Proceedings of the 9th International Conference on
Fixed Point Theory and its Applications, pp. 145–159

FIXED POINT THEOREMS IN THE NEW ERA OF THE KKM THEORY

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Abstract. In 2006, we introduced the concept of abstract convex spaces $(E, D; \Gamma)$ on which we can construct the KKM theory and study multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$ and $\mathfrak{B}$. Abstract convex spaces satisfying the KKM principle $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$ is called a *KKM space*. It is known that abstract convex spaces satisfying the partial KKM principle $1_E \in \mathfrak{KC}(E,E)$ is really adequate to establish the essential part of the KKM theory. Consequently, the KKM theory steps in the new era and becomes the study of such new spaces. In this survey, we show that several well-known fixed point theorems for G-convex spaces can be extended to KKM uniform spaces or abstract convex uniform spaces, and that our new theorems subsume numerous known results.

1. Introduction

One of the earliest equivalent formulations of the Brouwer fixed point theorem is a theorem of Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) in 1929 [19], which is the closed version of the following [26]:

The KKM theorem. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is,* $\mathrm{co}\, A \subset G(A)$ *for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

The KKM theory, first called by the author in 1992 [21], is the study of applications of various equivalent formulations of the KKM theorem and their generalizations. At the beginning, the basic theorems in the KKM theory and their applications were established for convex subsets of topological vector spaces mainly by Ky Fan in 1961-84 [3–9]. Then, the theory had been extended to convex spaces by Lassonde in 1983 [20], and to c-spaces (or H-spaces) by Horvath in 1984-93 [11–14] and others; see [41]. Since 1993, it was extended to generalized convex (G-convex) spaces in a sequence of papers of the author in 1993-2005 [26–29, 31, 42]. Its basic theorems have many applications to various equilibrium problems in nonlinear analysis and many other fields.

2000 *Mathematics Subject Classification.* Primary 47H04, 47H10. Secondary 46A16, 46A55, 52A07, 54C60, 54H25, 55M20, 91A06, 91B50.

Key words and phrases. Abstract convex space, KKM space, generalized (G-) convex space, (partial) KKM principle, map class $\mathfrak{B}$.

In the last decade, a number of authors have tried to imitate, modify, or generalize G-convex spaces and published a large number of papers. Most of them adopted artificial terminology and concepts without giving any proper examples or justifications. We found that most of such 'new' spaces are subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ or spaces having a family $\{\phi_A\}_{A\in\langle D\rangle}$ of singular simplexes, where $\langle D\rangle$ is the set of nonempty finite subsets of a set D. We noticed that this kind of spaces can be made into G-convex spaces; see [33, 34].

In order to destroy such unnecessary concepts and to upgrade the KKM theory, in 2006-09, we proposed a new concept of abstract convex spaces $(E, D; \Gamma)$; see [30, 33, 35–40]. Abstract convex spaces satisfying the KKM principle $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$ is called a *KKM space*. It is known that KKM spaces and spaces satisfying the partial KKM principle $1_E \in \mathfrak{KC}(E, E)$ are really adequate to establish the essential part of the KKM theory. Moreover, in the frame of such new spaces, certain broad classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps (having the KKM property) are also studied. Now the KKM theory becomes the study of KKM spaces or spaces satisfying the partial KKM principle.

In this survey, we introduce the birth of the new KKM space and the basic results in the KKM theory; and show that well-known fixed point theorems for G-convex spaces can be extended to KKM uniform spaces or abstract convex uniform spaces. In fact, we are concerned with the transition from [31] to [38] and show that new theorems in [38, 39] subsume lots of known theorems mainly due to the author.

2. Theory of the KKM spaces

Since the concept of G-convex spaces appeared in 1993, many authors have tried to imitate, modify, or generalize the concept and published a large number of papers of the same nature. In fact, in the last decade, there have appeared authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplexes. Such example are L-spaces, spaces having property (H), FC-spaces, convexity structures satisfying the H-condition, M-spaces and another L-spaces, simplicial spaces, L^*-spaces, and others. Some authors claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. Some authors also tried to generalize the KKM theorem for their own settings. They introduced various types of generalized KKM maps; for example, generalized KKM maps on L-spaces, generalized R-KKM maps, and many other KKM type maps using artificial terminology. We found that most of such spaces are subsumed in the concept of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, which can be made into G-convex spaces; see [33, 34]

In order to destroy such unnecessary concepts and to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the

KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [33, 35–40].

