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SEVERAL EPISODES IN RECENT STUDIES ON THE KKM THEORY

Sehie Park

The National Academy of Sciences, ROK, Seoul 137-044; and
Department of Mathematical Sciences
Seoul National University
Seoul 151–747, Korea
E-mail : shpark@math.snu.ac.kr

Abstract. Recently there have appeared a very large number of papers on the KKM theory. In this review, we select several of them and give comments; namely, Lin et al. [21], Chen et al. [4,5], Wen [44,45], Fang et al. [9], Hou [13], Xiang et al. [46,47], Gonzáles et al. [3,10], Kulpa et al. [18], and Park [35].

1. Introduction

In the last decade, there have appeared a very large number of papers on the KKM theory. Since the appearance of generalized convex (simply, G-convex) spaces in 1993, the concept has been challenged by several authors who aimed to obtain more general concepts. In fact, a number of modifications or imitations of the concept has followed; for example, L-spaces, spaces having property (H), FC-spaces, pseudo-H-spaces, another L-spaces, M-spaces, GFC-spaces, simplicial spaces, and others. It is known that most of such examples belong to the class of ϕ_A-spaces and are particular forms of G-convex spaces. Recently all of the above mentioned spaces are unified to the class of abstract convex spaces; see [32,39].

In our previous reviews [33-35], we presented quite critical comments on many of recent works on the KKM theory in order to improve the theory itself. Continuing this line, in the present review, we give comments on some other papers, namely, of Lin et al. [21], Chen et al. [4,5], Wen [44,45], Fang et al. [9], Hou [13], Xiang et al. [46,47], Gonzáles et al. [3,10], Kulpa et al. [18], and Park [35].

For the preliminaries, see [39,40] and references therein.

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2. Generalized convex spaces

In this section, we follow mainly [28-31,35] and references therein.

Definition. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of E and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J := \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A := \Gamma(A)$. When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

We add particular subclasses or variants of G-convex spaces as follows:

Definition. A *space X having a family* $\{\phi_A\}_{A \in \langle D \rangle}$ or simply a ϕ_A-*space*

$$(X, D; \{\phi_A\}_{A \in \langle D \rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$.

Proposition 2.1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Therefore, G-convex spaces and ϕ_A-spaces are essentially the same; see [35-38].

In the abstract of a recent paper [35], the author showed "that FC-spaces due to Ding are particular types of L-spaces due to Ben-El-Mechaiekh et al., and hence particular types of G-convex spaces. Some counter-examples are given and related matters are also discussed."

In fact, the paper begins with the following paragraph:

"Since the concept of generalized convex spaces (simply, G-convex spaces) in the KKM theory appeared in 1993 [27], a number of modifications or imitations have followed. Such examples are L-spaces due to Ben-El-Mechaiekh et al. [1], spaces having property (H) due to Huang, FC-spaces due to Ding, and others. In the present short note, we show that all of such examples are particular forms of G-convex spaces contrary to the routine claim of Ding that the class of FC-spaces contains L-spaces and G-convex spaces as true subclasses. We believe that reputed journals should clarify such incorrect statements in their publications."

We gave lots of examples of G-convex spaces $(X, D; \Gamma)$ in many of our previous works. But there are many peoples who do not know the proper role of D and the difference between $(X, D; \Gamma)$ and $(X; \Gamma)$. Here we give some proper examples of D listed in [35].

Exmaple 1. (1) The original KKM principle [16] is for the triple $(\Delta_n \supset V; \mathrm{co})$, where V is the set of vertices and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation. This triple can be regarded as $(\Delta_n, N; \Gamma)$, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_A := \mathrm{co}\{e_i \mid i \in A\}$ for each $A \subset N$.

(2) Fan's celebrated KKM lemma [8] is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E.

(3) A *convex space* $(X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [20] for $X = D$. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$.

(4) For some other examples, see [11,12,22].

These examples are the origins of our G-convex space $(X, D; \Gamma)$. Note that any KKM type theorem on $(X; \Gamma)$ can not generalize the KKM principle and the Fan lemma.

