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A SIMPLE PROOF OF THE SION MINIMAX THEOREM

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Abstract. For convex subsets X of a topological vector space E, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies generalized forms of the Sion minimax theorem.

The von Neumann-Sion minimax theorem is fundamental in convex analysis and in game theory. von Neumann [8] proved his theorem for simplexes by reducing the problem to the 1-dimensional cases. Sion's generalization [7] was proved by the aid of Helly's theorem and the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [5]. In a recent paper, Kindler [4] proved Sion's theorem by applying the 1-dimensional KKM theorem (i.e., every interval in $\mathbb{R}$ is connected), the 1-dimensional Helly theorem (i.e., any family of pairwise intersecting compact intervals in $\mathbb{R}$ has nonempty intersection), and Zorn's lemma (or other method).

In this short note, for convex subsets X of a topological vector space E, we show that a KKM principle implies a Fan-Browder type fixed point theorem and that this theorem implies a generalized form of the Sion minimax theorem.

Definition. If a multimap $G : X \multimap X$ satisfies

$$\operatorname{co} A \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all finite subset } A \text{ of } X,$$

then G is called a KKM map.

Definition. A multimap $T : X \multimap X$ is called a Fan-Browder map provided that

(a) for each $x \in X$, $T(x)$ is convex; and
(b) $X = \bigcup_{y \in N} \operatorname{Int} T^{-}(y)$ for some finite subset N of X.

Here, Int denotes the interior with respect to X and, for each $y \in X$, $T^{-}(y) := \{x \in X \mid y \in T(x)\}$.

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For a convex subset X of a topological vector space E, let us consider the following statements:

(A) The KKM principle. *For any closed-valued KKM map $G : X \multimap X$, the family $\{G(x)\}_{x \in X}$ has the finite intersection property.*

(B) The Fan-Browder fixed point theorem. *Any Fan-Browder map $T : X \multimap X$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Recall that (A) originates from the Knaster-Kuratowski-Mazurkiewicz theorem [5] and holds by Fan's lemma [3], and (B) from Fan [3] and Browder [1].

Theorem 1. *The statement* (A) *implies* (B).

Proof. Define a map $G : X \multimap X$ by $G(x) := X \setminus \operatorname{Int} T^{-}(x)$ for each $x \in X$. Then each $G(x)$ is (relatively) closed, and

$$\bigcap_{y \in N} G(y) = X \setminus \bigcup_{y \in N} \operatorname{Int} T^{-}(y) = X \setminus X = \emptyset$$

by (b). Therefore, the family $\{G(x)\}_{x \in X}$ does not have the finite intersection property, and hence, G is not a KKM map by (A). Thus, there exists a finite subset A of X such that $\operatorname{co} A \not\subset G(A) = \bigcup\{X \setminus \operatorname{Int} T^{-}(y) \mid y \in A\}$. Hence, there exists an $x_0 \in \operatorname{co} A$ such that $x_0 \in \operatorname{Int} T^{-}(y) \subset T^{-}(y)$ for all $y \in A$; that is, $A \subset T(x_0)$. Therefore, $x_0 \in \operatorname{co} A \subset T(x_0)$ by (a). □

Theorem 2. *Let X and Y be nonempty convex subsets of two topological vector spaces, and $f, s, t, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be four functions,*

$$\mu := \inf_{y \in Y} \sup_{x \in X} f(x, y) \text{ and } \nu := \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Suppose that

(2.1) *$f(x, y) \le s(x, y) \le t(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2.2) *for each $r < \mu$ and $y \in Y$, $\{x \in X \mid s(x, y) > r\}$ is convex; for each $r > \nu$ and $x \in X$, $\{y \in Y \mid t(x, y) < r\}$ is convex;*

(2.3) *for each $r > \nu$, there exists a finite subset $\{x_i\}_{i=1}^{m}$ of X such that $Y = \bigcup_{i=1}^{m} \operatorname{Int} \{y \in Y \mid f(x_i, y) > r\}$; and*

(2.4) *for each $r < \mu$, there exists a finite subset $\{y_j\}_{j=1}^{n}$ of Y such that $X = \bigcup_{j=1}^{n} \operatorname{Int} \{x \in X \mid g(x, y_j) < r\}$.*

Then we have $\mu \le \nu$, that is,

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \le \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

