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SOME EQUILIBRIUM PROBLEMS IN KKM SPACES

SEHIE PARK

Abstract. The partial KKM principle is an abstract form of the original KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its 'open' version. We show that how the fundamental existence theorems on equilibrium problems can be extended to abstract convex spaces satisfying the partial KKM principle. Precisely, most of important results in the KKM theory hold for such spaces without assuming any linear structure. Such results are the Fan type KKM theorem, the minimax theorem and the intersection lemma of von Neumann, the Nash equilibrium theorem, various fixed point theorems, Fan's minimax inequality, variational inequalities, best approximation theorems, existence theorems for solutions of generalized quasi-equilibrium problems, and others.

1. Introduction

By an *equilibrium problem*, Blum and Oettli [1] understood the problem of finding:

(EP) $$\widehat{x} \in X \quad \text{such that} \quad f(\widehat{x}, y) \leq 0 \quad \text{for all} \quad y \in X,$$

where X is a given set and $f : X \times X \to \overline{\mathbb{R}}$ is a given function.

We can consider more general problems as follows:

A *quasi-equilibrium problem* is to find

(QEP) $$\widehat{x} \in X \quad \text{such that } \widehat{x} \in S(\widehat{x}) \quad \text{and} \quad f(\widehat{x}, z) \leq 0 \quad \text{for all} \quad z \in S(\widehat{x}),$$

where X and f are as above and $S : X \multimap X$ is a given multimap.

A *generalized quasi-equilibrium problem* is to find

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(GQEP) $\quad \widehat{x} \in X$ and $\widehat{y} \in T(\widehat{x})$ such that $\widehat{x} \in S(\widehat{x})$

and $f(\widehat{x}, \widehat{y}, z) \leq 0$ for all $z \in S(\widehat{x})$,

where X and S are the same as above, Y is another given set, $T : X \multimap Y$ is another multimap, and $f : X \times Y \times X \to \overline{\mathbb{R}}$ is a given function.

These problems contain as special cases, for instance, optimization problems, problems of the Nash type equilibrium, complementarity problems, fixed point problems, and variational inequalities, as well as many others. There are many generalizations of these problems; see [23,25,29–32,35]. Many authors also work on such problems replacing extended real-valued functions by other functions whose ranges are some partially ordered sets, order complete Riesz spaces, or some ordered vector spaces. But we are not concerned with them here.

In this survey, we study some equilibrium problems, quasi-equilibrium problems, and generalized quasi-equilibrium problems in abstract convex spaces. We show that how the fundamental theorems on equilibrium problems can be extended to abstrct convex spaces. Precisely, most of important results in the KKM theory hold without assuming the linearity in topological vector spaces. Such examples are the Fan type KKM theorem, the minimax theorem and the intersection lemma of von Neumann, the Nash equilibrium theorem, various fixed point theorems, Fan's minimax inequality, variational inequalities, best approximation theorems, existence theorems for solutions of generalized quasi-equilibrium problems, and others. This paper contains some improved versions of corresponding results in [34–36] and their proofs will appear in our forthcoming works.

2. Abstract convex spaces

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition 2.1 ([33–36])**.** An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. We give examples of abstract convex spaces:

1. The triple $(\Delta_n \supset V; \mathrm{co})$ in the original KKM theorem [21]; where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

2. A triple $(X \supset D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma :=$co. Fan's celebrated KKM lemma [5] is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of E.

3. A *convex space* $(X \supset D; \Gamma)$ [27, 28] is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$; see [22]. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

4. A triple $(X \supset D; \Gamma)$ is called an *H-space* by Park [27] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a c-space by Horvath [14] or an H-space by Bardaro and Ceppitelli in 1988. Hyperconvex metric spaces due to Aronszajn and Panitchpakdi, Hyperbolic spaces due to Reich and Shafrir, and many others are c-spaces.

5. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see [27, 28, 32] and references therein. When $X = D$, G-convex spaces are called L-spaces.

