KKM theory for generalized convex spaces. Korean J. Comp. Appl. Math. 7: 1–28.

Park, S. 2002. Remarks on acyclic versions of generalized von Neumann and Nash equilibrium theorems. Appl. Math. Letters 15: 641–647.

Park, S. 2003. Coincidence, almost fixed point, and minimax theorems on generalized convex spaces. J. Nonlinear Convex Anal. 4: 151–164.

Park, S. 2007. Fixed point theorems for better admissible multimaps on almost convex sets. J. Math. Anal. Appl. 329(1): 690–702.

Park, S. 2008a. Applications of fixed point theorems on almost convex sets. J. Nonlinear Convex Anal. 9(1): 45–57.

Park, S. 2008b. Equilibrium existence theorems in KKM spaces. Nonlinear Anal. 69: 4352–4364.

Park, S. 2010a. From the KKM principle to the Nash equilibria. Inter. J. Math. Stat. 6(S10): 77–88.

Park, S. 2010b. Generalizations of the Nash equilibrium theorem in the KKM theory. Fixed Point Theory Appl. 2010: Article ID 234706, 23pp.

Park, S. 2010c. The KKM principle in abstract convex spaces: Equivalent formulations and applications. Nonlinear Anal. 73: 1028–1042.

Park, S. 2010d. A simple proof of the Sion minimax theorem. Bull. Korean Math. Soc. 47(5): 1037–1040.

Park, S. 2010e. On the von Neumann - Sion minimax theorem in KKM spaces. Appl. Math. Letters 23: 1269–1273.

Park, S. 2011a. Remarks on some basic concepts in the KKM theory. Nonlinear Anal. 74: 2439–2447.

Park, S. 2011b. The fixed point method versus the KKM method. In: Contributions to Game Theory and Management IV (Eds. L. A. Petrosyan, N. A. Zenkevich), Graduate School of Management, St. Petersburg Univ., St. Petersburg, Russia, pp. 347–360.

Park, S. 2011c. On S.-Y. Chang's inequalities and Nash equilibria. J. Nonlinear Convex Anal. 12(3): 455–471.

Park, S., Bae, J. S. and Kang, H. K. 1994. Geometric properties, minimax inequalities, and fixed point theorems on convex spaces. Proc. Amer. Math. Soc. 121: 429-440.

Ricceri, B. and Simons, S. (Eds.) 1989. Minimax Theory and Applications. Kluwer Acad. Publ., Dordrecht/Boston/London.

Simons, S. 1986. Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed-point theorems. Proc. Symp. Pure Math. Amer. Math. Soc. 45(2): 377–392.

Sion, M. 1958, On general minimax theorems. Pacific J. Math. 8: 171–176.

Tuy, H. 1974. On a general minimax theorem. Soviet Math. Dokl. 15(6): 1689–1692.

von Neumann, J. 1928. Zur Theorie der Gesellschaftsspiele. Math. Ann. 100: 295–320.

von Neumann, J. 1937. Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. Ergebnisse eines Mathematischen Kolloquiums 8: 73-83. [= A model of general economic equilibrium. Review of Econ. Studies 13: 1–9 (1945).]

von Neumann, J. 1953. Communication on the Borel notes. Econometrica 21: 124–125.