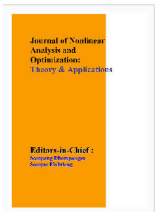

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A variant of the Nash equilibrium theorem in generalized convex spaces

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ABSTRACT: The existence theorem of pure-strategy Nash equilibrium due to H. Lu [Economics Letters 94 (2007) 459–462] is extended to generalized convex spaces. Consequently, our version can be applied to a broad class of abstract strategy spaces.

KEYWORDS: Generalized (G-) convex space; H-space; Acyclic map.

1. Introduction

In 1928, J. von Neumann obtained his celebrated minimax theorem, which is one of the fundamental results in the theory of games developed by himself.

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem [1,2] on equilibrium points of non-cooperative games. The following formulation is given by Fan [3, Theorem 4]:

Theorem 1.1. [3] *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j \neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

The original form of this theorem in [1,2] was for Euclidean spaces and its proofs were based on the Brouwer or Kakutani fixed point theorem. Since then there have appeared numerous generalizations and applications; see [4] and the references therein. Recently, H. Lu [5] obtained an existence theorem of pure-strategy Nash equilibrium where player's pure strategy spaces are topological vector spaces.

In the present paper, we show that such strategy spaces can be replaced by generalized convex spaces or G-convex spaces which are quite well-known in the fixed point theory and the KKM theory. Consequently, we obtained a very general version of Lu's existence theorem and our version can be applied to a broad class of abstract strategy spaces.

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Sections 2 and 3 are preliminaries on generalized convex spaces and fixed points of compositions of acyclic maps due to the present author. In Section 4, we give our main result which generalize Lu's theorem to G-convex spaces. Finally, we introduce some related generalizations of the Nash theorem.

2. Generalized convex spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [6-9]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$.

For details on G-convex spaces, see [10-15], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

Example 2.3. The original KKM theorem is for the triple $(\Delta_n \supset V; \mathrm{co})$, where V is the set of vertices and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation. This triple can be regarded as $(\Delta_n, N; \Gamma)$, where $N := \{0, 1, \dots, n\}$ and $\Gamma_A := \mathrm{co}\{e_i \mid i \in A\}$ for each $A \subset N$.

Example 2.4. Fan's celebrated KKM lemma is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E.

Example 2.5. A *convex space* $(X \supset D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Example 2.6. If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a C-space (or an H-space) due to Horvath. The hyperconvex metric spaces due to Aronszajn and Panitchpakti are examples of C-spaces.

Example 2.7. For other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, any continuous image of a G-convex space, L-spaces and B'-simplicial convexity due to Ben-El-Mechaiekh et al., Takahashi's convexity in metric spaces, Kulpa's simplicial structures, generalized H-spaces of Verma or Stachó, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares, FC-spaces of Ding, GFC-spaces of Khahn et al., and others.

Let $\{(X_i, D_i; \Gamma_i)\}_{i \in I}$ be any family of G-convex spaces. Let $X := \prod_{i \in I} X_i$ be equipped with the product topology and $D := \prod_{i \in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D \rangle$, define $\Gamma_A := \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then the following is known:

Lemma 2.8. *$(X, D; \Gamma)$ is a G-convex space.*

Definition 2.9. Let E be a topological space and $(X, D; \Gamma)$ a G-convex space. A multimap $T : E \multimap X$ is called a *Φ-map* provided that there exists a multimap $S : E \multimap D$ satisfying
(a) for each $z \in E$, $M \in \langle S(z) \rangle$ implies $\Gamma_M \subset T(z)$; and
(b) $E = \bigcup \{\operatorname{Int} S^-(y) \mid y \in D\}$.

A *continuous selection* $f : E \to X$ of a map $T : E \multimap X$ is a continuous function such that $f(z) \in T(z)$ for all $z \in E$.
The following is given in [12]:

Lemma 2.10. *Let E be a Hausdorff space, $(X, D; \Gamma)$ a G-convex space, and $T : E \multimap X$ a Φ-map. Then for any nonempty compact subset K of E, $T|_K$ has a continuous selection $f : K \to X$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D \rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A p$ for some $A \in \langle D \rangle$ with $|A| = n + 1$.*

From now on, we consider only G-convex spaces $(X, D; \Gamma)$ satisfying $X \supset D$.