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Example. In [33–40], we gave plenty of examples of abstract convex spaces as follows:

1. The original KKM theorem [19] is for the triple $(\Delta_n \supset V; \mathrm{co})$, where Δ_n is the standard n-simplex, V the set of its vertices $\{e_i\}_{i=0}^n$, and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation.

2. A triple $(X \supset D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma :=$co. Fan's celebrated KKM lemma [3] is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of E.

3. A *convex space* $(X \supset D; \Gamma)$ [23–26] is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept reduces to the one due to Lassonde for $X = D$; see [20]. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

4. A triple $(X \supset D; \Gamma)$ is called an *H-space* by Park [26, 41] if X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a c-space by Horvath [13] or an H-space by Bardaro and Ceppitelli in 1988. Horvath [11–14] obtained remarkable results on c-spaces.

5. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see [26–29, 31, 42] and references therein.

6. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular

n-simplexes) for $A \in \langle D \rangle$ with $|A| = n+1$. Every ϕ_A-space can be made into a G-convex space; see [33, 34].

7. A convexity space $(E, \mathcal{C})$ in the classical sense is an abstract convex space whenever E is given a topology; see [44], where the bibliography lists 283 papers.

8. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and for each $A \in \langle X \rangle$, $\Gamma_A := conv_H(A)$, where $conv_H(A)$ is the intersection of all closed convex subsets of H that contain A; see Kirk and Panyanak [18]. Then $(H, X; \Gamma)$ is an abstract convex space.

9. According to Horvath [15], a convexity on a topological space X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

10. A $\mathbb{B}$-space due to Briec and Horvath [1] is an abstract convex space.

Note that each of 3-10 has a large number of concrete examples.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp., $F \in \mathfrak{KO}(E, Z)$].

A *KKM space* $(E, D; \Gamma)$ is an abstract convex space satisfying the *KKM principle* $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$. Every KKM space satisfies the *partial KKM principle* $1_E \in \mathfrak{KC}(E, E)$.

In our recent works [36, 39, 43], we studied the elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces.

Example. We give examples of KKM spaces:

1. Every G-convex space is a KKM space; see [27].

2. A connected linearly ordered space $(X, \leq)$ can be made into a KKM space; see [38].

3. The extended long line L^* is a KKM space $(L^* \supset D; \Gamma)$ with the ordinal space $D := [0, \Omega]$. But L^* is not a G-convex space [39].

4. Suppose X is a closed convex subset of a complete $\mathbb{R}$-tree H, and $\Gamma_A := conv_H(A)$ for each finite $A \in \langle X \rangle$. Then $1_H \in \mathfrak{KC}(H, H)$ (Kirk and Panyanak [18]). Later we found that $1_H \in \mathfrak{KO}(H, H)$; see [40]. Therefore, $(H, X; \Gamma)$ is a KKM space.

5. For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property [15], the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [40].

6. A $\mathbb{B}$-space due to Briec and Horvath is a KKM space [1, Corollary 2.2].

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}\\ \Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}\\ \Longrightarrow \text{Space satisfying the partial KKM principle}\\ \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

In the KKM theory [36, 39, 40, 43], it is routine to reformulate the (partial) KKM principle to the following equivalent forms:

Fan type matching property
An intersection property
Geometric or section properties
The Fan-Browder type fixed point theorem
Existence theorem of maximal elements, and others

Any of such statements can be used to characterize the KKM spaces.