Proof. Suppose that there exists a real c such that

$$\nu := \sup_{x} \inf_{y} g(x, y) < c < \inf_{y} \sup_{x} f(x, y) =: \mu.$$

Define a map $T : X \times Y \multimap X \times Y$ by

$$T(x, y) := \{\bar{x} \in X \mid s(\bar{x}, y) > c\} \times \{\bar{y} \in Y \mid t(x, \bar{y}) < c\}$$

for each $(x,y) \in X \times Y$. Then each $T(x,y)$ is convex by (2.2). Moreover, for each $(\bar{x},\bar{y}) \in X \times Y$, we have

$$\begin{aligned} T^-(\bar{x},\bar{y}) &= \{x \in X \mid s(x,\bar{y}) > c\} \times \{y \in Y \mid t(\bar{x},y) < c\} \\ &\supset \{x \in X \mid f(x,\bar{y}) > c\} \times \{y \in Y \mid g(\bar{x},y) < c\} \\ &\supset \operatorname{Int}\{x \in X \mid f(x,\bar{y}) > c\} \times \operatorname{Int}\{y \in Y \mid g(\bar{x},y) < c\}. \end{aligned}$$

Therefore, by (2.3) and (2.4), $X \times Y$ is covered by

$$\{\operatorname{Int} T^-(x_i,y_j) \mid 1 \le i \le m, 1 \le j \le n\}.$$

Hence, T is a Fan-Browder map. Since $X \times Y$ is a convex subset of a topological vector space, (A) and (B) hold. Therefore, by (B), we have an $(x_0,y_0) \in X \times Y$ such that $(x_0,y_0) \in T(x_0,y_0)$. Therefore, $t(x_0,y_0) < c < s(x_0,y_0)$, a contradiction. □

Recall that a extended real-valued function $f : X \to \overline{\mathbb{R}}$ on a topological space X is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

For a convex set X, a extended real-valued function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in E \mid f(x) > r\}$ [resp., $\{x \in E \mid f(x) < r\}$] is convex for each $r \in \overline{\mathbb{R}}$.

Theorem 3. *Let X and Y be compact convex subsets of topological vector spaces, and $f,s,t,g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions satisfying*

(3.1) $f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ *for each* $(x,y) \in X \times Y$;

(3.2) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and $t(x,\cdot)$ is quasiconvex on Y; and*

(3.3) *for each $y \in Y$, $s(\cdot,y)$ is quasiconcave and $g(\cdot,y)$ is u.s.c. on X.*

Then we have

$$\min_{y\in Y} \sup_{x\in X} f(x,y) \le \max_{x\in X} \inf_{y\in Y} g(x,y).$$

Proof. Note that $y \mapsto \sup_{x\in X} f(x,y)$ is l.s.c. on Y and $x \mapsto \inf_{y\in Y} g(x,y)$ is u.s.c. on X. Therefore, the both sides of the inequality exist. Then all the requirements of Theorem 2 are satisfied. □

For $f = s = t = g$ in Theorem 3, we have the following Sion minimax theorem [7]:

Theorem 4. *Let X and Y be compact convex subsets of topological vector spaces and $f : X \times Y \to \mathbb{R}$ a real function such that*

(4.1) *for each $x \in X$, $f(x,\cdot)$ is l.s.c. and quasiconvex on Y; and*

(4.2) *for each $y \in Y$, $f(\cdot,y)$ is u.s.c. and quasiconcave on X.*

Then

(i) *f has a saddle point $(x_0,y_0) \in X \times Y$; and*

(ii) *we have*

$$\min_{y\in Y} \max_{x\in X} f(x,y) = \max_{x\in X} \min_{y\in Y} f(x,y).$$

Proof. It is well known and easy to see that the minima and maxima in Theorem 4 exist under our topological assumptions. Hence, there exists an $(x_0, y_0) \in X \times Y$ such that

$$\min_{y\in Y}\max_{x\in X} f(x,y) = \max_{x\in X} f(x,y_0) \geq f(x_0,y_0) \geq \min_{y\in Y} f(x_0,y) = \max_{x\in X}\min_{y\in Y} f(x,y).$$

Moreover, all the requirements of Theorem 3 with $f = g$ are satisfied. Therefore, the $\geq$'s in the above should be $=$ and we have the conclusion. □

Remark 1. von Neumann [8] obtained Theorem 4 when X and Y are subsets of Euclidean spaces and f is continuous.

2. (A) also holds for open-valued KKM maps, and (B) also holds when T^- has closed values. In this case, (A) implies (B) also.

3. For other simple proof of the Sion minimax theorem, see [4].

4. Theorem 2 is motivated from [2, Theorem 8], which is for $f = s = t = g$.

5. For the history of the KKM theory, see [6].

6. All the results in this paper can be extended to abstract convex spaces without assuming any linear structure.

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