For the definition of a G-convex space, at first, we assumed $X \supset D$ and an additional condition that

$$(*) \qquad \text{for each } A, B \in \langle D \rangle,\ A \subset B \text{ implies } \Gamma_A \subset \Gamma_B.$$

This monotonicity was removed since 1998 in [26] and the restriction $X \supset D$ since 1999 in [27]; see also [28-31,42,43]. However, note that most of useful examples of G-convex spaces satisfy $(*)$, but, examples not satisfying $(*)$ seem to be artificial:

Exmaple 2. Let $\Delta_3 = \mathrm{co}\, V$ where $V = \{e_0, e_1, e_2, e_3\}$.

(1) In the KKM principle, $(*)$ holds.

(2) Let $(\Delta_3, V; \Gamma)$ be a G-convex space given by $\Gamma\{e_0, e_1\} := \mathrm{co}\{e_0, e_1, e_2\}$ and $\Gamma(N) := \mathrm{co}\, N$ for all other $N \in \langle V \rangle$. Then Γ violates the isotonicity $(*)$.

In order to give another justification of the necessity of using the triple $(X, D; \Gamma)$ instead of the pair (E, Γ), we give examples:

Exmaple 3. (1) The well-known Sperner theorem and Alexandorff-Pasynkoff's theorem on $n+1$ closed sets covering the n-simplex can be derived by applying the KKM principle to the triple $(\Delta_n, V; \mathrm{co})$; see [41] and references therein. No other proof of these theorems using a pair (E, Γ) appeared yet.

(2) In Shapley's generalization of the KKM principle, a triple $(\Delta_n, N; \Gamma)$ appears, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_S := \Delta^S = \mathrm{co}\,\{e_i \mid i \in S\}$ for each $S \in \langle N \rangle$; see [27] and references therein.

(3) Let $\mathcal{C} := \mathcal{C}[0,1]$ be the class of all real continuous functions on $[0,1]$ and $\mathcal{P} := \mathcal{P}[0,1]$ the subclass of all polynomials $p(x)$ on $x \in [0,1]$ with real coefficients. Let $\varepsilon > 0$. For each $f \in \mathcal{C}$, choose a fixed $p_f \in \mathcal{P}$ which is ε-near to f, that is, $\max_{x \in [0,1]} |f(x) - p_f(x)| < \varepsilon$. Let $\Gamma : \langle \mathcal{C} \rangle \multimap \mathcal{P}$ be defined by $\Gamma_A := \mathrm{co}\,\{p_{f_i}\}_{i=0}^n \in \mathcal{P}$ for each $A = \{f_i\}_{i=0}^n \in \langle \mathcal{C} \rangle$. Moreover, let $\phi_A : \Delta_n \to \Gamma_A$ be a linear map such that $e_i \mapsto p_{f_i}$. Then $(X, D; \Gamma) := (\mathcal{P}, \mathcal{C}; \Gamma)$ is a G-convex space satisfying condition $(*)$ and $X \subsetneq D$.

(4) Similarly, by choosing a proper subset D of $\mathcal{C}$, we can obtain G-convex spaces $(X, D; \Gamma)$ satisfying $X \not\subseteq D$ or $X \not\supseteq D$. This is why we assumed X and D are not comparable in general.

(5) Since there are various forms of the Stone-Weierstrass approximation theorem, we can construct a large number of examples similar to the ones in (3) or (4).

(6) For a closed convex subset X of a complete $\mathbb{R}$-tree M, and $\Gamma_A := conv_M(A)$ for each $A \in \langle X \rangle$, the triple $(M \supset X; \Gamma)$ satisfies the partial KKM principle; see Kirk and Panyanak [15]. Later we found that $(M \supset X; \Gamma)$ is a KKM space.