Moreover, from the partial KKM principle we have a whole intersection property of the Fan type. From this, we can deduce the following:

Theorem 2.3 ([39]). *Let $(X, D; \Gamma)$ be a space satisfying the partial KKM principle, K a nonempty compact subset of X, and $G : D \multimap X$ a map such that*

(1) $\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ *[that is, G is transfer closed-valued];*

(2) $\overline{G}$ *is a KKM map; and*

(3) *either*

(i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

From this theorem, in [36, 39, 43], we deduced the following equivalent formulations of the partial KKM principle:

Analytic alternatives (a basis of various equilibrium problems)

Consequently, for a compact abstract convex space $(X;\Gamma)$, we deduced 15 theorems from the partial KKM principle. Moreover, we noticed there that, for a compact G-convex space $(X;\Gamma)$, each of these 15 theorems and their corollaries is equivalent to the original KKM theorem.

Further applications of the partial KKM principle are given [35, 36, 43] as follows:

Best approximations (under certain restrictions)
The von Neumann type minimax theorem
The von Neumann type intersection theorem
The Nash type equilibrium theorem
The Himmelberg fixed point theorem for KKM spaces
Weakly KKM maps [37]

3. The Fan-Browder type fixed point theorems

The KKM principle is equivalent to the Fan-Browder type fixed point theorem:

Theorem 3.1 ([39]). *An abstract convex space $(X, D;\Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X,\ T : X \multimap X$ satisfying*
(1) *$S(z)$ is open for each $z \in D$;*
(2) *for each $y \in X,\ \mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(3) *$X = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$,*
T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

An abstract convex space $(X, D;\Gamma)$ is a KKM space iff the above condition also holds for any map $S : D \multimap X$ such that
(1)$'$ *$S(z)$ is closed for each $z \in D$*
instead of (1).

From Theorem 3.1, we can deduce various abstract forms of known results as follows:

Corollary 3.2. *Let $(X, D;\Gamma)$ satisfy the partial KKM principle and $S : D \multimap X,\ T : X \multimap X$ be two maps such that*
(1) *$S(z)$ is open for each $z \in D$;*
(2) *for each $y \in X,\ \mathrm{co}_\Gamma S^-(y) \subset T^-(y)$;*
(3) *for each $y \in X,\ S^-(y) \neq \emptyset$; and*
(4) *$X \setminus S(z_0)$ is compact for some $z_0 \in D$.*
Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

An abstract convex space $(X, D;\Gamma)$ is said to be *compact* if X is compact.

Corollary 3.3. *Let $(X, D;\Gamma)$ be a compact abstract convex space satisfying the partial KKM principle and $S : X \multimap D,\ T : X \multimap X$ two maps such that*

(1) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T^-(y)$*; and*
(2) $X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.
Then T *has a fixed point* $x_0 \in X$.

Corollary 3.4. *Let* $(X, D; \Gamma)$ *be a compact abstract convex space satisfying the partial KKM principle and* $S : X \multimap D$ *a map such that*
(1) *for each* $x \in X$, $S(x)$ *is nonempty; and*
(2) *for each* $z \in D$, $S^-(z)$ *is open.*
Then there exists an $\widehat{x} \in X$ *such that* $\widehat{x} \in \mathrm{co}_\Gamma\, S(\widehat{x})$.

The following simplified form of Corollary 3.3 or 3.4 is also a Fan-Browder type fixed point theorem:

Corollary 3.5. *Let* $(X; \Gamma)$ *be a compact abstract convex space satisfying the partial KKM principle and* $T : X \multimap X$ *a map satisfying*
(1) *for each* $x \in X$, $T(x)$ *is* Γ*-convex; and*
(2) $X = \bigcup\{\mathrm{Int}\, T^-(y) \mid y \in X\}$.
Then T *has a fixed point.*

Remark. (1) For a convex subset X of a Hausdorff topological vector space E, if $T^-(y)$ itself is open, then Corollary 3.5 reduces to Browder's result [2].

(2) Note that Browder's result is a reformulation of Fan's geometric lemma [3] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

(3) Browder [2] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems.