3. Fixed points of compositions of acyclic maps

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a topological vector space, convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely in each stage.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is called an *acyclic map* whenever F is u.s.c. with compact acyclic values.

In the proof of the main result of this paper, as in [5], we can apply a fixed point theorem due to Gorniewicz. But there are more general fixed point theorems on compositions of acyclic maps.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following theorems are only few examples of our previous works; for more general results, see [10,14,16].

Theorem 3.1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

A nonempty subset X of a topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.
It is well-known that every nonempty convex subset of a locally convex Hausdorff topological vector space is admissible. Other examples of admissible topological vector spaces are ℓ^p, $L^p(0, 1)$, H^p for $0 < p < 1$, and many others; see [10,14,16] and references therein.

Theorem 3.2. *Let E be a Hausdorff topological vector space and X an admissible convex subset of E. Then any compact map $T \in \mathbb{V}_c(X, X)$ has a fixed point.*

4. Existence of pure-strategy Nash equilibrium

We follow [5]. Let $I := \{1, \ldots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple $\Lambda := \{X_1, \ldots, X_n; u_1, \ldots, u_n\}$, where the nonempty set X_i

is the ith player's pure strategy space and $u_i : X := \prod_{i=1}^n X_i \to \mathbb{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Let $X_{-i} := \prod_{j\in I\setminus\{i\}} X_j$ and denote by x and x_{-i} an element of X and X_{-i}, resp. A strategy n-tuple $(x_1^*, \ldots, x_n^*)$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*) \text{ for all } y_i \in X_i \text{ and } i \in I.$$

As in [17], we define an aggregate payoff function $U : X \times X \to \mathbb{R}$ as follows:

$$U(x,y) := \sum_{i=1}^{n} [u_i(y_i, x_{-i}) - u_i(x)] \text{ for any } x = (x_i, x_{-i}), y = (y_i, y_{-i}) \in X.$$

The following is given in [5, Proposition 1]:

Lemma 4.1. *Let Λ be a non-cooperative game, K a nonempty subset of X, and $x^* = \{x_1^*, \ldots, x_n^*\} \in K$. Then the following are equivalent:*
(a) *x^* is a Nash equilibrium;*
(b) *$\forall i \in I$, $\forall y_i \in X_i$, $u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*)$;*
(c) *$\forall y \in X$, $U(x^*, y) \leq 0$.*

Note that (c) implies $U(x^*, y) \leq 0$ for all $y \in D \subset X$.

Recall that a real-valued function $f : X \to \mathbb{R}$ on a topological space is *lower* [resp., *upper*] *semicontinuous* (*l.s.c.*) [resp., *u.s.c.*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$ is open for each $r \in \mathbb{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is convex for each $r \in \mathbb{R}$.

Now we have our main result:

Theorem 4.2. *Let $I = \{1, \ldots, n\}$ be a set of players, $(X, D; \Gamma) = \prod_{i=1}^n (X_i, D_i; \Gamma_i)$ a Hausdorff product G-convex space, K a nonempty compact subset of X, and Λ a non-cooperative game. Suppose that*

(i) *the function $U : X \times X \to \mathbb{R}$ satisfies that*

$$\{(x,y) \in X \times X \mid U(x,y) > 0\}$$

is open;

(ii) *for each $x \in K$, $\{y \in X \mid U(x,y) > 0\}$ is Γ-convex [that is, $M \in \langle\{y \in D \mid U(x,y) > 0\}\rangle$ implies $\Gamma_M \subset \{y \in X \mid U(x,y) > 0\}$];*

(iii) *for each $y \in X$, the set $\{x \in K \mid U(x,y) \leq 0\}$ is acyclic.*

Then there exists a point $x^* \in K$ such that x^* is an equilibrium point for the non-cooperative game.