6. A *ϕ_A-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D \rangle$ with $|A| = n + 1$. Every ϕ_A-space can be made into a G-convex space; see [33]. Recently, ϕ_A-spaces are called GFC-spaces by Khanh et al. [18]. When $X = D$, ϕ_A-spaces are called FC-spaces or simplicial spaces.

7. A convexity space $(E, \mathcal{C})$ in the classical sense is an abstrct convex space whenever E is given a topology; see [40], where the bibliography lists 283 papers.

8. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [19]. Then $(H \supset X; \Gamma)$ is an abstract convex space.

9. According to Horvath [16], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

10. A $\mathbb{B}$-space due to Briec and Horvath [3] is an abstract convex space.

Note that each of 3-10 has a large number of concrete examples.

Definition 2.2. For an abstract convex space $(X \supset D; \Gamma)$, a real function $f : X \to \mathbb{R}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is Γ-convex for each $r \in \mathbb{R}$.

Recall that a real function $f : X \to \mathbb{R}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp. $\{x \in X \mid f(x) < r\}$] is open for each $r \in \mathbb{R}$.

3. The KKM theorems

Definition 3.1. Let $(E, D; \Gamma)$ be an abstract convex space. A multimap $G : D \multimap E$ is called a *KKM map* if

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle.$$

Example. Granas [9] gave examples of KKM maps as follows:

(i) *Variational problems.* Let C be a convex subset of a vector space E and $\phi : C \to \mathbb{R}$ is a convex function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid \phi(y) \leq \phi(x)\} \quad \text{for } x \in C$$

is a KKM map.

(ii) *Best approximation.* Let C be a convex subset of a vector space E, p a seminorm on E, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid p(f(y) - y) \leq p(f(y) - x)\} \quad \text{for } x \in C$$

is a KKM map.

(iii) *Variational inequalities.* Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, C a convex subset of H, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C \mid \langle f(y), y - x \rangle \leq 0\} \quad \text{for } x \in C$$

is a KKM map.

Example. For a ϕ_A-space $(X, D; \{\phi_A\})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D \rangle \text{ and } J \in \langle A \rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$; see [33].

Definition 3.2. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

In our recent works [34–36,38], we studied foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the (partial) KKM principle.

Example. We give examples of KKM spaces:

1. Every G-convex space is a KKM space [28].
2. A connected linearly ordered space $(X, \leq)$ can be made into a KKM space [36].
3. The extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see [36]. But L^* is not a G-convex space.
4. For a closed convex subset X of a complete $\mathbb{R}$-tree H with $\Gamma_A := conv_H(A)$ for each $A \in \langle X \rangle$, the triple $(H \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [19]. Later we found that $(H \supset X; \Gamma)$ is a KKM space [37].
5. For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [16, 37].
6. A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [3, Corollary 2.2].

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}
\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\
\Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\
\Longrightarrow \text{Space satisfying the partial KKM principle} \\
\Longrightarrow \text{Abstract convex space.}
\end{gathered}$$

It is not known yet whether there is a space satisfying the partial KKM principle that is not a KKM space.

The original KKM theorem is the closed version of the following well-known fact; see [21, 27, 28]:

The KKM theorem. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is,* $\operatorname{co} A \subset G(A)$ *for each $A \subset D$) with closed* [*resp., open*] *values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

The following is a Fan type KKM theorem for KKM spaces [36]:

Theorem 3.3. *An abstract convex space $(X, D; \Gamma)$ is a KKM space if and only if for any map $G : D \multimap X$ satisfying*

(1) *G has closed* [*resp., open*] *values; and*

(2) *G is a KKM map,*

$\{G(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

The KKM theory, first called by the author [24], is the study of applications of various equivalent formulations of the KKM theorem. At the beginning, the theory was mainly devoted to study on convex subsets of topological vector spaces by Ky Fan [5–8]. Later, it has been extended to convex spaces by Lassonde [22], and to c-spaces (or H-spaces) by Horvath [12–15], and others. Later the KKM theory is extended to G-convex spaces in a sequence of papers of the author; see [27, 28, 32] and references therein. Recently, the theory is further extended to abstract convex spaces or the KKM spaces [33–36].