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of the origin 0 of E and for any finite subset $\{y_1, y_2, \dots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \dots, z_n\}$ of Y, such that $z_i - y_i \in V$ for each $i = 1, \dots, n$, and $\mathrm{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

Exmaple 4. Let Y be an almost convex dense subset of a subset D of E. Let V be a given neighborhood of 0. For each $A := \{x_0, x_1, \dots, x_n\} \in \langle D \rangle$, choose a subset $B := \{y_0, y_1, \dots, y_n\} \in \langle Y \rangle$ such that $y_i - x_i \in V$ for each $i = 0, 1, \dots, n$ and $\mathrm{co}\, B \subset Y$. Define a continuous function $\phi_A : \Delta_n \to \mathrm{co}\, B$ given by

$$\phi_A : u = \sum_{i=0}^{n} \lambda_i(u) e_i \mapsto \phi_A(u) := \sum_{i=0}^{n} \lambda_i(u) y_i$$

for $u \in \Delta_n$. Then $(Y, D; \{\phi_A\}_{A \in \langle D \rangle})$ can be made into a G-convex space. Note that $Y \subset D$.

Moreover, we give an interesting example of G-convex spaces in the following generalization of Fan's KKM lemma:

Lemma ([Yuan [48], p.6]). *Let X and Y be nonempty sets in a t.v.s. E and let $F : X \multimap Y$ be such that*

(a) *for each $x \in X$, $F(x)$ is closed in Y;*

(b) *for each $A \in \langle X \rangle$, $\mathrm{co}\, A \subset F(A)$; and*

(c) *there exists $x \in X$ such that $F(x)$ is compact.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Note that $\mathrm{co}\, A \subset F(A) \subset Y$ and hence $X \subset Y$. Moreover, $(Y \supset X;\ \mathrm{co})$ is a G-convex space.

3. Lin-Yao's pseudo H-spaces [19,21]

In 2003 [21], its authors introduced the following:

Definition 1 ([21]). Let X be a topological space, D be a nonempty set. The triple (X, D, q) is said to be a pseudo H-space if for each nonempty finite subset A of D, the restricted mapping $q : \Delta^{|A|-1} \to 2^X$ is upper semi-continuous with nonempty compact values, where $\Delta^{|A|-1}$ is an $(|A|-1)$-simplex with vertices $\{e_1, e_2, \dots, e_{|A|}\}$. If $D = X$, the triple (X, D, q) is written by (X, q).

Its authors incorrectly observed that a G-convex space $(X, D; \Gamma)$ with $|D| < \infty$ is an example of pseudo H-space and gave no other proper example. Therefore, they might obtain some statements on their spaces, but it seems to be not practical.

Now, by defining $\Gamma_A := q(\Delta^{|A|-1})$ for each nonempty finite subset A of D, then (X, D, q) can be an abstract convex space $(X, D; \Gamma)$ defined by Park in 2006.

Therefore the basic theorems in the recently developed abstract convex space theory can be applied.

Moreover, the following is given:

Definition 2 ([21]). Let (X, D, q) be a pseudo H-space. A mapping $F : D \to 2^X$ is a q-map if for each nonempty finite subset A of D, $q(\Delta^{|A|-1}) \subset \bigcup_{x\in A} F(x)$ and $q(\Delta^{|J|-1}) \subset \bigcup_{x\in J} F(x)$ for all nonempty finite subset J of A, where $\Delta^{|J|-1}$ is the convex hull of $\{e_{i_1}, e_{i_2}, \dots, e_{i_{k+1}}\}$ if $A = \{a_1, a_2, \dots, a_{n+1}\}$, $J = \{a_{i_1}, a_{i_2}, \dots, a_{i_{k+1}}\}$.

The authors of [21] then showed that, under a strong restriction, a q-map can be a KKM map and that $1_X \in \mathfrak{KC}(X, X)$. From this KKM theorem, they deduced routine intersection result, Fan-Browder type fixed point theorems, a selection theorem, a Ky Fan type minimax inequality, and an application to abstract economies. However, they fail to give any proper example of their space which is not a G-convex space.

Later, the authors of [19] incorrectly stated that if the map q is single-valued and we set $\Gamma(A) = q(\Delta^{|A|-1})$ for each nonempty finite subset A of X, then (X, D, Γ) forms a G-convex space. Note also that their Example 1 can not be an example of their space.