The Fan-Browder type fixed point theorem is used by Borglin and Keiding in 1976 and Yannelis and Prabhakar in 1983 to the existence of maximal elements in mathematical economics. We give a generalization of their result as follows:

Corollary 3.6. *Let* $(X, D; \Gamma)$ *be a compact abstract convex space satisfying the partial KKM principle and* $S : X \multimap D$, $T : X \multimap X$ *two maps such that*
(1) $S^-(z)$ *is open for each* $z \in D$*;*
(2) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*
(3) *for each* $x \in X$, $x \notin T(x)$.
Then there exists an $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$.

4. The Zima-Hadžić type fixed point theorems

We begin with the following:

Definition 4.1. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is the one with a basis $\mathcal{U}$ of a uniform structure of E.

We consider various subclasses or subsets of abstract convex uniform spaces as in [38]:

Definition 4.2. A *KKM uniform space* $(E, D; \Gamma; \mathcal{U})$ is a KKM space with a basis $\mathcal{U}$ of a uniform structure of E.

A KKM uniform space $(E \supset D; \Gamma; \mathcal{U})$ is called an *$L\Gamma$-space* if D is dense in E and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] := \{x \in E \mid A \cap U[x] \neq \emptyset\}$$

around a given Γ-convex subset $A \subset E$ is Γ-convex.

A singleton is not necessarily Γ-convex in an $L\Gamma$-space.

Example. 1. In particular, for G-convex spaces or H-spaces $(E \supset D; \Gamma; \mathcal{U})$ (where each Γ_A is contractible), we can define LG-spaces [28] or LH-spaces, resp.

2. For a c-space $(X; \Gamma)$, an $L\Gamma$-space reduces to an *l.c.*-space [13, 14]. Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an *l.c.*-space $(X; \Gamma)$ with $\Gamma_A = \operatorname{co} A$ for $A \in \langle X \rangle$. For other examples, see [13, 14].

3. A G-convex space $(X \supset D; \Gamma)$ is called a *metric LG-space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of metric *l.c.*-spaces due to Horvath [13].

4. Any hyperconvex metric space (H, d) is a complete metric *l.c.*-space $(H; \Gamma)$; see Horvath [14].

Definition 4.3. An abstract convex uniform space $(E \supset D;\ \Gamma;\ \mathcal{U})$ is said to be *locally Γ-convex* if D is dense in E and, for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \subset U$ and, for each $x \in E$, $\operatorname{co}_\Gamma(V[x] \cap D) \subset U[x]$, that is,

$$N \in \langle V[x] \cap D \rangle \Rightarrow \Gamma_N \subset U[x].$$

Definition 4.4. For an abstract convex uniform space $(E \supset D; \Gamma;\ \mathcal{U})$, a subset X of E is said to be *of the Z type* or *of the Zima-Hadžić type* if $D \cap X$ is dense in X and for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that, for each $N \in \langle D \cap X \rangle$ and any Γ-convex subset A of X, we have

$$A \cap V[z] \neq \emptyset \ \ \forall z \in N \Rightarrow A \cap U[x] \neq \emptyset \ \ \forall x \in \Gamma_N.$$

Definition 4.5. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a *Φ-map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $\operatorname{co}_\Gamma G(x) \subset H(x)$ [that is, $H(x)$ is Γ-convex relative to $G(x)$]; and

(ii) $X = \bigcup \{\operatorname{Int} G^-(y) \mid y \in D\}$.

Definition 4.6. In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\mathrm{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

Definition 4.7. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying conditions

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

A subset X of E is *admissible* (in the sense of Klee) iff every compact subset K of X is Klee approximable into E.

The following summarizes the mutual relations among the various subclasses of abstract convex uniform spaces; see [31, 38]:

Lemma 4.8. *In the class of abstract convex uniform spaces $(X, D; \Gamma; \mathcal{U})$, the following hold:*

(1) *Any $L\Gamma$-space is of the Zima-Hadžić type.*

(2) *Every nonempty subset of an $L\Gamma$-space is locally Γ-convex whenever every singleton is Γ-convex.*

(3) *Any nonempty subset of a locally Γ-convex space is a Φ-set.*

(4) *Any Zima-Hadžić type subset of an abstract convex uniform space such that every singleton is Γ-convex is a Φ-set.*