A milestone of the history of the KKM theory was erected by Ky Fan [5]. He extended the original KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space. Further applications were followed by himself [6–8] and many other authors; see [9, 27].

Corollary 3.4 (Fan [5])**.** *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x) \subset Y$ be given such that the following two conditions are satisfied:*

(1) *The convex hull of any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\bigcup_{i=1}^{n} F(x_i)$.*

(2) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

4. The von Neumann type minimax theorem

An abstract convex space $(X, D; \Gamma)$ is said to be *compact* if X is compact.

Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of abstract convex spaces. Let $X := \prod_{i \in I} X_i$ be equipped with the product topology and $D = \prod_{i \in I} D_i$. For each

$i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(X, D; \Gamma)$ is an abstract convex space.

Note that for the case $X_i = D_i$ for each i is a G-convex space, the product of Γ-convex subsets is also Γ-convex in the product G-convex space; see [41].

In the framework of our KKM theory on abstract convex spaces, we have the following generalization of the von Neumann–Sion minimax theorem.

Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces. For their product as above, we define $\Gamma_{X \times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y \rangle$.

Theorem 4.1. *Let $(E; \Gamma) := (X \times Y; \Gamma_{X \times Y})$ be the product abstract convex space, $f, s, t, g : X \times Y \to \overline{\mathbb{R}}$ be four functions, $\mu := \inf_{y \in Y} \sup_{x \in X} f(x, y)$, and $\nu := \sup_{x \in X} \inf_{y \in Y} g(x, y)$. Suppose that*

(1) *$f(x, y) \le s(x, y) \le t(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each $r < \mu$ and $y \in Y$, $\{x \in X \mid s(x, y) > r\}$ is Γ_1-convex; for each $r > \nu$ and $x \in X$, $\{y \in Y \mid t(x, y) < r\}$ is Γ_2-convex;*

(3) *for each $r > \nu$, there exist $\{x_i\}_{i=1}^m \subset X$ such that $Y = \bigcup_{i=1}^m \mathrm{Int}\, \{y \in Y \mid f(x_i, y) > r\}$; and*

(4) *for each $r < \mu$, there exist $\{y_j\}_{j=1}^n \subset Y$ such that $X = \bigcup_{j=1}^n \mathrm{Int}\, \{x \in X \mid g(x, y_j) < r\}$.*

If $(E; \Gamma)$ satisfies the partial KKM principle, then we have

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \le \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Corollary 4.2 (Park [34])**.** *Let $(X; \Gamma)$ and $(Y; \Gamma')$ be compact abstract convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(1) *$f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. and $g(x, \cdot)$ is quasiconvex on Y; and*

(3) *for each $y \in Y$, $f(\cdot, y)$ is quasiconcave and $g(\cdot, y)$ is u.s.c. on X.*

If $X \times Y$ satisfies the partial KKM principle, then we have

$$\min_{y \in Y} \sup_{x \in X} f(x, y) \le \max_{x \in X} \inf_{y \in Y} g(x, y).$$

If $f = g$ and if X and Y are convex spaces, Corollary 4.2 reduces to the following Sion's generalization of the von Neumann minimax theorem:

Corollary 4.3 (Sion [39])**.** *Let X, Y be compact convex sets in topological vector spaces. Let f be a real function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x, y)$ is an l.s.c. quasiconvex function on Y, and*

(2) *for each fixed $y \in Y$, $f(x, y)$ is a u.s.c. quasiconcave function on X,*

then we have

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Kakutani [17] expressed the von Neumann theorem in 1928 as follows:

Corollary 4.4 (von Neumann [42]). *Let $f(x, y)$ be a continuous real function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$. If for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x\in K} \min_{y\in L} f(x, y) = \min_{y\in L} \max_{x\in K} f(x, y).$$

5. The von Neumann-Fan type intersection theorem

Given a Cartesian product $X = \prod_{i\in I} X_i$ of sets, let $X^i := \prod_{j\neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$[y_i, x^i] := (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

In our KKM theory, we have the following generalization of the von Neumann-Fan intersection theorem in [35]:

Theorem 5.1. *Let $\{(X_i; \Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X; \Gamma) = (\prod_{i=1}^n X_i; \Gamma)$ satisfies the partial KKM principle and, for each i, let A_i and B_i be subsets of X satisfying the following:*

(1) *for each $y \in X$, $B_i(y) := \{x \in X \mid [y_i, x^i] \in B_i\}$ is open; and*

(2) *for each $x \in X$, $\emptyset \neq \mathrm{co}_\Gamma B_i(x) \subset A_i(x) := \{y \in X \mid [y_i, x^i] \in A_i\}$.*

Then we have $\bigcap_{i=1}^n A_i \neq \emptyset$.

If each X_i is a compact G-convex space, so is X.

Corollary 5.2 (Fan [6]). *Let $X_1, X_2, \ldots, X_n$ be $n\,(\geq 2)$ compact convex sets each in a real Hausdorff topological vector space. Let $E_1, E_2, \ldots, E_n$ be n subsets of $X = \prod_{i=1}^n X_i$ having the following two properties:*

(a) *For each i and every point $x_i \in X_i$, the section $E_i(x_i)$ formed by all points $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ of X^i such that $(x_1, \ldots, x_n) \in E_i$ is open in X^i.*

(b) *For each i and every point $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ of X^i, the section $E_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ formed by all points $x_i \in X_i$ such that $(x_1, \ldots, x_n) \in E_i$ is nonempty and convex.*

Then the intersection $\bigcap_{i=1}^n E_i$ is nonempty.

6. The Nash equilibrium theorem

From Theorem 5.1, we can deduce the following generalization of the Nash equilibrium theorem for spaces satisfying the partial KKM principle in [35]:

Theorem 6.1. *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*
(0) *$f_i(x) \leq g_i(x)$ for each $x \in X$;*
(1) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on X_i;*
(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on X_i; and*
(4.3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*
Then there exists a point $\widehat{x} \in X$ such that

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \text{for all } i.$$

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem on equilibrium points of non-cooperative games. The following is formulated by Fan [6, Theorem 4]:

Corollary 6.2 (Nash)**.** *Let $X_1, X_2, \ldots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \ldots, f_n$ be n real-valued continuous functions defined on $X = \prod_{i=1}^n X_i$. If for each $i = 1, 2, \ldots, n$ and for any given point $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \prod_{j\neq i} X_j$, $f_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ is a quasiconcave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \ldots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \ldots, \widehat{x}_n) \qquad (1 \leq i \leq n).$$

The point $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)$ in the conclusion of Corollary 6.2 is called the *Nash equilibrium*. This concept is a natural extension of the local maxima (for the case $n = 1$, $f = f_1$) and of the saddle points (for the case $n = 2$, $f_1 = -f$, $f_2 = f$).

7. The Fan-Browder type fixed point theorems

From the KKM principle, we can show the following:

Theorem 7.1. *Let $(X, D;\Gamma)$ be a compact space satisfying the partial KKM principle and $S : X \multimap D$, $T : X \multimap X$ two maps such that*
(1) *$S^-(z)$ is open for each $z \in D$; and*
(2) *for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$.*
Then either
(i) *T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$; or*
(ii) *there exists an $x_1 \in X$ such that $S(x_1) = \emptyset$.*

Theorem 7.1 for the case (i) is *a Fan-Browder type fixed point theorem* and for the case (ii) *a maximal element theorem.*

From Theorem 7.1, we obtain the following Fan-Browder type theorems.

Corollary 7.2. *Let* $(X, D; \Gamma)$ *be a compact abstract convex space satisfying the partial KKM principle and* $S : X \multimap D$, $T : X \multimap X$ *two maps such that*
(1) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; *and*
(2) $X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.
Then T *has a fixed point* $x_0 \in X$.

Corollary 7.3. *Let* $(X, D; \Gamma)$ *be a compact space satisfying the partial KKM principle and* $S : X \multimap D$ *a map such that*
(1) *for each* $x \in X$, $S(x)$ *is nonempty; and*
(2) *for each* $z \in D$, $S^-(z)$ *is open.*
Then there exists an $\widehat{x} \in X$ *such that* $\widehat{x} \in \mathrm{co}_\Gamma S(\widehat{x})$.