4. C.-M. Chen et al. [4,5]

In 2005 [5], its authors introduced a new family of maps $Q(X, Y)$, studied its properties, and obtained some fixed point theorems about this family. They defined as follows:

"Let X be a subset of a Hausdorff t.v.s. E and Y a Hausdorff t.v.s., we define a new class $Q(X, Y)$ of set-valued maps from X into Y as follows. $T \in Q(X, Y)$ implies that for any compact convex subset K of X and any continuous function $f : T(K) \to K$, the composition $f(T|_K) : K \to 2^K$ has a fixed point.

Subclasses of $Q(X, Y)$ are the class of continuous functions $C(X, Y)$, the class of the Kakutani maps $K(X, Y)$ (with convex values and codomains being convex spaces), the class of the acyclic maps $V(X, Y)$ (with acyclic values), and the class of the approachable maps $\mathcal{A}_0(X, Y)$ (whose domains and codomains are subsets of t.v.s.), and so forth."

This is clearly an incorrect imitation of the class $\mathfrak{B}$ due to S. Park; see [24,25,27, and many others] and references therein. Such imitations also appear in papers of Ding and others.

To the best of our knowledge, at present, *no subclass of* $Q(X, Y)$ *is known.* If the Schauder conjecture holds, then $C(X, Y) \subset Q(X, Y)$, where C denotes the class of continuous functions. (A recent resolution of the conjecture turns out to have gaps in its proof.) Moreover, it is regretful to find no new correct result in [5].

In 2007 [4], Chen introduced a generalized *R-KKM* mapping and a new class R_{ψ_N}-$KKM(X, Y)$, and claimed some fixed point theorems, matching theorems, coincidence theorems, and minimax inequalities on L-convex spaces.

Since 1998, the present author has worked with the concept of generalized convex (simply, G-convex) spaces $(X, D; \Gamma)$, where D is a nonempty set. When D is a subset of X, such a space is called an L-convex space in [4], where it is incorrectly stated that L-convex spaces were introduced by H. Ben-El-Mechaiekh, P. Deguire and A.

Granas [2]. Recall that particular forms of G-convex spaces $(X;\Gamma) = (X, X;\Gamma)$ are called L-spaces by H. Ben-El-Mechaiekh et al. [1].

The concept of G-convex spaces is used only in the definition of a generalized *R-KKM* mapping; all of the other usages of L-convex spaces in this paper are simply for L-spaces in the sense of Ben-El-Mechaiekh et al. [1]. Note that Chen's generalized *R-KKM* mappings are different from those of other authors. Because of this and the incorrect application of the KKM theorem, Theorems 1-3 of [3] cannot be considered as proved. Moreover, in the definition of the class R_{ψ_N}-$KKM(X, Y)$, the role of G-convex spaces (the author's L-convex spaces) is not clear. This remark also applies to Theorems 4-23 [4].

Chen's aim in this paper seems to generalize some well-known results of the KKM theory in his own setting, but without giving any proper examples or any suggested applications. Moreover, note that a number of references are obsolete or incorrect.

5. Wen [44,45]

In 2006 [44], a Fan type matching theorem for transfer compactly open covers of a hyperconvex metric space is given and used to obtain a Fan-Browder coincidence theorem, a Fan-type best approximation theorem, and a Brouwer-Schauder-Rothe-type fixed point theorem.

All of the results in this paper are modifications, using an artificial terminology, of known results. The author frequently claims that his results improve or generalize corresponding ones of other authors without giving any proper concrete example.

In 2008 [45], its author claims that a new KKM theorem is established in L-convex spaces and applied to a matching theorem, Fan-Browder type coincidence or fixed point theorems, a maximal element theorem, and some equilibrium existence theorems for economics and games.

Here, its author's L-convex spaces is the L-spaces due to Ben-El-Mechaiekh et al. [1]. Wen's misconception that L-spaces contain G-convex spaces caused such repetitions of particular cases of already well-known results.

6. Fang et al. [9]

In this section, motivated by Fang and Huang [9], we give another example of spaces satisfying the partial KKM principle (that is, an abstract form of the KKM theorem):

Definition. A Φ_A-*space*

$$(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of l.s.c. maps $\Phi_A : \Delta_n \multimap X$ for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Note that any Φ_A-space is an abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \mathrm{Im}\,\Phi_A$ for $A \in \langle D\rangle$; see [32-34].