(5) *Every G-convex Φ-space is admissible. More generally, every nonempty compact Φ-subset of a G-convex space is Klee approximable.*

The open version of the KKM principle is also useful to deduce very general fixed point theorems. The following example is the main result of this section:

Theorem 4.9 ([38]). *Let $(X \supset D; \Gamma; \mathcal{U})$ be a KKM and Hausdorff uniform space and $T : X \multimap X$ a compact u.s.c. map with nonempty closed Γ-convex values. If $T(X)$ is of the Zima-Hadžić type, then T has a fixed point $x_* \in T(x_*)$.*

Corollary 4.10. *Let $(X \supset D; \Gamma; \mathcal{U})$ be a Hausdorff $L\Gamma$-space and $T : X \multimap X$ a compact u.s.c. map with closed Γ-convex values. Then T has a fixed point $x_0 \in X$.*

Example. 1. For an LG-space $(X, D; \Gamma)$, this is given in [28]. Many particular forms were stated there.

2. The extended long line L^* is a compact $L\Gamma$-space; see [38]. Now it has *the fixed point property.* This is a proper example of Corollary 4.10 which is not an LG-space.

Corollary 4.11 (Himmelberg [10]). *Let X be a convex subset of a locally convex Hausdorff topological vector space E and $T : X \multimap X$ a compact u.s.c. multimap with nonempty closed convex values. Then T has a fixed point $x_0 \in T(x_0)$.*

Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Rhee, and others. For the literature, see [26].

More early, in order to give simple proofs of von Neumann's intersection lemma and the minimax theorem, Kakutani obtained the following generalization of the Brouwer theorem to multimaps:

Corollary 4.12 (Kakutani [16]). *If $x \mapsto \Phi(x)$ is a u.s.c. point-to-set mapping of an r-dimensional closed simplex S into the family of closed convex subsets of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary 4.13 (Kakutani [16]). *Corollary 4.12 is also valid even if S is an arbitrary bounded closed convex set in an Euclidean space.*

As Kakutani noted, Corollary 4.13 readily implies von Neumann's 1937 Lemma, and later it is known that those two results are equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

According to Debreu (A commentary on the Kakutani fixed point theorem, in *Shizuo Kakutani: Selected Papers* [17]), "Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved by elementary means."

5. Fixed point theorems on abstract convex spaces

The unified fixed point theory in generalized convex uniform spaces in [31] can be extended to abstract convex uniform spaces or KKM uniform spaces; see [38]. In this section, we introduce results on two typical multimap classes $\mathfrak{KC}$ and $\mathfrak{B}$.

We have the following Horvath type fixed point theorem [14]:

Theorem 5.1 ([38]). *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, and $F \in \mathfrak{KC}(E, E)$ be a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost* *fixed point property* (*that is, for any* $V \in \mathcal{U}$, $F|_X$ *has a* V*-fixed point* $x_V \in X$ *satisfying* $F(x_V) \cap (x_V + V) \neq \emptyset$).

Further if $(E, \mathcal{U})$ *is Hausdorff and if* F *is closed, then it has a fixed point.*

From Theorem 5.1, as in [36], we have the following generalization of the Schauder-Tychonoff-Hukuhara fixed point theorem:

Corollary 5.2. *Let* $(E, D; \Gamma; \mathcal{U})$ *be an abstract convex Hausdorff uniform space satisfying the partial KKM principle. If* $f : E \to E$ *is a continuous function such that* $\overline{f(E)}$ *is a compact* Φ*-set in* E, *then* f *has a fixed point.*

Note that Theorem 5.1 and Corollary 5.2 contain a large number of known fixed point theorems since there are so many Φ-sets.

Corollary 5.3. *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be an abstract convex Hausdorff uniform space and* $F \in \mathfrak{KC}(E, E)$ *be a closed compact map. If every singleton of* X *is* Γ*-convex and* $F(X)$ *is of the Zima-Hadžić type, then* F *has a fixed point.*

Note that most of results and examples in the preceding section are consequences of Corollary 5.3.

Definition 5.4. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This definition works for G-convex spaces or ϕ_A-spaces.