The following is a simplified form of Corollary 7.2 or 7.3:

Corollary 7.4. *Let* $(X; \Gamma)$ *be a compact space satisfying the partial KKM principle and* $T : X \multimap X$ *a map such that*
(1) *for each* $x \in X$, $T(x)$ *is* Γ*-convex; and*
(2) $X = \bigcup\{\mathrm{Int}\, T^-(y) \mid y \in X\}$.
Then T *has a fixed point.*

From Corollary 7.4, we can easily obtain the following in 1968:

Corollary 7.5 (Browder [4])**.** *Let* K *be a nonempty compact convex subset of a Hausdorff topological vector space. Let* T *be a map of* K *into* 2^K*, where for each* $x \in K$, $T(x)$ *is a nonempty convex subset of* K*. Suppose further that for each* y *in* K, $T^-(y)$ *is open in* K*. Then there exists* x_0 *in* K *such that* $x_0 \in T(x_0)$.

Note that Browder's result is a reformulation of Fan's geometric lemma [5] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

Browder [4] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Fan-Browder theorem, see [27].

The Fan-Browder type fixed point theorem is used by Borglin and Keiding [2] and Yannelis and Prabhakar [44] to the existence of maximal elements in mathematical economics. We give a generalization of their result as follows:

Corollary 7.6. *Let* $(X, D; \Gamma)$ *be a compact space satisfying the partial KKM principle and* $S : X \multimap D$, $T : X \multimap X$ *two maps such that*

(1) $S^{-}(z)$ *is open for each* $z \in D$*;*

(2) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*

(3) *for each* $x \in X$, $x \notin T(x)$.

Then there exists an $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$.

8. The Fan type minimax inequalities

From Theorem 7.1, we obtain the following analytic alternative which is a basis of various equilibrium problems:

Theorem 8.1. *Let* $(X, D; \Gamma)$ *be a compact space satisfying the partial KKM principle,* $f : D \times X \to \mathbb{R}$ *and* $g : X \times X \to \mathbb{R}$ *two real functions, and* $\alpha, \beta \in \mathbb{R}$. *Suppose that*

(1) $\{y \in X \mid f(z, y) > \alpha\}$ *is open for each* $z \in D$*; and*

(2) *for each* $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \alpha\} \subset \{x \in X \mid g(x, y) > \beta\}$.

Then either

(a) *there exists a* $\widehat{y} \in X$ *such that* $f(z, \widehat{y}) \le \alpha$ *for all* $z \in D$*; or*

(b) *there exists an* $\widehat{x} \in X$ *such that* $g(\widehat{x}, \widehat{x}) > \beta$.

From Theorem 8.1, we immediately have the following generalized form of the Fan minimax inequality:

Theorem 8.2. *Under the hypothesis of Theorem 8.1, if* $\alpha = \beta = \sup\{g(x, x) \mid x \in X\}$, *then*

(c) *there exists a* $\widehat{y} \in X$ *such that*

$$f(z, \widehat{y}) \le \sup_{x \in X} g(x, x) \quad \text{for all } z \in D; \text{ and}$$

(d) *we have the minimax inequality*

$$\inf_{y \in X} \sup_{z \in D} f(z, y) \le \sup_{x \in X} g(x, x).$$

In 1972, Fan established a minimax inequality from Corollary 3.4:

Corollary 8.3 (Fan [8])**.** *Let* X *be a compact convex set in a topological vector space. Let* f *be a real function defined on* $X \times X$ *such that*

(i) *for each fixed* $x \in X$, $f(x, y)$ *is an l.s.c. function of* y *on* X;

(ii) *for each fixed* $y \in X$, $f(x, y)$ *is a quasiconcave function of* x *on* X.

Then we have the minimax inequality

$$\min_{y \in X} \sup_{x \in X} f(x, y) \le \sup_{x \in X} f(x, x).$$

Fan gave applications of his inequality as follows:

A variational inequality (extending Hartman-Stampacchia [10] and Browder [4]).