Remark. We give examples of Φ_A-spaces and similar ones:

1. Every ϕ_A-space is a Φ_A-space. Hence G-convex spaces and the so-called FC-spaces are Φ_A-spaces.

2. A similar concept adopting u.s.c. maps with nonempty compact values instead of l.s.c. maps is called a *pseudo H-space* in [21]; see Section 3.

3. In [9], the concept of Φ_A-spaces are implicitly given without any practical examples.

Definition. For a Φ_A-space $(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\Phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Remark. This KKM map is called a *generalized L-KKM map* in [9], where it is said to contain the so-called generalized R-KKM maps. The author of [7] claimed as follows: "The class of generalized *R-KKM* mappings includes those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses."

Proposition 6.1. *A KKM map $T : D \multimap X$ on a Φ_A-space $(X, D; \{\Phi_A\})$ is a KKM map on a new abstract convex space $(X, D; \Gamma^T)$.*

Proof. Define $\Gamma^T : \langle D\rangle \multimap X$ by $\Gamma^T_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma^T)$ becomes an abstract convex space. Note that $\Gamma_A \subset T(A)$ for each $A \in \langle D\rangle$ and hence $T : D \multimap X$ is a KKM map on the abstract convex space $(X, D; \Gamma^T)$.

The following is a KKM theorem for Φ_A-spaces. The proof is just a simple modification of the corresponding one in [23,28,32]:

Theorem 6.2. *For a Φ_A-space $(X, D; \{\Phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n + 1$, we have $\Phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

$(*)$ $\bigcap_{z\in M} G(z)$ is compact for some $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} G(z) \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\Phi_N(e_i) \subset G(z_i)$ for $0 \le i \le n$. Then $e_i \mapsto \Phi_N^{-}G(z_i)$ is a closed valued map since Φ_N is l.s.c. Moreover, $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \Phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM theorem, $\Delta_n \supset \bigcap_{i=0}^{n} \Phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\Phi_N(\Delta_n) \cap \big(\bigcap_{z\in N} G(z)\big) \neq \emptyset$.

The second conclusion is clear.

7. Hou [13]

In 2007 [13], Hou Jicheng claimed that some recent results on KKM theorems in Park and Kim [43] and Kalmoun and Rihai [14] are false by giving a counterexample. We show that the counterexample in [13] does not work and hence Hou's claim is false. Consequently, the results are correct.

In this section, as in [13] all topological spaces are assumed to be Hausdorff.

In [6], Ding introduced the following notions. Let X and Y be two topological spaces. A multimap $G : X \multimap Y$ is said to be *transfer compactly closed-valued* [resp., *transfer compactly open-valued*] on X if for every $x \in X$ and each nonempty compact subset K of Y, $y \notin Gx \cap K$ [resp., $y \in Gx \cap K$] implies that there exists a point $x' \in X$ such that $y \notin \mathrm{cl}_K(Gx' \cap K)$ [resp., $y \in \mathrm{int}_K(Gx' \cap K)$], where $\mathrm{cl}_K(Gx' \cap K)$ and $\mathrm{int}_K(Gx' \cap K)$ is the closure and the interior of $Gx' \cap K$ in K, resp. Note that $\mathrm{cl}_K(Gx' \cap K) = \overline{Gx' \cap K}$.

The following in [43, Theorem 3] is the main target of Hou [13]:

Theorem 7.1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$ an admissible map. Let $G : D \multimap Y$ be a map such that*

(1) for each $x \in D$, Gx is transfer compactly closed in Y;

(2) for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset \overline{G}(N)$; and

(3) there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{\overline{G}x : x \in L_N \cap D\} \subset K$.

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Hou [13] claimed that the proofs of Theorem 7.1 and three of its variants due to Kalmoun and Rihai [14] are not correct. However, he failed to indicate which part of the proofs are incorrect.

Instead, Hou [13] gave a counterexample showing the above theorems for the case F is a constant map which belongs trivially to the admissible class $\mathfrak{A}_c^\kappa(X, X)$.