We have the following main result in this section:

Theorem 5.5 ([38])**.** *Let* $(E, D; \Gamma; \mathcal{U})$ *be an abstract convex uniform space,* $X \subset Y$ *subsets of* E, *and* $F : Y \multimap Y$ *a map such that* $F|_X \in \mathfrak{B}(X, Y)$ *and* $F(X)$ *is Klee approximable into X. Then F has the almost fixed point property.*

Further if $(E, \mathcal{U})$ *is Hausdorff,* F *is closed, and* $\overline{F(X)}$ *is compact in* Y, *then* F *has a fixed point* $x_0 \in Y$ (*that is,* $x_0 \in F(x_0)$).

In [31] and others, we gave some of our previous results which are direct consequences of Theorem 5.5 as follows. From now on, for simplicity, all topological spaces are assumed to be *Hausdorff* unless explicitly stated otherwise.

Since 1992, we introduced and supplied a lot of examples of the admissible class $\mathfrak{A}_c^\kappa$ of multimaps in [22, 23].

In 1993 [22] and 1994 [23], we obtained the following with a different method:

Corollary 5.6 ([22]). *Let X be a compact convex subset of a t.v.s. E on which its dual E^* separates points. Then any map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

Corollary 5.7 ([23]). *Let X be a convex subset of a locally convex t.v.s. E and $F \in \mathfrak{A}_c^\sigma(X,X)$. If F is compact, then it has a fixed point.*

In 1997, the author introduced the 'better' admissible class $\mathfrak{B}$ of multimaps. We noticed that $\mathfrak{A}_c^\kappa \subset \mathfrak{B}$.

Corollary 5.8 ([24]). *Let X be a convex subset of a locally convex t.v.s. E. Then every compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Corollary 5.9 ([25]). *Let E be a t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Moreover, in [25], we listed more than sixty papers in chronological order, from which we could deduce particular forms of Corollary 5.9.

The following form of the main theorem of [29] in 2004 follows from Theorem 5.5:

Corollary 5.10 ([29]). *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

In case $X = Y = E$, Theorem 5.5 reduces to the following main theorem of [31] in 2007:

Corollary 5.11 ([31]). *Let $(X,D;\Gamma;\mathcal{U})$ be a G-convex uniform space and $F \in \mathfrak{B}(X,X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*

Further if F is closed and compact, then F has a fixed point.

Corollary 5.11 contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces. Such subclasses are those of admissible spaces, Φ-spaces, sets of the Zima-Hadžić type, locally G-convex spaces, and LG-spaces; see [31]. Mutual relations among those subclasses and some related results on approximable maps, Kakutani maps, acyclic maps, Φ-maps, and others are investigated in [31]

The following two results are simple consequences of Corollary 5.11:

Corollary 5.12. *Let $(X,D;\Gamma;\mathcal{U})$ be an admissible G-convex space. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

Corollary 5.13. *Let $(X,D;\Gamma;\mathcal{U})$ be a compact admissible G-convex space. Then any map $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.*

The following is a consequence of Theorem 5.5:

Corollary 5.14. *Let X and Y be subsets of a t.v.s. E such that $X \subset Y$ and $F : Y \multimap Y$ a map.*

(1) *If $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X, then $F|_X$ has the almost fixed point property.*

(2) *Further if F is closed and $F|_X$ is compact, then F has a fixed point.*

Note that, in (1), E is not necessarily Hausdorff. Corollary 5.14 would be better than [32, Theorem 2.2]. In [32], it should be $\mathfrak{B} = \mathfrak{B}^p$.

Finally, recall that there are several hundred published works on the KKM theory and analytical fixed point theory; and we can cover only an essential part of them. For the more historical background for the related fixed point theory, the reader can consult with [26]. For more involved or generalized versions of the results in this paper, see [20, 23, 32] for convex spaces, [11–14, 41] for H-spaces, [26–29, 31, 42] for G-convex spaces, and [33–40] for abstract convex spaces and references therein.

Added in Proof. Recently, we published more detailed basic KKM theory in [43].

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