A geometric formulation of the inequality (equivalent to the Fan-Browder fixed point theorem).

Separation properties of upper demicontinuous multimaps, coincidence and fixed point theorems.

Properties of sets with convex sections (Fan [6]).

A fundamental existence theorem in potential theory.

9. The Himmelberg type fixed point theorems

In this section, we show that the open version of the KKM theorem is also useful to deduce very general fixed point theorems for topological vector spaces or KKM spaces. For simplicity, we give only one example.

We introduce particular subclasses or subsets of KKM uniform spaces.

Definition 9.1. A *KKM uniform space* $(E, D; \Gamma; \mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a Hausdorff uniform structure of E.

A KKM uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *$L\Gamma$-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

Example. 1. In particular, for G-convex spaces or H-spaces $(E \supset D; \Gamma; \mathcal{U})$ (where each Γ_A is contractible), we can define LG-spaces [32] or LH-spaces ($l.c.$-spaces), resp.

2. For a c-space $(X; \Gamma)$, an $L\Gamma$-space reduces to an $l.c.$-space [14, 15]. Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an $l.c.$-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples, see [14, 15]. A singleton is not necessarily Γ-convex in an $L\Gamma$-space.

3. A G-convex space $(X \supset D; \Gamma)$ is called a *metric LG-space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of metric $l.c.$-spaces due to Horvath [14].

4. (Horvath [15]) Any hyperconvex metric space (H, d) is a complete metric $l.c.$-space $(H; \Gamma)$.

The following is the main result of this section:

Theorem 9.2. *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be an* $L\Gamma$*-space and* $T : X \multimap X$ *a compact u.s.c. map with closed* Γ*-convex values. Then* T *has a fixed point* $x_0 \in X$*.*

Particular forms of Theorem 9.2 were obtained by von Neumann [43], Kakutani [17], Himmelberg [11], Horvath [15], and Park [32]. Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Rhee, and others. For the literature, see [27].

10. Variational inequalities

From Theorem 8.2, we obtain the following type of variational inequalities:

Theorem 10.1. *Let* $(X; \Gamma)$ *be a compact space satisfying the partial KKM principle and* $p, q : X \times X \to \mathbb{R}$ *and* $h : X \to \mathbb{R}$ *functions satisfying*

(1) $p(x, y) \leq q(x, y)$ *for each* $(x, y) \in X \times X$*, and* $q(x, x) \leq 0$ *for all* $x \in X$*;*
(2) *for each* $x \in X$*,* $q(x, \cdot) + h(\cdot)$ *is quasiconcave on* X*; and*
(3) *for each* $y \in X$*,* $p(\cdot, y) - h(\cdot)$ *is l.s.c. on* X*.*

Then there exists a $y_0 \in X$ *such that*

$$p(x, y_0) + h(y_0) \leq h(x) \quad \text{for all } x \in X.$$

In 1966, the following variational inequality appeared:

Corollary 10.2 (Hartman-Stampacchia [10])**.** *Let* K *be a compact convex subset in* $\mathbb{R}^n$ *and* $f : K \to \mathbb{R}^n$ *a continuous map. Then there exists* $u_0 \in K$ *such that*

$$(f(u_0), v - u_0) \geq 0 \quad \text{for all } v \in K,$$

where $(\cdot, \cdot)$ *denotes the scalar product in* $\mathbb{R}^n$*.*

Using this result, its authors obtained existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions of Dirichlet boundary value problems associated with certain nonlinear elliptic differential functional equations.

Later Corollary 10.2 is known to be equivalent to the Brouwer fixed point theorem. Corollary 10.2 was extended as follows:

Corollary 10.3 (Browder [4])**.** *Let* E *be a locally convex Hausdorff topological vector space,* K *a compact convex subset of* E*, and* T *a continuous mapping of* K *into* E^**. Then there exists an element* u_0 *of* K *such that*

$$(T(u_0), u - u_0) \geq 0$$

for all $u \in K$*.*

Here, E^* is the topological dual of E equipped with an adequate topology and $(\ ,\ )$ denotes the pairing between elements of E^* and elements of E. This theorem is later extended and improved by many authors by pointing out that the local convexity is superfluous.