In order to show that Hou's example does not work, we note the following well-known fact:

Lemma 7.2. *Let X and Y be two topological spaces. A multimap $G : X \multimap Y$ is transfer compactly closed-valued if and only if for any compact subset K of Y, we have*

$$K \cap \bigcap_{x \in X} Gx = K \cap \bigcap_{x \in X} \overline{Gx}.$$

We show that Hou's counterexample does not work: In his example, X is a convex subset of a t.v.s. E containing the origin θ, $K = \{\theta\}$, $F : X \multimap X$ is a constant map $F(x) = \{\theta\}$, and a map $G : X \multimap X$ is defined. He claimed G is transfer compactly closed-valued. He also showed $\bigcap_{x \in X} Gx = \emptyset$, the empty set, and $\bigcap_{x \in X} \overline{Gx} = \{\theta\}$. Hence we have

$$\emptyset = \overline{F(X)} \cap K \cap \bigcap_{x \in X} Gx \neq \overline{F(X)} \cap K \cap \bigcap_{x \in X} \overline{Gx} = \{\theta\},$$

which shows that G is not transfer compactly closed-valued by Lemma 7.2 since $\overline{F(X)} \cap K$ is compact. This is a contradiction.

From the beginning G did not satisfy condition (1) of Theorem 7.1.

Remark. (1) In the definition of transfer compactly closed-valuedness of a map $G : X \multimap Y$, if we replace the topology of Y by its compactly generated extension, then G has simply transfer closedness. Therefore "compactly" can be easily eliminated and does not generalize anything.

(2) There are large number of papers treating equivalent conditions of the transfer compactly closedness or some similar concepts. If we choose Lemma 7.2 instead of the original definition of Ding [6], then we have more clear situation and could more easily destroy such unnecessary papers like [13].

8. Xiang et al. [46,47]

In [46] and [45], their authors claimed some relationships among the abstract convexity, the selection property, and the fixed point property. They showed that if a convexity structure $\mathcal{C}$ defined on a topological space has the selection property [resp., the weak selection property] then $\mathcal{C}$ satisfies the H-condition [resp., H_0-condition]. Moreover, they showed that, in a *l.c.* compact metric space, the selection property implies the fixed point property. Note that their terminology has their own meaning.

In this section, our aim is to show that all results in [46,47] are either consequences of known ones or can be stated in more general forms in the frame of G-convex spaces.

In [46,47], for a pair $(Y, \mathcal{C})$, where Y is a topological space and $\mathcal{C}$ is a family of subsets of Y, $\mathcal{C}$ is called a *convexity structure* if

(1) the empty set $\emptyset$ is in $\mathcal{C}$;

(2) $\mathcal{C}$ is stable for intersections, that is, if $\mathcal{D} \subset \mathcal{C}$ is nonempty, then $\bigcap_{A \in \mathcal{D}} A$ is in $\mathcal{C}$.

It would be better to define a convexity structure as for a convexity space in the classical sense (that is, we assume $Y \in \mathcal{C}$ instead of $\emptyset \in \mathcal{C}$). Note that the case $\mathcal{C} = \{\emptyset\}$ will be meaningless since there would be the only convex subset $\emptyset$.

A pair $(X, \mathcal{C})$ is said to have the *weak selection property* with respect to a topological space S if every multivalued map $F : S \to 2^X$ admits a single-valued continuous selection whenever F has nonempty convex values and preimages relatively open in X. [Such F is usually called *a Browder map.*]

Its authors say that a pair $(Y, \mathcal{C})$ satisfies the H_0-*condition* if the convexity structure $\mathcal{C}$ has the following property:

(H_0) For each finite subset $\{y_0, \dots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \mathrm{conv}\{y_0, \dots, y_n\}$ such that $f(\Delta_J) \subset \mathrm{conv}\{y_j \mid j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \dots, n\}$.

Recall that any pair $(Y, \mathcal{C})$ satisfying the (H_0)-condition becomes a particular form $(Y; \mathrm{conv})$ of a G-convex space $(Y, D; \Gamma)$.