Proof. Suppose the conclusion does not hold. Then, by Lemma 4.1, for each $x \in K$, there exists a point $y \in D$ such that $U(x,y) > 0$. We define two multimaps $S : K \multimap D$ and $T : K \multimap X$ as follows:

$$T(x) := \{y \in X \mid U(x,y) > 0\} \text{ and } S(x) := \{y \in D \mid U(x,y) > 0\}$$

for each $x \in K$. Then each $T(x)$ is nonempty and, for each $x \in K$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$ by (ii). Moreover, for each $x \in K$, there exists $y \in D$ such that $x \in S^-(y) = \{x \in K \mid U(x,y) > 0\}$. Note that this $S^-(y)$ is open since $S^-(y)$ is homeomorphic to

$$\{(x,y) \in K \times \{y\} \mid U(x,y) > 0\} = \{(x,y) \in X \times Y \mid U(x,y) > 0\} \cap (K \times \{y\}).$$

This is relatively open in $K \times \{y\}$ which is homeomorphic to K.

Therefore $T : K \multimap X$ is a Φ-map on the compact subset K of X and, by Lemma 2.2, has a continuous selection $f : K \to X$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D\rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$ for some $A \in \langle D\rangle$ with $|A| = n+1$.

Here we define a multimap $F : X \multimap K$ by

$$F(y) := \{x \in K \mid U(x,y) \leq 0\} \text{ for } y \in X.$$

Then, by (i), $\{(x,y) \mid U(x,y) \le 0\}$ is closed in $X \times X$ and hence

$$\mathrm{Gr}(F) := \{(x,y) \mid U(x,y) \le 0\} \cap (X \times K)$$

is closed in $X \times K$ as the intersection of two closed sets. Hence F is a closed compact map with acyclic values by (iii) and hence an acyclic map. Then it is well-known that $pF\phi_A : \Delta_n \multimap \Delta_n$ has a fixed point $a_0 = pF\phi_A(a_0)$; see Theorem 3.2. Let $y_0 := \phi_A(a_0) \in \Gamma_A \subset X$. Then $y_0 = \phi_A(a_0) \in \phi_A pF(y_0) = fF(y_0)$ and hence $y_0 = f(x_0)$ for some $x_0 \in F(y_0) \subset K$, that is, $U(x_0, y_0) \le 0$.

On the other hand, $x_0 = f(y_0) \in T(y_0)$ since f is a selection of T. Then, by the definition of T, we have $U(x_0, y_0) > 0$, which is a contradiction. □

Remark. Note that condition (i) can be replaced by one of the following:
(i)$'$ the function $U(x,y)$ is lower semicontinuous on $X \times X$.

(i)$''$ $\forall i \in I$, the function $u_i : X \to \mathbb{R}$ is continuous.

For the case (i)$'$, when $X = D$ is a topological vector space, Theorem 4.1 reduces to [5, Theorem 1]. Note that Nash's original theorem is a simple consequence of Theorem 4.1 under the case (i)$''$.

5. Other Nash type theorems

There are a large number of generalizations of the Nash theorem based on fixed point theorems. For example, based on a generalization of the Kakutani fixed point theorem due to Fan [18] and Glicksberg [19], certain generalizations of the Nash theorems were obtained; see [20,21].

Instead of the fixed point technique, we can apply the KKM theory. The first proof of the Nash theorem by the KKM method was given by Fan [3]. Applying the KKM method, we obtained some of the most general forms of the Nash theorem as follows:

Theorem 5.1. [22] *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*
(0) *$f_i(x) \le g_i(x)$ for each $x \in X$;*
(1) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on X_i;*
(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on X_i; and*
(3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*
Then there exists a point $\widehat{x} \in X$ such that

$$g_i(\widehat{x}) \ge \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad for\ all\ i = 1, 2, \ldots, n.$$

Theorem 5.2. [23] *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying $(0)-(3)$. Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \ge \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad for\ all\ \ i \in I.$$

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