11. Best approximations

A simple consequence of Theorem 10.1 is the following well-known existence result on best approximations originated from Fan [7]:

Theorem 11.1. *Let X be a compact convex subset of a topological vector space E and $f : X \to E$ a continuous function. Then for any continuous seminorm p on E, there exists a point $y_0 \in X$ such that*

$$p(y_0 - f(y_0)) \leq p(x - f(y_0)) \quad \textit{for all } x \in X.$$

Theorem 11.1 implies the following generalization of the Schauder fixed point theorem:

Corollary 11.2 (Fan [7])**.** *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$\|y_0 - f(y_0)\| = \min_{x \in X} \|x - f(y_0)\|.$$

(*In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.*)

Fan also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. Those are known as best approximation theorems and applied to obtain generalizations of the Brouwer fixed point theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps in Fan [7].

12. Generalized quasi-equilibrium problems

In this section, we deal with existence of solutions of certain quasi-equilibrium problems in abstract convex spaces satisfying the partial KKM principle without any linear structure or in topological vector spaces (t.v.s.).

We obtained the following in [34]:

Theorem 12.1. *Let $(X; \Gamma)$ be a compact space satisfying the partial KKM principle, and let $S : X \multimap X$ be a map with nonempty Γ-convex values and open fibers such that $\overline{S} : X \multimap X$ is u.s.c. Suppose that $\psi : X \times X \to \mathbb{R}$ is a continuous function such that $\psi(x, \cdot)$ is quasiconvex and*

$$\psi(x, x) \geq 0 \qquad \textit{for all } x \in X.$$

Then there exists an $\widehat{x} \in X$ *such that*

$$\widehat{x} \in S(\widehat{x}) \quad and \quad \psi(\widehat{x}, x) \geq 0 \qquad for\ all\ x \in S(\widehat{x}).$$

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [20]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Example. Every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy space H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any compact convex locally convex subset of a t.v.s. is admissible.

We obtained the following fixed point theorem [26] which generalizes the Himmelberg theorem [11]:

Theorem 12.2. *Let E be a Hausdorff t.v.s. and X an admissible convex subset of E. Then any compact acyclic map $F : X \multimap X$ has a fixed point $x \in X$.*

From Theorem 12.2, we deduced the following existence theorem [30] for a generalized quasi-equilibrium problem:

Theorem 12.3. *Let X and Y be admissible convex subsets of Hausdorff t.v.s. E and F, respectively, $S : X \multimap X$ a compact closed map, $T : X \multimap Y$ a compact acyclic map, and $\phi : X \times Y \times X \to \mathbb{R}$ a u.s.c. function. Suppose that*

(1) *the function $m : X \times Y \to \mathbb{R}$ defined by*

$$m(x,y) = \max_{s \in S(x)} \phi(x,y,s) \quad for\ (x,y) \in X \times Y$$

is l.s.c.; and

(2) *for each $(x,y) \in X \times Y$, the set*

$$M(x,y) = \{u \in S(x) \mid \phi(x,y,u) = m(x,y)\}$$

is acyclic.

Then there exists an $(\widehat{x}, \widehat{y}) \in X \times Y$ such that

$$\widehat{x} \in S(\widehat{x}),\ \widehat{y} \in T(\widehat{x}),\ and\ \phi(\widehat{x}, \widehat{y}, \widehat{x}) \geq \phi(\widehat{x}, \widehat{y}, s) \quad for\ all\ s \in S(\widehat{x}).$$

Moreover, in [31], Theorem 12.3 is applied to deduce collectively fixed point theorems, intersection theorems for sets with convex sections, and quasi-equilibrium theorems. For related results, see [23–31] and references therein.

Added in Proof. Recently, we published more detailed basic KKM theory in [38].

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Sehie Park
The National Academy of Sciences, Republic of Korea, Seoul 137–044;
Department of Mathematical Sciences, Seoul National Univ., Seoul 151–747, Korea
E-mail address: shpark@math.snu.ac.kr