Its authors are mainly concerned with pairs $(Y, \mathcal{C})$ satisfying the (H_0)-condition. Therefore all results in [46] are consequences of the G-convex space theory. Moreover, in the end of [46], the authors incorrectly state that the G-convexity structure (due to Park) is a pair $(Y, \mathcal{C})$ satisfying the (H_0)-condition.

Moreover, recall that H. Komiya [17] already obtained almost same results to this paper. Moreover, it can be shown that all results in [46,47] are either consequences of known ones or can be stated in more general forms in the frame of G-convex spaces.

9. Gonzáles et al. [3,10]

In 2008, Cain and González [3] considered relationship among some subclasses of the class of G-convex spaces and introduced a subclass of new L-spaces. In [3, Theorems 3.2 and 3.4], it was shown that their L-spaces are G-convex spaces.

We can consider a ψ_A-space $(X, D; \{\psi_A\}_{A\in\langle D\rangle})$, similar to a ϕ_A-space, where $\psi_A : [0,1]^n \to X$ is continuous for each $A \in \langle D\rangle$ with $|A| = n+1$. Such types of spaces are given by Michael, Llinares, and Cain and Gonzáles [3]. For each $n \geq 0$, considering continuous functions $g_n : \Delta_n \to [0,1]^n$ given by

$$g_n : u = \sum_{i=0}^{n} \lambda_i(u)e_i \mapsto (\lambda_0(u), \ldots, \lambda_{n-1}(u))$$

for $u \in \Delta_n$ and by putting $\phi_A := \psi_A g_n$, a ψ_A-space becomes a ϕ_A-space.

In [10], it is repeated to show that G-convex spaces and L-spaces satisfy the partial KKM principle. They added that L-spaces satisfy the properties of the Fan type minimax inequality, Fan-Browder type fixed point, and the Nash type equilibrium. All of such results are already known for more general G-convex spaces.

From the introduction: "New versions and generalizations of the KKM theorem are created to help obtain some kind of abstract convexities in different topological spaces. L-spaces were introduced in [3] as an alternative simpler version of the so-called MC-space. Using a KKM theorem in the context of L-spaces, we obtain a Ky Fan inequality that allows us to prove existence of fixed points. Then as a direct applications, existence of Nash equilibria in compact L-spaces is also proved."

In Section 2, its authors present some KKM theorems for G-spaces. Then they used these theorems to obtain similar results for L-spaces. Their G-space is the G-convex space due to S. Park and H. Kim [42]. In the first author's previous cowork [3, Theorems 3.2 and 3.4], it was shown that L-spaces are G-convex spaces. Recall that there are plenty of literature devoting to KKM theorems for G-convex spaces.

It is routine to deduce a Fan type minimax inequality, a Browder type fixed point theorem, and a Nash type equilibrium theorem from a KKM type theorem. This was done in Section 3 (Theorems 15-17) for compact L-spaces. However, this is already known for more general G-convex spaces and other spaces; for example; see S. Park [30,31].

10. Simplicial spaces of Kulpa and Szymanski [18]

Definition ([18])**.** A collection $\mathcal{S}$ of singular simplexes in a topological space X is called a simplicial structure on X if: (a) For any finite subset $\{a_0, a_1, \ldots, a_n\}$ of (not necessarily distinct) points of X, there exists $\sigma \in \mathcal{S}$ such that $\sigma : [e_0, e_1, \ldots, e_n] \to X$ and $\sigma(e_i) = a_i$ for each $i = 0, 1, \ldots, n$; (b) If $\sigma \in \mathcal{S}$, then the restriction of σ to any face of the domain of σ belongs to $\mathcal{S}$.

A simplicial space $(X, \mathcal{S})$ is a topological space X together with a simplicial structure $\mathcal{S}$.

The simplicial spaces extend Bielawski's simplicial convexity and are particular to ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ for the case $X = D$.

Definition ([18])**.** Let $\mathcal{F} = \{F(x) \mid x \in X\}$ be a family of subsets of a simplicial space Y indexed by elements of a topological space X. We say that $\mathcal{F}$ is a convex open (simply, CO) family if $F(x)$ is a non-empty convex subset of Y for each $x \in X$ and $F^-(y)$ is an open subset of X for each $y \in Y$.

Recall that the corresponding multimap $F : X \multimap Y$ is usually called a Φ-map or a Fan-Browder map.

In Section 3 in [18], the existence of certain CO family is given (Theorem 5). Section 4 [18] deals with families of real functions with certain compatibility and their properties. In Section 5.1 [18], the authors derive two double-sided *Infimum Principles* (Finite Version and General Version) from Theorem 4. These are applied to the Tychnoff and Schauder fixed point theorems and a general minimax principle on simplicial spaces. Section 5.2 deals with one-sided Infimum Principle derived from Theorem 4. This principle is applied to generalizations of the Ky Fan minimax inequality and the Nash equilibrium theorem. In Section 5.3, the authors obtained another Infimum Principles related set-valued maps. Among a slew of possible consequences of the last Infimum Principles, the authors state only those of utmost significance, that is, the Kakutani fixed point theorem and the Gale-Nikaido-Debreu theorem.

Here the most simple form of Infimum Principles asserts that for continuous real functions $f, g : X \times Y \to \mathbb{R}$, where X and Y are compact convex subsets of topological vector spaces, if $f(x, \cdot)$ and $g(\cdot, y)$ are quasi-convex, then there exists a point (a, b) such that $f(a, b) = \inf_{x\in X} f(x, b)$ and $g(a, b) = \inf_{y\in Y} g(a, y)$ [18].

Remark on Infimum Principles. Although simplicial spaces and Theorems 3 and 4 are very particular ones in the KKM theory, the Infimum Principles in this paper are very deep results and applicable to generalize important results like fixed point theorems due to Schauder, Tychnoff, Kakutani, and Fan-Browder; minimax theorems; the Nash equilibrium theorem; the Gale-Nikaido-Debreu theorem; and the Ky Fan minimax inequality. Such method supplies surely a new scope in the KKM theory and is not comparable to previous methods mentioned in Section 3.

Finally, in Section 6 [18], its authors suggested a way of extending their results to a wider class of topological spaces that contains, in particular, the class of L-spaces due to Ben-El-Mechaiekh et al. [1] and defined an L^*-*structure* on a topological space X by means of a map $L : \langle X\rangle \multimap X$ that satisfies the following condition:

(**) If $A \in \langle X \rangle$ and $\{U_x \mid x \in A\}$ is an open cover of X, then there exists $B \subset A$ such that $L(B) \cap \{U_x \mid x \in B\} \neq \emptyset$.

Accordingly, the authors call (X, L) an L^*-*space*, and a non-empty subset Y of X to be L^*-*convex* if for each non-empty finite subset A of Y, $L(A) \subset Y$.

It follows from Theorem 3 on Indexed Families in [18] that any L-space is an L^*-space. The authors wrote that it is very conceivable that the converse statement is not true, though they have to admit they do not have any counterexample at this moment.

They also restated their Theorem 4 [18] in terms of L^*-spaces as Theorem 12. Finally, the authors stated that "Theorem 12 enables transferring some results of the paper [18] from *simplicial spaces* to L^*-spaces. In particular, it gives a possibility of eliminating singular simplexes from Nash's equilibrium theorem, and the Nash theorem becomes a set-theoretic schema. However this line of research is not going to be elaborated here."

Remarks on L^*-spaces.

(1) An L^*-*space* (X, L) is a particular case of a KKM-space $(E, D; \Gamma)$ for $X = E = D$ and $L = \Gamma$ in [39].

(2) Theorem 12 of [18] is a particular case of one of our results on KKM spaces. Therefore, according to Kulpa and Szymanski, our result enables transferring some results of [18] from simplicial spaces to KKM spaces.

(3) L-spaces due to Ben-El-Mechaiekh et al. [1] is a G-convex space which in turn a KKM space. To the best of our knowledge, the only known example of a KKM space which is not a G-convex space is the extended long line $(L^*, D; \Gamma)$ with the ordinal space $D := [0, \Omega]$; see the references in [39].

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